This is a consolidation of material from several places in the course notes.

The isomorphism between ordered and unordered simplicial chains can be reformulated in an abstract setting that will be needed later. We begin by defining a category **SCPairs** whose objects are pairs of simplicial complexes $(\mathbf{K}, \mathbf{K}_0)$ and whose morphisms are given by subcomplex inclusions $(\mathbf{L}, \mathbf{L}_0) \subset (\mathbf{K}, \mathbf{K}_0)$; in other words, $\mathbf{L}_0$ is a subcomplex of both $\mathbf{L}$ and $\mathbf{K}_0$ while $\mathbf{L}$ is also a subcomplex of $\mathbf{K}$. A *homology theory* on this category is a covariant functor h_* valued in some category of modules together with a natural transformation

$$\partial(\mathbf{K}, \mathbf{L}) : h_*(\mathbf{K}, \mathbf{L}) \longrightarrow h_{*-1}(\mathbf{L})$$

such that

- (a) one has long exact homology sequences,
- (b) if $\mathbf{K}$ is a simplex and $\mathbf{v}$ is a vertex of $\mathbf{K}$ then $h_*(\{\mathbf{v}\}) \to h_*(\mathbf{K})$ is an isomorphism,
- (c) if $\mathbf{K}$ is 0-dimensional with vertices $\mathbf{v}_j$ then the associated map from $\oplus_j \; h_*(\{\mathbf{v}_j\})$ to $h_*(\mathbf{K})$ is an isomorphism,
- (d) if $\mathbf{K}$ is obtained from $\mathbf{M}$ by adding a single simplex $\mathbf{S}$, then $h_*(\mathbf{S}, \partial\mathbf{S}) \to h_*(\mathbf{M}, \mathbf{K})$ is an isomorphism,
- (e) if $\mathbf{K}$ is complex consisting only of a single vertex then $h_0(\mathbf{K})$ is the underlying ring R and $h_j(\mathbf{K}) = 0$ if $j \neq 0$.

A *natural transformation* from one such theory (h_*, ∂) to another (h'_*, ∂') is a natural transformation of θ of functors that is compatible with the mappings ∂ and ∂'; specifically, we want

$$\theta(\mathbf{L}) \circ \partial \quad = \quad \partial' \circ \theta(\mathbf{K}, \mathbf{L}) \ .$$

These conditions imply the existence of a commutative ladder diagram as in Theorem 6, where the rows are the long exact sequences determined by the two abstract homology theories. The definition is set up so that the proof of the next result is formally parallel to the proof of Theorem I.1.7:

THEOREM I.1.8. *Suppose we are given a natural transformation of homology theories θ as above such that $\theta(\mathbf{K})$ is an isomorphism if $\mathbf{K}$ consists of just a single vertex. Then $\theta(\mathbf{K}, \mathbf{L})$ is an isomorphism for all pairs $(\mathbf{K}, \mathbf{L})$.*■

Relating simplicial to singular homology

As noted at the beginning of Section II.1, if $(P, \mathbf{K})$ is a simplicial complex, then for each free generator $\mathbf{v}_0 \; \cdots \; \mathbf{v}_q$ of $C_q(P, \mathbf{K})$ there is a unique affine (hence continuous) map $T : \Delta_q \to P$ which sends a point $(t_0, \; \cdots \; , t_q) \in \Delta_q$ to $\sum_j \; t_j \, \mathbf{v}_j \in P$. One can think of these as linear simplices in P, and accordingly this construction assigns a singular simplex in P to each free generator. By construction, this actually defines a chain map $\theta_\#$ from $C_*(P, \mathbf{K})$ onto a chain subcomplex of $S_q(P)$, and the inclusion is augmentation preserving. Note that if $(\mathbf{K}, \mathbf{L})$ is a pair consisting of a simplicial complex and a subcomplex with underlying space pair (P, Q), then the construction also yields a chain map from $\theta_{\#} : C_*(\mathbf{K}, \mathbf{L})$ to $S_*(P, Q)$, with a commutative diagram involving short exact sequences of simplicial and singular chain complexes which goes from the unordered chain complex short exact sequence

$$0 \;\to\; C_*(\mathbf{L}) \;\to\; C_*(\mathbf{K}) \;\to\; C_*(\mathbf{K}, \mathbf{L}) \;\to\; 0$$

to the singular chain complex short exact sequence

$$0 \;\to\; S_*(Q) \;\to\; S_*(P) \;\to\; S_*(P, Q) \;\to\; 0\ .$$

THEOREM II.4.1. *Let $(X, \mathbf{K})$ be a simplicial complex, let $(A, \mathbf{L})$ determine a subcomplex, and let $\theta_* : H_*(\mathbf{K}, \mathbf{L}) \to H_*(X, A)$ be the natural transformation from simplicial to singular homology that was described previously. Then θ_* is an isomorphism.*

Proof. The idea is to apply Theorem I.1.8 on natural transformations of homology theories on simplicial complex pairs. In order to do this, we must check that singular homology for simplicial complexes satisfies the five properties $(a) - (e)$ listed shortly before the statement of I.1.8 on page 15. Property (c), which gives the homology of a finite set, is verified in Proposition IV.1.4, and Properties (a), (b), (d) and (e) — which involve long exact sequences, the homology of a contractible space (more precisely, a simplex), excision for adjoining a single simplex, and the homology of a point — are respectively established in Theorem II.2.2, Corollary II.2.5, Theorem II.3.8, and the discussion following the problem stated after Corollary 1.1.4. Since all these properties hold, Theorem I.1.8 implies that the map θ_* must be an isomorphism for all simplicial complex pairs.■