Real and complex projective spaces – 1

> Generations of mathematicians are growing up who are on the whole splendidly trained, but suddenly find that, after all, they do need to know what a projective plane is.
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> I. Kaplansky, *Linear Algebra and Geometry: A Second Course* (Dover, New York, 2003), p. *vii*.

Although many books on algebraic topology define and use projective spaces as examples to illustrate the basic constructions in the subject, usually the reasons for interest in such spaces are discussed to a very limited extent and their rich geometric structure receives little if any attention. This is certainly understandable because the time and space devoted to background material and other subjects must be limited. However, at certain points a little additional information on the underlying geometry of such spaces can be helpful for motivating some of the constructions in algebraic topology and making them easier to understand, and this document is an attempt to fill in some basic geometrical properties of projective spaces.

The obvious first question is how and why mathematicians started working with projective spaces, and the answer goes back to the theory of perspective drawing which was developed mainly during the 14th and 15th centuries. Among other things, perspective geometry provides a theoretical explanation of a basic empirical fact about vision; namely, when we see parallel lines it generally looks as if they meet at some point very far away on the horizon. This suggest the potential usefulness of thinking that parallel lines meet in some point at infinity.

(Sources: http://en.wikipedia.org/wiki/Projective_space http://en.wikipedia.org/wiki/Complex_projective_space)

One also expects that different families of parallel lines will meet at different points at infinity depending upon their directions. One can do this in a purely formal manner by adding one point at infinity to each line in a plane such that two lines have the same point at infinity if and only if the lines are parallel, and as in the citation below such a notational convention is sometimes very useful.

> Extending the space ... [is often a] fruitful method for extracting understandable results from the bewildering chaos of special cases: *projective geometry* and n – *dimensional geometry* paved the way for the modern concepts [upon which algebraic geometry is based].
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> J. Dieudonné, The *historical development of algebraic geometry*. American Mathematical Monthly **88** (1972), 827 – 866.

Given the importance of coordinates studying geometrical problems, it is natural to ask if one can find a reasonable extension of ordinary (so – called) Cartesian coordinates to points at infinity. The following description of the standard way of doing so is taken from the article **http://en.wikipedia.org/wiki/Real_projective_plane**; there are a few editorial changes to clarify possible misprints and similar issues.

> **Homogeneous coordinates.** The points in the [projective] plane can be represented by **homogeneous coordinates**. A point has homogeneous coordinates $[x : y : z]$, where the coordinates $[x : y : z]$ and $[tx : ty : tz]$ are considered to represent the same point, for all nonzero values of t. The points with coordinates $[x : y : 1]$ are the usual **real plane**, called the **finite part** of the projective plane, and points with coordinates $[x : y : 0]$, called **points at infinity** or **ideal points**, constitute a line called the **line at infinity**. (The homogeneous coordinates $[0 : 0 : 0]$ do not represent any point.)
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> The lines in the plane can also be represented by homogeneous coordinates. A projective line corresponding to an ordinary line $ax + by + c = 0$ in $\mathbf{R}^2$ has homogeneous coordinates $(a : b : c)$. Thus, these coordinates have the equivalence relation $(a : b : c) = (da : db : dc)$ for all nonzero values of d. Hence a different equation of the same line $dax + dby + dc = 0$ gives the same homogeneous coordinates. A point $[x : y : z]$ lies on a line $(a : b : c)$ if $ax + by + c = 0$. Therefore, lines with coordinates $(a : b : c)$ where a, b are not both 0 correspond to the lines in the usual **real plane**, because they contain points that are not at infinity. The projective line with coordinates $(0 : 0 : 1)$ is the line at infinity, since the only points on it are those with $z = 0$.

Higher dimensional projective spaces can be defined similarly; in $\boldsymbol{n}$ – dimensional projective geometry there are sets of $(\boldsymbol{n}+\mathbf{1})$ homogeneous coordinates, and as before two sets of homogeneous coordinates define the same point if and only if one set is a nonzero multiple of the other. In another direction, if one generalizes ordinary coordinate space to the standard $\boldsymbol{n}$ – dimensional coordinate vector space over an arbitrary field $\mathbf{F}$, then the associated **affine** $\boldsymbol{n}$ – space can be extended to a **projective** $\boldsymbol{n}$ – space over $\mathbf{F}$ by a formally identical construction. In this course we shall be particularly interested in the case where this new field $\mathbf{F}$ is the **complex numbers**. One motivation for working with complex projective spaces is that they provide an extremely convenient and powerful setting for studying questions about solutions to systems of algebraic equations in several variables (the starting point of **algebraic geometry**); the previously cited quote from Dieudonné's paper reflects this fact.

