Rational vs. real homology and cohomology

We want to compare $H_*(X, A; \mathbb{Q})$ and $H_*(X, A; \mathbb{R})$ and likewise for cohomology.

<u>ALGEBRA</u> If G is an abelian group, then
$G \otimes \mathbb{R} \cong (G \otimes \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$.
$G \otimes \mathbb{Q} \cong G_{(0)}$.

<u>Prop.</u> If $0 \to W \to V \to U \to 0$ is a short exact sequence of $\mathbb{Q}$-vector spaces, then $0 \to W \otimes_{\mathbb{Q}} \mathbb{R} \to V \otimes_{\mathbb{Q}} \mathbb{R} \to U \otimes_{\mathbb{Q}} \mathbb{R} \to 0$

is also short exact.

<u>Idea of proof</u>: Over $\mathbb{Q}$ all short exact sequences are equivalent to split ones:

$$0 \to W \xrightarrow{\text{injection}} W \oplus U \xrightarrow{\text{projection}} U \to 0$$

$$w \rightsquigarrow (w, 0)$$
$$(w, u) \rightsquigarrow u$$

(hence $V \cong W \oplus U$)

<u>Cor:</u> If (C_*, d_*) is a chain complex of $\mathbb{Q}$-vector spaces, then

$$H_*(C_* \otimes_{\mathbb{Q}} \mathbb{R}, d_* \otimes_{\mathbb{Q}} id_{\mathbb{R}}) \cong H_*(C_*, d_*) \otimes_{\mathbb{Q}} \mathbb{R}.$$

<u>Cor:</u> If (X, A) is a pair of spaces, then $H_*(X, A; \mathbb{R}) \cong H_*(X, A; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$

$$\dim_{\mathbb{R}} H^q(X, A; \mathbb{R}) = \dim_{\mathbb{Q}} H^q(X, A; \mathbb{Q})$$

(FINITE OR TRANSFINITE)

all q.

In other words, <u>real homology and cohomology are completely determined by their rational counterparts.</u>