EXERCISES FOR MATHEMATICS 205A

FALL 2014

The references denote sections of the text for the course:

J. R. Munkres, *Topology* (Second Edition), Prentice-Hall, Saddle River NJ, 2000. ISBN: 0–13–181629–2.

Solutions to nearly all these exercises are given in the files `math205Asolutions`n`.pdf` for $n \leq 6$ (in the course directory). Other web sites with solutions to exercises in Munkres (including some not given in our files) are `http://www.math.ku.dk/~moller/e03/3gt/3gt.html` and `http://dbfin.com/?=munkres` (there are no guarantees that everything on these sites is free of errors).

I. Foundational material

I.1 : Basic set theory

(Munkres, §§ 1, 2, 3, 9)

Munkres, § 9, pp. 64–65: 2ac, 5

Additional exercise

1. Let X be a set and let A, $B \subset X$. The *symmetric difference* $A \oplus B$ is defined by the formula

$$A \oplus B \quad = \quad (A - B) \ \cup \ (B - A)$$

so that $A \oplus B$ consists of all objects in A or B but not both. Prove that $\oplus$ is commutative and associative on the set of all subsets of X, that $A \oplus \emptyset = A$ for all A, that $A \oplus A = \emptyset$ for all A, and that one has the following distributivity relation for A, B, $C \subset X$:

$$A \cap (B \oplus C) \quad = \quad (A \cap B) \ \oplus \ (A \cap C)$$

[*Hint:* It might be helpful to draw some Venn diagrams.]

I.2 : Products, relations and functions

(Munkres, §§ 5, 6, 8)

Munkres, § 6, p. 44: 4a

Additional exercises

1.* Let X and Y be sets, suppose that A and C are subsets of X, and suppose that B and D are subsets of Y. Verify the following identities:

(i) $A \times (B \cap D) = (A \times B) \cap (A \times D)$

(*ii*) $A \times (B \cup D) = (A \times B) \cup (A \times D)$
(*iii*) $A \times (Y - D) = (A \times Y) - (A \times D)$
(*iv*) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
(*v*) $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$
(*vi*) $(X \times Y) - (A \times B) = (X \times (Y - B)) \cup ((X - A) \times Y)$

2. Let $f : A \to B$ be a 1–1 and onto mapping of sets, let g be the inverse function to f, and let $C \subset B$. Prove that the image set $g[C]$ is equal to the inverse image set $f^{-1}[C]$; this result implies that the two possible meanings of $f^{-1}[C]$ (image under f^{-1}, inverse image under f) yield the same subset.

3. Given a set X and a binary relation $\mathcal{R}$ on X, define a new binary relation $\mathcal{R}^{\#}$ on X such that $x\ \mathcal{R}^{\#}\ y$ if and only if $x = y$ or there is a finite sequence $v_0, \cdots, v_m$ such that $v_0 = x$, $v_m = y$ and for each i we have either $v_i\ \mathcal{R}\ v_{i+1}$ or $v_{i+1}\ \mathcal{R}\ v_i$. Prove that $\mathcal{R}^{\#}$ is an equivalence relation on X, and if $\mathcal{S}$ is an equivalence relation such that $x\ \mathcal{S}\ y$ whenever $x\ \mathcal{R}\ y$, then we also have $x\ \mathcal{S}\ y$ whenever $x\ \mathcal{R}^{\#}\ y$. — The latter implies that $\mathcal{R}^{\#}$ is the minimal equivalence relation on X such that x and y are equivalent whenever $x\ \mathcal{R}\ y$, and it is called *the equivalence relation generated by* $\mathcal{R}$.

4. The game of chess is played on an 8×8 board with squares alternately colored black and white (or some other pair of contrasting colors). A chess player is likely to notice very quickly that a bishop can move to any square of the same color it currently occupies but cannot more to a square of the opposite color. The goal of the exercise is to give a mathematical proof of this assertion.

