EXERCISES FOR MATHEMATICS 205B

WINTER 2012

File Number 03

DEFAULT HYPOTHESES. Unless specifically stated otherwise, all spaces are assumed to be Hausdorff and locally arcwise connected.

1. Suppose that $(P, \mathbf{K})$ is an n-dimensional simplical complex. If $0 \leq m \leq n$, define the m-skeleton $(P_m, \mathbf{K}_m)$ to be the subcomplex consisting of all simplices in $\mathbf{K}$ of dimension $\leq m$.

(a) Explain why $(P_1, \mathbf{K}_1)$ is a graph, show that P is (arcwise) connected if and only if P_1 is, and explain why P is a finite union of pairwise disjoint connected subcomplexes $(P_\alpha, \mathbf{K}_\alpha)$.

(b) Suppose that we are given a finite set of chain complexes $\{C_*^\alpha, d_*^\alpha\}$. If $C_* = \oplus_\alpha\ C_*^\alpha$ and $d = \oplus_\alpha\ d_*^\alpha$, show that $\{C_*, d_*\}$ is a chain complex and that $H_*(C)$ is isomorphic to the direct sum $\oplus_\alpha\ H_*(C^\alpha)$.

(c) In the setting of (a) and (b), prove that the homology groups of $(P, \mathbf{K})$ are isomorphic to the direct sum of the homology groups of the subcomplexes $(P_\alpha, \mathbf{K}_\alpha)$.

2. (a) Suppose that $(P, \mathbf{K})$ is the union of two connected subcomplexes $(P_1, \mathbf{K}_1)$ and $(P_2, \mathbf{K}_2)$ and that the intersection of these subcomplexes is a single vertex. Prove that $H_q(\mathbf{K})$ is isomorphic to $H_q(\mathbf{K}_1) \oplus H_q(\mathbf{K}_2)$ if $q > 0$ and $H_0(\mathbf{K}) \cong \mathbb{Z}$.

(b) Using (a) and finite induction, for each $n > 0$ construct a connected n-dimensional simplicial complex $\mathbf{K}$ such that $H_q(\mathbf{K}) \neq 0$ for all q such that $1 \leq q \leq n$.

3. (a) Let $\mathbf{K}$ be the subcomplex of the standard simplex Δ_3 consisting of all edges and the face opposite the first vertex $\mathbf{e}_0$. Compute the homology groups of $\mathbf{K}$ using any valid method (exact sequences are very useful).

(b) Let $\mathbf{K}$ be the $(n-1)$-skeleton of the standard simplex Δ_n. Compute the homology groups of $\mathbf{K}$.

4. Let (A_*, d_*^A) and (B_*, d_*^B) be chain complexes, and let $f, g : (A_*, d_*^A) \to (B_*, d_*^B)$ be chain maps. A *chain homotopy* from f to g is a sequence of maps $D_q : A_q \to B_{q+1}$ such that $d^B \circ D\ +\ D \circ d^A = g - f$. To chain maps f, g are said to be chain homotopic if there is a chain homotopy from f to g.

(a) Prove that "chain homotopic" is an equivalence relation.

(b) Prove that if f and g are chain homotopic, then the induced homology maps f_* and g_* are equal.

(c) Prove that if f and g are as in (b) and $h : B_* \to C_*$ is a map of chain complexes, then $h \circ f$ is chain homotopic to $h \circ g$. Dually, prove that if $\varphi : W_* \to A_*$ is a chain map, then $f \circ \varphi$ is chain homotopic to $g \varphi g$.

5. (a) Suppose that (C_*, d_*) is a chain complex of R-modules for some ring R, and let $u \in H_q(C)$ be a nonzero class. Prove that there is a chain complex C' which contains C as a subcomplex and has the property that u maps to zero under the map from $H_q(C)$ to $H_q(C')$ induced by inclusion. [*Hint:* Define $C'_k = C_k$ if $k \neq q+1$, $C'_{q+1} = C_{q+1} \oplus R$, and define d' on the latter so that it maps the extra generator of the latter to a representative for u.]

(b) Let $f : A \to B$ be a module homomorphism, and define a chain complex with $C_1 = A$, $C_0 = B$, $d_1 = f$, and all other modules and boundary homomorphisms equal to zero. Compute the homology groups of (C_*, d_*). In particular, show that at most one homology group is zero if f is either 1–1 or onto.

(c) Let G_q be a sequence of finitely generated abelian groups such that $G_q = 0$ for $q < 0$ and at most finitely many groups G_q are nonzero. Construct a chain complex (C_*, d_*) such that (i) $C_q = 0$ for $q < 0$ and for $q > n$ for some $n > 0$, (ii) C_q is finitely generated free abelian for all q, (iii), we have $H_q(C) = G_q$. [*Hint:* First show that it suffices to prove this for a complex with one nonzero G_q where the latter is cyclic; for example, use direct sums. Next, find very simply chain complexes whose homologies are given by such sequences G_q.]

