

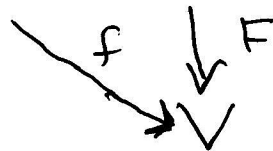
More details for alg-top-notes.pdf,

Section VII.5

Prop. 0 $A = \text{abel. gp.}$, $V = \text{rational vector space}$,
 $f: A \rightarrow V$ additive homomorphism. Then
there is a unique $F: A_{(0)} \rightarrow V$ ($A_{(0)}$ a \mathbb{Q} vsp.

map) s.t. $A \xrightarrow{j_A} A_{(0)}$

[Universal
Mapping
Property]



commutes.

Proof. Let $F[a, m] = \frac{1}{m} f(a)$. If F is well-defined, then one can check that F is a \mathbb{Q} vsp map, and it is unique because if $G: j_A = f$ then $G[a, 1] = f(a) = F[a, 1]$; since the elements $[a, 1]$ span $A_{(0)}$, it follows that $G = F$ everywhere — To see that F is well-defined, note that $[a, m] = [a', m'] \Rightarrow m(m'a - ma') = 0$, some $m \neq 0$, so that $m(m'f(a) - mf(a')) = 0$ in V which yields $\frac{1}{m} f(a) = \frac{1}{m'} f(a')$. ■

Note If $f: G \rightarrow H$ is a homomorphism, then $f_{(0)}$ is the map in Prop 0 arising from $G \xrightarrow{f} H \xrightarrow{j_H} H_{(0)}$.

Complement to Prop 0. Suppose that

$h: A \rightarrow W$ is a homomorphism into a \mathcal{Q} v.s.p. such that for each hom $f: A \rightarrow V$ as in Prop 0, there is a unique \mathcal{Q} linear $F: W \rightarrow V$ s.t. $f = F \circ h$. Then there is a unique \mathcal{Q} v.s.p. iso $G: A_{(0)} \rightarrow W$ such that $G \circ j_A = h$.

Proof This is a fairly standard argument about Universal Mapping Properties: We have unique maps $\left\{ \begin{matrix} G: A_{(0)} \rightarrow W \\ K: W \rightarrow A_{(0)} \end{matrix} \right\}$ such that $\left\{ \begin{matrix} G \circ j_A = h \\ K \circ h = j_A \end{matrix} \right\}$.

Therefore we have $\left\{ \begin{matrix} K \circ G \circ j_A = j_A = Id_{A_{(0)}} \circ j_A \\ G \circ K \circ h = h = Id_W \circ h \end{matrix} \right\}$, and

by the uniqueness statement in the Universal Mapping Property we have $\left\{ \begin{matrix} K \circ G = Id_{A_{(0)}} \\ G \circ K = Id_W \end{matrix} \right\}$, so G is an iso. and $G^{-1} = K$. ■

Most of the conclusions in Thm. 1 follow quickly from the results on Universal Mapping Properties:

(ii) j_A is an iso. if A is a Q.v.s.p.

PROOF $1_A: A \rightarrow A$ has the Universal Mapping Property.

(iii) $A \rightarrow A_{(0)}$ is a natural transformation of covariant functors.

PROOF $f: A \rightarrow B$ hom. $\Rightarrow f_{(0)}$ is the unique hom $A_{(0)} \rightarrow B_{(0)}$ such that this square commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ j_A \downarrow & & \downarrow j_B \\ A_{(0)} & \xrightarrow{f_{(0)}} & B_{(0)} \end{array} \quad \left(\begin{array}{l} \text{i.e., associated to} \\ A \rightarrow B \rightarrow B_0 \end{array} \right).$$

If $f = id_A$, then the unique map is $id_{A_{(0)}}$. Given $g: B \rightarrow C$, the identity $(g \circ f)_{(0)} = g_{(0)} \circ f_{(0)}$ follows from the commutativity of the following diagram:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ j_A \downarrow & & j_B \downarrow & & \downarrow j_C \\ A_{(0)} & \xrightarrow{f_{(0)}} & B_{(0)} & \xrightarrow{g_{(0)}} & C_{(0)} \end{array} \circ$$

$$(iv) (A \oplus B)_{(0)} \cong A_{(0)} \oplus B_{(0)}.$$

PROOF. Let $p: A \oplus B \rightarrow A$, $q: A \oplus B \rightarrow B$ denote coordinate projections, and let $\varphi: (A \oplus B)_{(0)} \rightarrow A_{(0)} \oplus B_{(0)}$ have coordinate projections $p_{(0)}$ + $q_{(0)}$ respectively; i.e., $\varphi[a, b; m] = ([a, m], [b, m])$.

φ is 1-1: Say $\varphi[a, b; m] = (0, 0)$, so $[a, m] = 0$ + $[b, m] = 0$. This implies $ma = 0$ + $m'b = 0$ for $m, m' \neq 0$. Hence $m'm a = 0$ + $mm'b = 0$, so that $[a, b; m] = 0$.

φ is onto: Given $[a, m], [b, m'] \in A_0, B_0$, we have $\varphi([m'a, mb; mm']) = ([m'a, mm'], [mb, mm']) = ([a, m], [b, m'])$. ■