Here is the formal setting: Model the chessboard mathematically by the set

$$B \quad = \quad \{1,2,3,4,5,6,7,8\} \times \{1,2,3,4,5,6,7,8\}$$

so that the squares correspond to ordered pairs of points (i, j) and the color of a square depends upon whether $i + j$ is even or odd. Define a binary relation $\mathcal{R}$ on B such that $(i, j)\ \mathcal{R}\ (p, q)$ if $p = i + \alpha$ and $q = j + \beta$ where $\alpha, \beta \in \{-1, 1\}$ and $(p, q) \in B$ (these correspond to a bishop moving one square in any permissible direction on an empty board), and let $\mathcal{E}$ be the equivalence relation generated by $\mathcal{R}$.

Here is the formal statement of the exercise: Prove that $\mathcal{E}$ has exactly two equivalence classes, so that the equivalence class of a point is determined by whether $i + j$ is even or odd.

5. Suppose that $\mathcal{R}_1$ is an equivalence relation on X, let $X/\mathcal{R}_1$ denote the set of equivalence classes for $\mathcal{R}_1$, and let $\mathcal{R}_2$ be an equivalence relation on $X/\mathcal{R}_1$. Define a binary relation $\mathcal{S}$ on X such that $x\ \mathcal{S}\ y$ if and only if the equivalence classes $[x]$ and $[y]$ of $x, y \in X$ with respect to $\mathcal{R}_1$ satisfy $[x]\ \mathcal{R}_2\ [y]$. Prove that $\mathcal{S}$ also defines an equivalence relation on X.

I.3 : Cardinal numbers

(Munkres, §§ 4, 7, 9)

Munkres, § 7, p. 51: 4**
Munkres, § 9, p. 62: 5
Munkres, § 11, p. 72: 8

Additional exercises

1.* Show that the set of *countable* subsets of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$.

2. Let α and β be cardinal numbers such that $\alpha < \beta$, and let X be a set such that $|X| = \beta$. Prove that there is a subset A of X such that $|A| = \alpha$.

Definition. If X is a set, a family $\mathcal{F}$ of sets is of *finite character* provided it has the following properties:

For each $A \in \mathcal{F}$, every finite subset of A belongs to $\mathcal{F}$.

If every finite subset of a given set A belongs to $\mathcal{F}$, then A belongs to $\mathcal{F}$.

3.* Prove the **Teichmnüller–Tukey Lemma** (sometimes simply known as *Tukey's Lemma*): Let $\mathbf{P}(X)$ denote the set of all subsets of the set X. If $\mathcal{F} \subset \mathbf{P}(X)$ is a nonempty family of finite character, then $\mathcal{F}$ has a maximal element with respect to inclusion.

FOOTNOTES. O. Teichmüller (1913–1943) is best known mathematically for his contributions to complex analysis including his work of certain fundamental objects named after him, but he is also remembered for his political views and related events. J. W. Tukey (1915–2000) is best known for contributions to the mathematical sciences in statistics and computer science, and especially for his work on the Fast Fourier Transform. In nonmathematical circles he is also known for coining the computer word *bit* as an abbreviation for *binary digit*; at one point he was also credited with inventing the word *software*, but further study indicated that the attribution was almost certainly incorrect (however, it is still occasionally seen or heard).

I.4 : The real number system

(Munkres, § 4)

Munkres, § 4, pp. 34–36: 9c [*Hint:* Use the conclusion from 9b.]

Additional exercises

1. Suppose that : $\mathbb{R} \to \mathbb{R}$ is a set-theoretic function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$; some obvious examples of this type are the functions $f(x) = c \cdot x$ for some fixed real number c. Does it follow that **every** such function f has this form? [*Hint:* Why is $\mathbb{R}$ a vector space over $\mathbb{Q}$? Recall that every vector space over a field has a (possibly infinite) basis by Zorn's Lemma.]

2. Let $(X, \leq)$ be a well-ordered set, and let $A \subset X$ be a nonempty subset which has an upper bound in X. Prove that A has a least upper bound in X. [*Hint:* If A has an upper bound, set of upper bounds for A has a least element β.]

3. Suppose that $\{a_n\}$ is a sequence of real numbers which converges to some limit value L in the extended real number system (so L may be $\pm\infty$), and let $\{a_{n(k)}\}$ be a subsequence of $\{a_n\}$. Prove that $\{a_{n(k)}\}$ also converges to L.

4. (i) Prove that the real number system has the **Cantor nested intersection property**: If we are given a sequence of closed intervals $\{[a_k, b_k]\}$ in the real numbers such that for each n we have $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$, then these is at least one point p which lies in all the intervals.