6. Given a simplicial complex $(P, \mathbf{K})$ with linearly ordered vertices and $P \subset \mathbb{R}^N$, the **cone** $C(\mathbf{K})$ has a underlying polyhedron $C(P) \subset \mathbb{R}^{N+1}$ consisting of all points $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ such that $x = (1-t)y$ for some $y \in P$ and $t \in [0, 1]$. If $P \subset \mathbb{R}^2$ this is just the usual geometric notion of a cone with base P and vertex point $\mathbf{e}_3$. The simplicial decomposition is given by the first few items below:

(a) Suppose that $A \subset \mathbb{R}^n$ is a simplex with vertices v_i. Prove that $C(A)$ is a simplex whose vertices are the last unit vector $\mathbf{e}_{N+1}$ and the points $(v_i, 0)$.

(b) Using (a) verify that if the simplices of $\mathbf{K}$ are given by A_α, then the simplices $C(A_\alpha)$ and their faces form a simplicial decomposition of $C(P)$, called the *standard cone decomposition* $C(\mathbf{K})$.

(c) Define an ordering of the vertices in $C(\mathbf{K})$ such that $\mathbf{e}_{N+1}$ is the first vertex and the remaining vertices, which correspond to the vertices of $\mathbf{K}$, the follow in the given order. Prove that the homology groups of $\big(C(P), C(\mathbf{K})\big)$ are isomorphic to the homology groups of a point. [*Hint:* Imitate the proof for a simplex.]

7. Given $(P, \mathbf{K})$ as above, define its **suspension** $\Sigma(P)$ to be the union of $C(P)$ with the image of $C(P)$ under the reflection map S on $\mathbb{R}^{N+1}$ which sends the unit vector $\mathbf{e}_{N+1}$ to $-\mathbf{e}_{N+1}$ and sends all other standard unit vectors to themselves (hence $\Sigma(P)$ is a union of an upper cone and a lower cone which meet in P).

(a) Explain why $\Sigma(P)$ has a canonical simplicial decomposition $\Sigma(\mathbf{K})$ in which the upper and lower cones are subcomplexes. — We order its vertices so that $\mathbf{e}_{N+1}$ and its negative are the first two in the list, and then we use the given ordering for the remaining vertices.

(b) Using a Mayer-Vietoris sequence for the decomposition of $\Sigma(\mathbf{K})$ into two cones, show that $H_q(\Sigma(\mathbf{K}))$ is isomorphic to $H_{q-1}(\mathbf{K})$ if $q \neq 0, 1$, it is isomorphic to $\mathbb{Z}$ if $q = 0$, and we have $H_1(\mathbf{K}) \oplus \mathbb{Z} \cong H_0(\mathbf{K})$.

8. Suppose we are given a commutative diagram as below, in which the rows are short exact sequences ($\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ sends x to $(x, 0)$, and $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is projection onto the second coordinate):

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \to & \mathbb{Z}\oplus\mathbb{Z} & \to & \mathbb{Z} & \to & 0 \\ & & \big\downarrow 0 & & \big\downarrow f & & \big\downarrow 0 & & \\ 0 & \to & \mathbb{Z} & \to & \mathbb{Z}\oplus\mathbb{Z} & \to & \mathbb{Z} & \to & 0 \end{array}$$

Does it follow that $f = 0$? Either prove this or give a counterexample. [*Hint:* Think about a nilpotent 2×2 matrix in Jordan form.]

9. Let $f : M \to N$ be a homomorphism of R-modules for some ring R, and define a short exact sequence of chain complexes

$$0 \ \to\ A_* \ \to\ B_* \ \to\ C_* \ \to\ 0$$

as below, in which the top row corresponds to dimension or degree k and the bottom row corresponds to dimension or degree $k - 1$. All other objects and maps are taken to be zero.

$$\begin{array}{ccccccccc} 0 & \to & 0 & \longrightarrow & M & \overset{=}{\longrightarrow} & M & \to & 0 \\ & & \big\downarrow & & \big\downarrow f & & \big\downarrow & & \\ 0 & \to & N & \overset{=}{\longrightarrow} & N & \longrightarrow & 0 & \to & 0 \end{array}$$

Prove that the connecting homomorphism $\partial : H_k(C) \to H_{k-1}(A)$ corresponds to f under the canonical isomorphisms from M to $H_k(C)$ and from N to $H_{k-1}(A)$.

10. The boundary of a triangular prism P_3 has a simplicial decomposition $\mathbf{K}$ with vertices A, B, C, D, E, F along with the 2-simplices ABC, ADE, ABE, BEF, BCF, ACF, ADF, DEF and their edges; geometrically, ABC and DEF are the bottom and top respectively, and the lateral edges are AD, BE and CF (see `exercises03a.pdf` for a drawing). Find a nontrivial cycle in $C_2(P_3, \mathbf{K})$ of the form $\sum_\alpha n_\alpha \sigma_\alpha$ where σ_α runs through all the standard free generators of the chain group and each n_α is ± 1.

