More details for <u>algtop-notes.pdf</u>, Section <u>VII</u>. 5

<u>Prop. 0</u> A = abel. gp., V = rational vector space, $f : A \longrightarrow V$ additive homomorphism. Then there is a unique $F : A_{(0)} \longrightarrow V$ (a $\mathbb{Q}$ vsp. map) s.t.

$$\begin{array}{ccc} A & \xrightarrow{j_A} & A_{(0)} \\ & \searrow^{f} & \downarrow F \\ & & V \end{array}$$

commutes.

[Universal Mapping Property]

<u>Proof.</u> Let $F[a,n] = \frac{1}{n} f(a)$. If F is well-defined, then one can check that F is a $\mathbb{Q}$ vsp map, and it is unique because if $G \circ j_A = f$ then $G[a,1] = f(a) = F[a,1]$; since the elements $[a,1]$ span $A_{(0)}$, it follows that $G = F$ everywhere. — To see that F is well-defined, note that $[a,n] = [a',n'] \Longrightarrow m(n'a - na') = 0$, some $m \neq 0$, so that $m(n'f(a) - n f(a')) = 0$ in V which yields $\frac{1}{n} f(a) = \frac{1}{n'} f(a')$. ∎

<u>Note</u> If $f: G \to H$ is a homomorphism, then $f_{(0)}$ is the map in Prop 0 arising from $G \xrightarrow{f} H \xrightarrow{j_H} H_{(0)}$.

<u>Complement to Prop 0.</u> Suppose that $h: A \to W$ is a homomorphism into a $\mathbb{Q}$ v.sp. such that for each hom $f: A \to V$ as in Prop. 0, there is a unique $\mathbb{Q}$ linear $F: W \to V$ s.t. $f = F \circ h$. Then there is a unique $\mathbb{Q}$ v.sp iso $G: A_{(0)} \to W$ such that $G \circ j_A = h$.

<u>Proof</u> This is a fairly standard argument about Universal Mapping Properties: We have unique maps $\left\{\begin{matrix} G: A_{(0)} \to W \\ K: W \to A_{(0)} \end{matrix}\right\}$ such that $\left\{\begin{matrix} G \circ j_A = h \\ K \circ h = j_A \end{matrix}\right\}$.

Therefore we have $\left\{\begin{matrix} K \circ G \circ j_A = j_A = Id_{A_{(0)}} \circ j_A \\ G \circ K \circ h = h = Id_W \circ h \end{matrix}\right\}$, and by the uniqueness statement in the Universal Mapping Property we have $\left\{\begin{matrix} K \circ G = Id_{A_0} \\ G \circ K = Id_W \end{matrix}\right\}$, so G is an iso. and $G^{-1} = K$. ∎

Most of the conclusions in Thm. 1 follow quickly from the results on Universal Mapping Properties:

<u>(ii)</u> j_G is an iso. if A is a $\mathbb{Q}$ v.sp.

<u>PROOF</u> $1_A : A \to A$ has the Universal Mapping Property.

<u>(iii)</u> $A \to A_{(0)}$ is a natural transformation of covariant functors.

<u>PROOF</u> $f : A \to B$ hom. $\Rightarrow f_{(0)}$ is the unique hom $A_{(0)} \to B_{(0)}$ such that this square commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j_A"'] & B \arrow[d, "j_B"] \\ A_{(0)} \arrow[r, "f_{(0)}"] & B_{(0)} \end{tikzcd} \qquad \left(\begin{array}{l}\text{i.e., associated to} \\ A \to B \to B_0\end{array}\right).$$

If $f = id_A$, then the unique map is $id_{A(0)}$. Given $g : B \to C$, the identity $(g \circ f)_{(0)} = g_{(0)} \circ f_{(0)}$ follows from the commutativity of the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j_A"'] & B \arrow[r, "g"] \arrow[d, "j_B"] & C \arrow[d, "j_C"] \\ A_{(0)} \arrow[r, "f_{(0)}"] & B_{(0)} \arrow[r, "g_{(0)}"] & C_{(0)} \end{tikzcd} .$$

<u>(iv)</u> $(A\oplus B)_{(0)} \cong A_{(0)} \oplus B_{(0)}$.

<u>PROOF.</u> Let $p: A\oplus B \to A$, $q: A\oplus B \to B$ denote coordinate projections, and let $\varphi: (A\oplus B)_{(0)} \to A_{(0)} \oplus B_{(0)}$ have coordinate projections $p_{(0)}$ + $q_{(0)}$ respectively; i.e., $\varphi[a, b; n] = ([a, n], [b, n])$.

<u>φ is 1-1</u>: Say $\varphi[a, b; n] = (0, 0)$, so $[a, n] = 0$ + $[b, n] = 0$. This implies $ma = 0$ + $m'b = 0$ for some $m, m' \neq 0$. Hence $m'ma = 0$ + $mm'b = 0$, so that $[a, b; n] = 0$.

<u>φ is onto:</u> Given $[a, n], [b, n'] \in A_0, B_0$, we have $\varphi([n'a, nb; nn']) = ([n'a, nn'], [nb, nn']) = ([a, n], [b, n'])$. ■