(ii) Suppose that the endpoints in (i) are all rational numbers. Does it follow that there is a rational number which lies in all the intervals? Prove this or give a counterexample.

II. Metric and topological spaces

II.1 : Metrics and topologies

(Munkres, §§ 12, 13, 14, 16, 20)

Munkres, § 13, p. 83: 3
Munkres, § 16, pp. 91–92: 1, 3

Additional exercises

0. (i) Given a finite set X, explain why there are only finitely many topologies on X.
(ii) List all the topologies on the set $\{0, 1\}$, and prove that the list contains all the possibilities.

1.** In the integers $\mathbb{Z}$ let p be a fixed prime. For each integer $a > 0$ let $U_a(n)$ be the set $\{\, x \mid x = n + kp^a,\ \text{some } k \in \mathbb{Z}\}$. Prove that the sets $U_a(n)$ form a basis for some topology on $\mathbb{Z}$. [*Hint:* Let $\nu_p(n)$ denote the largest nonnegative integer k such that p^k divides n and show that

$$\mathbf{d}_p(a, b) = \frac{1}{p^{\nu_p(a-b)}}$$

defines a metric on $\mathbb{Z}$].

2.* Let $A \subset X$ be closed and let $U \subset A$ be open in A. Let V be any open subset of X with $U \subset V$. Prove that $U \cup (V - A)$ is open in X.

3. Let E be a subset of the topological space X. Prove that every open subset $A \subset E$ is also open in X if and only if E itself is open in X.

4. A subset A of a metric space X is said to be *bounded* if there is some constant $K > 0$ such that $\mathbf{d}(a_1, a_2) \leq K$ for all $a_1, a_2 \in A$.
(i) Prove that for each $\varepsilon > 0$ and each x in a metric space X, the open neighborhood $N_\varepsilon(x)$ is bounded.
(ii) Prove that a subset of a bounded subset is also bounded, and that the union of two bounded sets is also bounded. Why does this not generalize to infinite unions of bounded subsets?

5. (i) Let $\mathbf{U}$ be the family of all subsets V in $\mathbb{R}$ such that V is empty, equal to $\mathbb{R}$, or an open ray of the form (a, ∞) for some $a \in \mathbb{R}$. Prove that $\mathbf{U}$ is a topology on $\mathbb{R}$ which is contained in the usual metric topology (sometimes $\mathbf{U}$ is called the *lower semicontinuous topology*).
(ii) Let $\mathbf{L}$ be the family of all subsets V in $\mathbb{R}$ such that V is empty, equal to $\mathbb{R}$, or an open ray of the form $(-\infty, a)$ for some $a \in \mathbb{R}$. Prove that $\mathbf{L}$ is a topology on $\mathbb{R}$ which is contained in the usual metric topology (sometimes $\mathbf{L}$ is called the *upper semicontinuous topology*).
(iii) Prove that $\mathbf{L} \cup \mathbf{U}$ is not a topology on $\mathbb{R}$. [*Hint:* Show that if $\mathbf{T}$ is a topology containing the union then $\mathbf{T}$ contains the metric topology, and the intersection of a nonempty proper subset of $\mathbf{L}$ with a nonempty proper subset of $\mathbf{U}$ is either empty or else it is not $\mathbf{L} \cup \mathbf{U}$.]

Note. Upper and lower semicontinuous functions play significant roles in real analysis and measure theory; see the article `http://en.wikipedia.org/wiki/Semi-continuity` for a few brief remarks on these concepts.

6. Let X be a topological space, let $\mathcal{B}$ be a base for the topology, and for each $x \in X$ let $\mathcal{B}_x$ be the subfamily of all $U \in \mathcal{B}$ such that $x \in U$ (the *associated neighborhood base at* x). Prove that the subfamilies $\mathcal{B}_x$ have the following properties:

(N1) $x \in U$ for each $U \in \mathcal{N}_x$.
(N2) If $U_1, U_2 \in \mathcal{B}_x$, then there is some $V \in \mathcal{B}_x$ such that $V \subset U_1 \cap U_2$.
(N3) If $V \in \mathcal{B}_x$ and $y \in V$ then there is some $W \subset V$ such that $W \in \mathcal{B}_y$.
(N4) U is open in X if and only if for each $x \in U$ there is some $V_x \in \mathcal{B}_x$ such that $x \in V_x \subset U$.