11. Suppose that we have a short exact sequence of chain complexes

$$0 \ \to\ A_* \ \to\ B_* \ \to\ C_* \ \to\ 0$$

and let $i : A_* \to B_*$ is the injection given by this sequence. In analogy with other situations, we say that i is a *chain complex retract* if there is a chain map $\rho : B_* \to A_*$ such that $\rho \circ i =$ identity on A_*. Prove that if i is a retract then there is an isomorphism

$$H_*(B) \ \cong\ H_*(A) \ \oplus\ H_*(B/A)$$

such that i_* maps $H_*(A)$ to the first factor of this direct sum decomposition. [*Hints:* First show that the existence of ρ_* implies that i_* is 1–1. Why does this imply that $\partial : H_{q+1}(B/A) \to H_q(A)$ is zero for all q and that $H_*(B) \to H_*(B/A)$ is onto? Using this map and ρ_* define a homomorphism from $H_*(B)$ to $H_*(A) \oplus H_*(B/A)$ and show that this map must be both 1–1 and onto.]

12. ($\star$) If G and H are abelian groups, then the set $\mathrm{Hom}(G, H)$ of homomorphisms from G to H is an abelian group with respect to the standard notion of addition (pointwise). If $\alpha : G_1 \to G_2$

and $\beta : H_1 \to H_2$ are homomorphisms, then $\alpha^* : \mathrm{Hom}(G_2, H) \to \mathrm{Hom}(G_1, H)$ is defined by $\alpha^*(f) = f \circ \alpha$ and $\beta_* : \mathrm{Hom}(G, H_1) \to \mathrm{Hom}(G, H_2)$ is defined by $\beta_*(f) = \beta \circ f$. Analogs of the standard distributivity laws for composites of linear transformations imply that α^* and β_* are abelian group homomorphisms.

(a) Suppose that $0 \to A \to B \to C$ is an exact sequence of abelian groups and G is an abelian group. Prove that

$$0 = \mathrm{Hom}(G, 0) \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, B) \to \mathrm{Hom}(G, C)$$

is exact.

(b) Suppose that $A \to B \to C \to 0$ is an exact sequence of abelian groups and G is an abelian group. Prove that

$$0 = \mathrm{Hom}(0, G) \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$$

is exact.

(c) Suppose that $0 \to A \to A \oplus C \to C \to 0$ is a split short exact sequence of abelian groups ($i.e.$, the map from A is the injection sending x to $(x, 0)$, and the map to C is projection onto the second coordinate). Prove that the two sequences

$$0 = \mathrm{Hom}(G, 0) \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, A \oplus C) \approx \mathrm{Hom}(G, A) \oplus \mathrm{Hom}(G, C) \to \mathrm{Hom}(G, C) \to 0$$

$$0 = \mathrm{Hom}(0, G) \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(A \oplus C, G) \approx \mathrm{Hom}(A, G) \oplus \mathrm{Hom}(C, G) \to \mathrm{Hom}(A, G) \to 0$$

are split short exact sequences.

REMARKS. In (a) and (b) it does not follow that either $\mathrm{Hom}(G, ...)$ or $\mathrm{Hom}(...,\ G)$ takes short exact sequences to short exact sequences. Counterexamples are given by the short exact sequence

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$$

with $G = \mathbb{Z}_2$ in either case. In particular, the identity map is not in the image of the homomorphism $\mathrm{Hom}(\mathbb{Z}_4, \mathbb{Z}_2) \to \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2)$ induced by the inclusion of $\mathbb{Z}_2$ in $\mathbb{Z}_4$, and it is also not in the image of the homomorphism $\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_4) \to \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2)$ induced by the onto mapping from $\mathbb{Z}_4$ to $\mathbb{Z}_2$.

13. Suppose that $(P, \mathbf{K})$ is a connected simplicial complex, and let $A, B \in \mathbf{K}$. Prove that there is a sequence of simplices $A = S_0, \cdots, S_p = B$ such that each intersection is nonempty. [*Hint:* Define a binary relation on simplices such that A and B are related if a chain of the given type exists. Why is this an equivalence relation, and why is the union of all simplices in a given equivalence class a closed and open subset of P?]

14. Suppose that $(P, \mathbf{K})$ is a connected 2-dimensional simplicial complex such that every edge lies on exactly two 2-simplices. Prove that the number of edges is divisible by 3. [*Hint:* Count the number of pairs (E, T) where E is an edge and T is a 2-simplex containing E. There are two ways of doing so — one by grouping the pairs with the same first coordinate and another by grouping the pairs with the same second coordinate.]