Solutions to Quiz 2

1. Suppose that two $n \times n$ matrices A and B are similar. Prove that their adjoints A^* and B^* are also similar when the scalars are either real or complex numbers (*i.e.*, the result is true in both cases).

SOLUTION

We are given that $B = P^{-1}AP$, and if we take adjoints of this equation we see that $B^* = P^*A^*(P^{-1})^*$. If $Q = (P^{-1})^*$ then $Q^{-1} = P^*$ and therefore $B^* = Q^{-1}A^*Q$,■

2. Give examples of matrices A and B of the same size such that $\exp(A+B) \neq \exp(A)\exp(B)$. In fact, there are examples in the 2×2 case where the entries are not at all complicated.

SOLUTION

Let E_{ii} denote the matrix whose (i,j) entry is 1 and all other entries are 0. Then $E_{ij}E_{jk} = E_{ik}$ and $E_{jk}E_{ij} = 0$ if $i \neq k$, so these sorts of matrices are candidates for examples. One can check these identities directly in the 2×2 cases where (i,j) is either $(1,2)$ or $(2,1)$.

If H is the 2×2 matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then $H^2 = I$, $H = E_{12} + E_{21}$, and $\exp H = (\cosh 1)I + (\sinh 1)H$. On the other hand, we know that $E_{12}^2 = 0 = E_{21}^2$, which means that $\exp E_{12} = I + E_{12}$ and $\exp E_{21} = I + E_{21}$. We then have

$$(\exp E_{12})(\exp E_{21}) \quad = \quad (I + E_{12})(I + E_{21}) \quad = \quad I + H + E_{11} \quad =$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and this is clearly not equal to $\exp H$. More precisely, the diagonal entries of this matrix are unequal but the diagonal entries of $\exp H$ are equal.■

3. Suppose that the nilpotent 5×5 matrix N is also an elementary Jordan matrix. Show that N is similar to its adjoint N^*.

SOLUTION

The conditions in the problem mean that left multiplication by N has the following effect on the standard unit vectors:

$$N\mathbf{e}_1 \;=\; \mathbf{0}\,, \qquad N\mathbf{e}_k \;=\; \mathbf{e}_{k-1} \quad \text{for} \quad k > 1$$

If we define a new ordered basis by $\mathbf{v}_k = \mathbf{e}_{6-k}$ for $k = 1, 2, 3, 4, 5$ then the matrix of left multiplication by N with respect to the new ordered basis is equal to N^*. Therefore N and N^* are similar matrices.■