Conversely, if we are given a set X and families $\mathcal{B}_X$ for each $x \in X$ such that **(N1)** – **(N3)** all hold, prove that there is a unique topology $\mathbf{T}$ on X such that the union $\cup_{x \in X} \mathcal{B}_x$ is a base for $\mathbf{T}$.

II.2 : Closed sets and limit points

(Munkres, § 17)

Munkres, § 17, pp. 100–102: 2, 8a, 8c^*, 19*, 20*

Additional exercises

0. Prove or give a counterexample to the following statement: If U and V are disjoint open subsets of a topological space X, then their closures are also disjoint.

1. Give an example to show that in a metric space the closure of an open ε disk about a point is not necessarily the set of all points whose distance from the center is $\leq \varepsilon$.

Definition. A subspace D of a topological space X is *dense* if $\overline{D} = X$; equivalently, it is dense if and only if for every nonempty open subset $U \subset X$ we have $U \cap D \neq \emptyset$.

2. For which spaces is X the only dense subset of itself?

3. Let U and V be open dense subsets of X. Prove that $U \cap V$ is dense in X.

4. A subspace A of a topological space X is said to be *locally closed* if for each $a \in A$ there is an open neighborhood U of a in X such that $U \cap A$ is closed in U. Prove that A is locally closed if and only if A is the intersection of an open subset and a closed subset.

5. (a) Suppose that D is dense in X, and let $A \subset X$. Give an example to show that $A \cap D$ is not necessarily dense in A.

(b) Suppose that $A \subset B \subset X$ and A is dense in B. Prove that A is dense in $\overline{B}$.

6. Let E be a subset of the topological space X. Prove that every closed subset $A \subset E$ is also closed in X if and only if E itself is closed in X.

7. Given a topological space X and a subset $A \subset X$, explain why the closure of the interior of A does not necessarily contain A.

8.* If U is an open subset of X and B is an arbitrary subset of X, prove that $U \cap \overline{B} \subset \overline{U \cap B}$.

9.* If X is a topological space, then the *Kuratowski closure axioms* are the following properties of the operation $A \to \mathbf{CL}(A)$ sending $A \subset X$ to its closure $\overline{A}$:
(C1) $A \subset \mathbf{CL}(A)$ for all $A \subset X$.
(C2) $\mathbf{CL}(\mathbf{CL}(A)) = \mathbf{CL}(A)$
(C3) $\mathbf{CL}(A \cup B) = \mathbf{CL}(A) \cup \mathbf{CL}(B)$ for all A, $B \subset X$.
(C4) $\mathbf{CL}(\emptyset) = \emptyset$.

Given an arbitrary set Y and a operation $\mathbf{CL}$ assigning to each subset $B \subset Y$ another subset $\mathbf{CL}(B) \subset Y$ such that **(C1)** – **(C4)** all hold, prove that there is a unique topology $\mathbf{T}$ on Y such that for all $B \subset Y$, the set $\mathbf{CL}(B)$ is the closure of B with respect to $\mathbf{T}$.

10. Suppose that X is a space such that $\{p\}$ is closed for all $x \in X$, andn let $A \subset X$. Prove the following statements:

(a) $\mathbf{L}(A)$ is closed in X.

(b) For each point $b \in \mathbf{L}(A)$ and open set U containing b, the intersection $U \cap A$ is infinite.

11.* Suppose that X is a set and that $\mathbf{I}$ is an operation on subsets of X such that the following hold:

(i) $\mathbf{I}(X) = X$.

(ii) $\mathbf{I}(A) \subset A$ for all $A \subset X$.

(iii) $\mathbf{I}(\mathbf{I}(A)) = \mathbf{I}(A)$ for all $A \subset X$.

(iv) $\mathbf{I}(A \cap B) = \mathbf{I}(A) \cap \mathbf{I}(B)$.

Prove that there is a unique topology $\mathbf{T}$ on X such that $U \in \mathbf{T}$ if and only if $\mathbf{I}(A) = A$.