It is often useful to know something about the symmetry and incidence structures of the projective space $\mathbf{FP}^{\boldsymbol{n}}$. We shall start with the latter. Actually, we shall start with the symmetry and incidence structures on coordinate affine $\boldsymbol{n}$ – space $\mathbf{F}^{\boldsymbol{n}}$ in order to motivate everything. In ordinary **2** – and **3** – dimensional Euclidean/Cartesian geometry, lines and planes are respectively given as translates (in group – theoretic language, the ***cosets***) of **1** – and **2** – dimensional vector subspaces, and in fact one can extend this definition to $\mathbf{F}^{\boldsymbol{n}}$ where n is an arbitrary field (furthermore, everything can be done over a division ring in which one drops the commutativity assumption on multiplication). Similarly, we can define the $\boldsymbol{k}$ – planes in $\mathbf{F}^{\boldsymbol{n}}$ to be the translates of the $\boldsymbol{k}$ – dimensional vector subspaces. One can then prove the following result:

THEOREM 1. *Suppose that* $\mathbf{P} = \mathbf{y} + \mathbf{V}$ *is a* $\boldsymbol{k}$ *– plane in* $\mathbf{F}^n$, *and let* $\mathbf{W}$ *be the vector subspace of* $\mathbf{F}^{n+1} = \mathbf{F}^n \times \mathbf{F}$ *spanned by* $\mathbf{P} \times \{1\}$. *Then the following hold*:

(1) $\mathbf{W}$ *is a* $(\boldsymbol{k}+1)$ *– dimensional vector subspace of* $\mathbf{F}^{n+1}$.

(2) *If* $\mathbf{H(W)}$ *is the set of all points in* $\mathbf{FP}^n$ *represented by points whose homogeneous coordinates lie in* $\mathbf{W}$, *then* $\mathbf{P} = \mathbf{H(W)} \cap \mathbf{F}^n$, *where* $\mathbf{F}^n$ *is viewed as the ordinary points of* $\mathbf{FP}^n$.■

Conversely, if $\mathbf{W}$ *is a* $(\boldsymbol{k}+1)$ *– dimensional vector subspace of* $\mathbf{F}^{n+1} = \mathbf{F}^n \times \mathbf{F}$ whose intersection *with* $\mathbf{F}^n \times \{1\}$ *is nonempty*, *then* $\mathbf{P} = \mathbf{H(W)} \cap \mathbf{F}^n$ *is either a point or a* $\boldsymbol{k}$ *– plane*.

This theorem indicates that one should define $\boldsymbol{k}$ – planes in $\mathbf{FP}^n$ to be sets of the form $\mathbf{H(W)}$, where $\mathbf{W}$ is a $(\boldsymbol{k}+1)$ – dimensional vector subspace of $\mathbf{F}^{n+1}$.

We then have the following symmetry result.

THEOREM 2. *Suppose that* $\mathbf{F}$ *is the real or complex numbers*, *and suppose that* $\mathbf{P}$ *and* $\mathbf{Q}$ *are* $\boldsymbol{k}$ *– planes in* $\mathbf{FP}^n$. *Then there is a homeomorphism* $\boldsymbol{h}$ *from* $\mathbf{FP}^n$ *to itself which sends* $\mathbf{P}$ *to* $\mathbf{Q}$.

SKETCH OF THE PROOF: Let $\mathbf{W}$ and $\mathbf{V}$ be the $(\boldsymbol{k}+1)$ – dimensional vector subspaces of $\mathbf{F}^{n+1}$ associated to $\mathbf{P}$ and $\mathbf{Q}$ respectively, and let $\mathbf{T}$ be an invertible $\mathbf{F}$ – linear transformation on $\mathbf{F}^{n+1}$ sending $\mathbf{W}$ to $\mathbf{V}$. One can then check that $\mathbf{T}$ passes to a self – homeomorphism on the quotient space $\mathbf{FP}^n$ and it sends $\mathbf{P}$ to $\mathbf{Q}$.■

This result gives us all we need for the present course, but it is definitely just the beginning of the geometrical study of projective spaces. Although Cayley's often repeated 19th century statement, "**Projective geometry is all geometry,**" is no longer broad enough to be accurate, the role of projective spaces in geometry and topology is, and for many reasons is certain to remain, fundamental to both subjects. Here are some references for further information.

http://math.ucr.edu/~res/progeom/

The files in this directory give most of the basic results in the subject up to and including theorems on quadric hypersurfaces, and they also discuss the topological classification of the latter. The files also contain extensive references for still further information.

http://math.ucr.edu/~res/math133/geometrynotes4a.pdf
http://math.ucr.edu/~res/math133/geometrynotes4b.pdf
http://math.ucr.edu/~res/math153/history08.pdf (see pp. 4 – 7)

These files discuss the role of perspective theory in the development of projective geometry and contain material not in the previously cited directory. The cited passage in the third document contains some additional historical remarks.

http://isaacsolomonmath.wordpress.com/2011/12/25/real-complex-projective-space-part-1/
http://isaacsolomonmath.wordpress.com/2011/12/26/real-complex-projective-space-part-2/

These links provide more detailed information on the topology of complex projective spaces.

http://math.ucr.edu/~res/math205A/gentopexercises2008.pdf

Exercises **V.1.3 – 4** on pages **12 – 13** of this file are about the topological structure of the real projective plane, and likewise for Exercise **V.2.3** on page **13.**