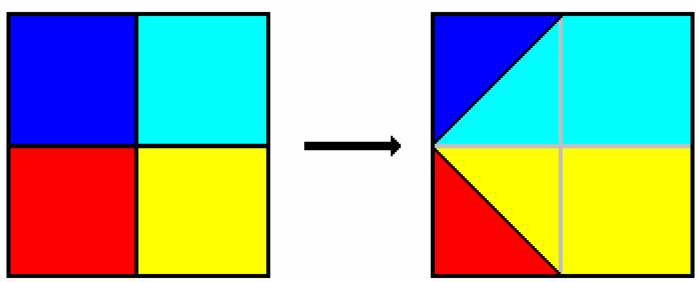
Complements of closed line segments in the plane

We would like to prove that the complement of a closed line segment in the coordinate plane $\mathbf{R}^2$ is homeomorphic to the complement of a point. It will suffice to define a continuous mapping **F** from the square $[0, 2] \times [0, 2]$ to itself is the identity on the boundary, preserves the second coordinate, is onto, and finally is **1 – 1** except on the segment $[0, 1] \times \{1\}$, which is mapped to the point $(0, 1)$. We can then extend **F** to all of $\mathbf{R}^2$ by taking the identity map off the square, and the restriction of the resulting map will define a homeomorphism **G** from $\mathbf{R}^2 - [0, 1] \times \{1\}$ to $\mathbf{R}^2 - \{(0,1)\}$. The assertion that map **G** is a homeomorphism will follow if we can show that **G** is a closed mapping, and this can be done as follows: Suppose that **A** is a closed subset of $\mathbf{R}^2 - [0, 1] \times \{1\}$. Then $\mathbf{B} = \mathbf{A} \cup [0, 1] \times \{1\}$ is a closed subset of $\mathbf{R}^2$ (Why**?**). We claim that **F[B]** is closed in $\mathbf{R}^2$; if so, then the elementary relationship $\mathbf{G[A]} = \mathbf{F[B]} - \{(0,1)\}$ implies that **G[A]** is closed in $\mathbf{R}^2 - \{(0,1)\}$. To complete the argument, we shall show that **F** is a closed mapping. Its restriction to the compact set $[0, 2] \times [0, 2]$ is closed and it is the identity (hence closed) on the closed set $\mathbf{R}^2 - (0, 2) \times (0, 2)$; since these two sets form a finite closed covering of $\mathbf{R}^2$, it follows that **F** is a closed mapping as required.

The drawing below illustrates how **F** can be constructed. In this picture, the square is cut into four smaller squares of side **1**, and the colors indicate how each of the smaller squares is mapped. On each horizontal segment of the form $[0, 2] \times \{t\}$, the restriction of the mapping to the sub–segments $[0,1] \times \{t\}$ and $[1, 2] \times \{t\}$ will be linear.



In this drawing, the smaller squares on the left hand side are $[0,1] \times [0,1]$ (*red*), $[1,2] \times [0,1]$ (*yellow*), $[0,1] \times [1,2]$ (*dark blue*) and $[1,2] \times [1,2]$ (*light blue*).

In order to make this mathematically rigorous, we need to give explicit formulas for the value of **F** on each of the four pieces.

On the square $[0, 1] \times [0, 1]$, we have $\mathbf{F}(s,t) = (s(1 - t), t)$.

On the square $[1, 2] \times [0, 1]$, we have $\mathbf{F}(s,t) = (s + st - 2t, t)$.

On the square $[0, 1] \times [1, 2]$, we have $\mathbf{F}(s,t) = (s(t - 1), t)$.

On the square $[1, 2] \times [1, 2]$, we have $\mathbf{F}(s,t) = (2t + 3s - st - 4, t)$.

It is also necessary to check that the definitions agree on the overlapping pieces of the four squares; however, this is just a sequence of routine algebraic computations. Note that there are **six** cases to be checked, corresponding to the six combinations of two squares from the original set of four.