12.* If X is a topological space and $A \subset X$ then the *exterior* of X, denoted by $\mathrm{Ext}(X)$, is defined to be $X - \overline{A}$. Prove that this construction has the following properties:

(a) $\mathrm{Ext}(A \cup B) = \mathrm{Ext}(A) \cap \mathrm{Ext}(B)$.

(b) $\mathrm{Ext}(A) \cap A = \emptyset$.

(c) $\mathrm{Ext}(\emptyset) = X$.

(d) $\mathrm{Ext}(A) \subset \mathrm{Ext}\left(\mathrm{Ext}\left(\mathbf{L}(A)\right)\right)$.

13. Let A_1 and A_2 be subsets of a topological space X, and let B be a subset of $A_1 \cap A_2$ that is closed in both A_1 and A_2 with respect to the subspace to topologies on each of these sets. Prove that B is closed in $A_1 \cup A_2$.

14.* Suppose that A is a closed subset of a topological space X and B is the closure of $\mathrm{Int}(A)$. Prove that $\overline{B} \subset A$ and $\mathrm{Int}(B) = \mathrm{Int}(A)$.

15. Let X be a topological space and let $A \subset Y \subset X$.

(a) Prove that the interior of A with respect to X is contained in the interior of A with respect to Y.

(b)* Prove that the boundary of A with respect to Y is contained in the intersection of Y with the boundary of A with respect to X.

(c)* Give examples to show that the inclusions in the preceding two statements may be proper (it suffices to give one example for which both inclusions are proper).

16. Let X be a topological space and let A be a subspace. In Exercise 18.9 on page 102 of Munkres, the *boundary* of A (in X), written $\mathrm{Bdy}\,(A, X)$ or often just $\mathrm{Bdy}\,(A)$. is defined to be $\overline{A} \cap \overline{X - A}$. In some settings the word "boundary" has another meaning, and to avoid ambiguities it is sometimes useful to use "frontier" as an alternative name and to write $\mathrm{Front}\,(A, X)$ or often just $\mathrm{Front}\,(A)$.

(i) Prove that $\mathrm{Bdy}\,(A) = \overline{A} \cap \overline{X - A} = \mathrm{Bdy}\,(X - A)$.

(ii) Prove that $x \in \mathrm{Bdy}\,(A)$ if and only if every open set containing x contains at least one point of A and at least one point of $X - A$. [*Hint:* We know that $x \in \overline{A}$ if and only if every open set containing x contains a point of A.]

(iii) Prove that x is a frontier/boundary point of A if and only if $x \in \overline{A} - \mathrm{Int}\,A$.

(iv) Prove that A is closed in X if and only if $\mathrm{Bdy}\,(A) \subset A$.

17. Let $A_1, \cdots, A_n$ be subsets of the topological space X. Prove that

$$\mathrm{Int}\left(\bigcap_{j=1}^{n} A_j\right) = \bigcap_{j=1}^{n} \mathrm{Int}\,(A_j)\ .$$

.

18. Let (X, d) be a topological space, and let C and Y be subsets of X. Prove the relative boundary relationship

$$\mathrm{Bdy}\,(C \cap Y, Y) \quad \subset \quad \mathrm{Bdy}\,(C, X)$$

where $\mathrm{Bdy}\,(A, B)$ denotes the boundary of A in B for $A \subset B \subset X$. Give an example where $X = \mathbb{R}$ with the usual metric and the containment is proper.

II.3 : Continuous functions

(Munkres, §§ 18, 21)

Munkres, § 18, pp. 111–112: 2, 3, 6^*, $8a$ with $Y = \mathbb{R}$, $9c$

Additional exercises

1. Give examples of continuous functions from $\mathbb{R}$ to itself that are neither open nor closed.

2. Let X be a topological space, and let $f,\ g : X \to \mathbb{R}$ be continuous. Prove that the functions $|f|$, $\max(f, g)$ [whose value at $x \in X$ is the larger of $f(x)$ and $g(x)$] and $\min(f, g)$ [whose value at $x \in X$ is the smaller of $f(x)$ and $g(x)$] are all continuous. [*Hints:* If $h : X \to \mathbb{R}$ is continuous, what can one say about the sets of points where $h = 0$, $h < 0$ and $h > 0$? What happens if we take $h = f - g$?]

3.* Let $f : X \to Y$ be a set-theoretic mapping of topological spaces.

(a) Prove that f is open if and only if $f[\,\mathrm{Int}(A)\,] \subset \mathrm{Int}\,(\,f[A]\,)$ for all $A \subset X$ and that f is closed if and only if $\overline{f[A]} \subset f\,\left[\,\overline{A}\,\right]$ for all $A \subset X$.

(b) Using this and other results from the course notes, prove that f is continuous and closed if and only if $f[\,\overline{A}\,] = \overline{f[A]}$ for all $A \subset X$ and f is is continuous and open if and only if $f^{-1}[\,\mathrm{Int}(B)\,] = \mathrm{Int}\,\left(\,f^{-1}[B]\,\right)$ for all $B \subset Y$.

4.* A mapping of topological spaces $f : X \to Y$ is said to be *light* if for each $y \in Y$ the subspace $f^{-1}[\,\{x\}\,]$ inherits the discrete topology (every subset is both open and closed). Prove that the composite of two continuous light mappings is also light (of course, it is also continuous).

NOTE. Basic results in complex function theory imply that every nonconstant complex analytic mapping from an open subset $U \subset \mathbb{C}$ to $\mathbb{C}$ is both light and open; one reference for these and other facts in complex analysis is the following classic text:

> L. V. Ahlfors, *Complex Analysis* (3^{rd} Ed.). McGraw-Hill, New York, 1979.
> (See Corollary 1 on p. 132 for openness, and see the third paragraph on p. 127 for lightness.)

A striking converse to the result in the previous paragraph was established by S. Stoïlow in the nineteen twenties: *If U is open in $\mathbb{C}$ and $f : U \to \mathbb{C}$ is continuous, light and open, then there are homeomorphisms h from U to itself and k from (the open set) $V = f[U]$ to itself such that $k^{-1} \circ$ "f" $\circ h$ is complex analytic.* Here "f" refers to the unique map from U to V defined using f. One reference for this theorem and related results is the following:

> G. T. Whyburn, *Topological Analysis* (Revised Edition), Princeton Mathematical Series No. 23. Princeton University Press, Princeton, NJ, 1964.

5. If $f(x,y) = (x^2 - y^2)/(x^2 + y^2)$ unless $x = y = 0$ and $f(0,0) = 0$, show that $f : \mathbb{R}^2 \to \mathbb{R}$ is not continuous at $(0,0)$. [*Hint:* Consider the behavior of f on straight lines through the origin.]

6. Let $f(x,y) = 2\,x^2y/(x^4 + y^2)$ unless $x = y = 0$ and $f(0,0) = 0$, and define $\varphi(t) = (t, at)$ and $\psi(t) = (t, t^2)$.

(a) Show that $\lim_{t\to 0} f\circ\varphi(t) = 0$; *i.e.*, f is continuous on every line through the origin.

(b) Show that $\lim_{t\to 0} f\circ\psi(t) \neq 0$ and give a rigorous argument to explain why this and the preceding part of the exercise imply f is not continuous at $(0,0)$.

7. Let X and Y be metric spaces, and let $r > 0$. A mapping $f : X \to Y$ is said to be a *similarity transformation with ratio of similitude* r if it is 1–1 onto and for all $(u,v) \in X$ we have

$$\mathbf{d}_Y\,(\,f(u), f(v)\,) \quad = \quad r \cdot \mathbf{d}_X(u,v)\ .$$

It follows immediately that a 1–1 onto map is an isometry if and only if it is a similarity transformation with ratio of similitude equal to 1.

(i) Prove that a similarity transformation is uniformly continuous.

(ii) Prove that if $f : X \to Y$ is a similarity transformation with ratio of similitude r, then f^{-1} is a similarity transformation with ratio of similitude r^{-1}.

(iii) Prove that if $f : X \to Y$ and $g : Y \to Z$ are similarity transformations with ratio of similitude r and s respectively, then $g\circ f$ is a similarity transformation with ratio of similitude $s\cdot r$.

Note. Further information about similarity transformations from $\mathbb{R}^n$ to itself is summarized in `metgeom.pdf` and `affine+convex.pdf`.

8. Suppose that $f : X \to Y$ is a homeomorphism of topological spaces, and that $A \subset X$ and $B \subset Y$ are such that $f[A] = B$. Prove that f maps $X - A$ homeomorphically onto $Y - B$.

9. Let $\mathbf{U}$ denote the topology in Exercise II.1.5, which consists of $\mathbb{R}$, the empty set and all intervals (b, ∞) where b is some real number. If $(X, \mathbf{T})$ is a topological space, a continuous function $f : (X, \mathbf{T}) \to (\mathbb{R}, \mathbf{U})$ is said to be *lower semi-continuous*.

(i) Let $(X, \mathbf{T})$ be a topological space, and let f be a real valued function on X. Prove that f is lower semi-continuous if and only if for each $x \in X$ and $\varepsilon > 0$ there is some open set $U_{\varepsilon,x}$ containing x such that $y \in U_{\varepsilon,x}$ implies $f(y) > f(x) - \varepsilon$.

(ii) Let (a,b) be an open interval in $\mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ be the function whose value is 1 on $[a,b]$ and zero elsewhere, and let $\mathbf{T}$ be the usual metric topology on $\mathbb{R}$. Prove that $f : (\mathbb{R}, \mathbf{T}) \to (\mathbb{R}, \mathbf{U})$ is lower semi-continuous.

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function (but not necessarily strictly increasing), and let $\mathbf{M}$ be the usual topology on $\mathbb{R}$. Prove that if f satisfies the one-sided continuity condition $f(x) = \text{L.U.B.}_{\,t<x}\ f(t)$ for all x, then $f : (\mathbb{R}, \mathbf{M}) \to (\mathbb{R}, \mathbf{U})$ is lower semi-continuous.

(iv) Let $a, b \in \mathbb{R}$ with $a \geq 0$, and let $f(x) = ax + b$, where $x \in \mathbb{R}$. Prove that $f : (\mathbb{R}, \mathbf{U}) \to (\mathbb{R}, \mathbf{U})$ is continuous.

(v) Let $a, b \in \mathbb{R}$ with $a \geq 0$, and let $f(x) = -x$, where $x \in \mathbb{R}$. Prove that $f : (\mathbb{R}, \mathbf{U}) \to (\mathbb{R}, \mathbf{U})$ is not continuous.

II.4 : Cartesian products

(Munkres, §§ 15, 19)

Munkres, § 18, pp. 111–112: 4, 10, 11

Munkres, § 20, pp. 126–129: 3*b*

Additional exercises

1.[**] ("A product of products is a product.") Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of nonempty sets, and let $\mathcal{A} = \cup\{\mathcal{A}_\beta \mid \beta \in \mathcal{B}\}$ be a partition of $\mathcal{A}$. Construct a bijective map of $\prod\{A_\alpha \mid \alpha \in \mathcal{A}\}$ to the set

$$\prod_\beta \{\prod\{A_\alpha \mid \alpha \in \mathcal{A}_\beta\}\ \}\ .$$

If each A_α is a topological space and we are working with product topologies, prove that this bijection is a homeomorphism.

2.[**] *Non-Hausdorff topology.* In topology courses one is ultimately interested in spaces that are Hausdorff. However, there are contexts in which certain types of non-Hausdorff spaces arise (for example, the Zariski topologies in algebraic geometry, which are defined in most textbooks on that subject or in `http://math.ucr.edu/~res/math145A-2014/math145Anotes07-08.pdf`, pp. 7.5–7.6).

(*a*) A topological space X is said to be *irreducible* if it cannot be written as a union $X = A \cup B$, where A and B are proper closed subspaces of X. Show that X is irreducible if and only if every pair of nonempty open subsets has a nonempty intersection. Using this, show that an open subset of an irreducible space is irreducible.

(*b*) Show that every set with the indiscrete topology is irreducible, and every infinite set with the finite complement topology is irreducible.

(*c*) Show that an irreducible Hausdorff space contains at most one point.

3. (*i*) Let $f : X \to Y$ be a map, and define the *graph* of f to be the set Γ_f of all points $(x, y) \in X \times Y$ such that $y = f(x)$. Prove that the map $x \to (x, f(x))$ is a homeomorphism from X to Γ_f if and only if f is continuous.

(*ii*) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that Γ_f is closed but f is not continuous. [*Hint:* Set $f(x) = 1/x$ for $x \neq 0$.]

4. Let X be a topological space that is a union of two closed subspaces A and B, where each of A and B is Hausdorff in the subspace topology. Prove that X is Hausdorff.

5.[*] Let A be some nonempty set, let $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ and $\{Y_\alpha \mid \alpha \in \mathcal{A}\}$ be families of topological spaces, and for each $\alpha \in A$ suppose that $f_\alpha : X_\alpha \to Y_\alpha$ is a homeomorphism. Prove that the product map

$$\prod_\alpha f_\alpha\ :\ \prod_\alpha X_\alpha \longrightarrow \prod_\alpha Y_\alpha$$

is also a homeomorphism. [*Hint:* What happens when you take the product of the inverse maps?]

6. Let X be a topological space and let $T : X \times X \times X \to X \times X \times X$ be the map that cyclically permutes the coordinates: $T(x, y, z) = (z, x, y)$ Prove that T is a homeomorphism. [*Hint:* What is the test for continuity of a map into a product? Can you write down an explicit formula for the inverse function?]

7. Let $\alpha, \beta \in \{1, 2, \infty\}$, and let $|\cdots|_\alpha$ and $|\cdots|_\beta$ be the norms for $\mathbb{R}^n$ that were described in Section II.1.

(*a*) Explain why there are positive constants m and M (depending upon α and β) such that

$$m \cdot |x|_\beta \quad \leq \quad |x|_\alpha \quad \leq \quad M \cdot |x|_\beta$$

for all $x \in \mathbb{R}^n$.

(b) Explain why the interior of the closed unit disk with $\mathbf{d}_\alpha$ radius 1 in $\mathbb{R}^n$ is the set of all x such that $|x|_\alpha < 1$ and the frontier is the set of all x such that $|x|_\alpha = 1$.

(c) Prove that there is a homeomorphism h from $\mathbb{R}^n$ to itself such that $|h(x)|_\beta = |x|_\alpha$ for all x. [*Hints:* One can construct this so that $h(x)$ is a positive multiple of x. It is necessary to be a little careful when checking continuity at the origin.]

(d) Prove that the hypercube $[-1,1]^n$ is homeomorphic to the unit disk of all points $x \in \mathbb{R}^n$ satisfying $\sum\; x_i^2 \;=\; 1$ such that the frontier of the hypercube is mapped onto the unit sphere.

8.* Suppose that X, X', Y and Y' are metric spaces and that $f : X \to X'$ and $g : Y \to Y'$ are similarity transformations whose ratios of similitude are both equal to $r > 0$ (see Exercise II.3.7 for definitions). Prove that $f \times g$ is a similarity transformation with ratio of similitude r, where the metrics on the products are the $\mathbf{d}^{\langle p \rangle}$ product metrics for $p = 1, 2$ or ∞ (there are three separate conclusions; it is possible to give a proof which handles all three simultaneously).

9. Suppose that $(X, \mathbf{T}_X)$ and $(Y, \mathbf{T}_Y)$ are topological spaces such that $\mathbf{T}_X$ and $\mathbf{T}_Y$ are the discrete topologies. Explain why the product topology on $X \times Y$ is also the discrete topology.

10. Suppose that X and Y are topological spaces with $A \subset X$ and $B \subset Y$. Assume that the topology on $X \times Y$ is the product topology.

(i) Prove that $\mathrm{Int}\,(A \times B) = \mathrm{Int}\,(A) \times \mathrm{Int}\,(B)$.

Note. Recall that an analog of (i) for closures was proved in the course notes.

(ii) Prove that $\mathrm{Bdy}\,(A \times B) \;=\; \big(\mathrm{Bdy}\,(A) \times \overline{B}\big) \;\cup\; \big(\overline{A} \times \mathrm{Bdy}\,(B)\,\big)$.

11. Specialize the preceding exercise to $X = Y = \mathbb{R}$ and $A = B = (0,1)$ or $A = B = [0,1]$, and use the statements in the exercise to prove that the closure of $(0,1) \times (0,1)$ is $[0,1] \times [0,1]$, the interior of $[0,1] \times [0,1]$ is $(0,1) \times (0,1)$, and the boundaries of both $(0,1) \times (0,1)$ and $[0,1] \times [0,1]$ are given (as one would intuitively expect) by $[0,1] \times [0,1] \;-\; (0,1) \times (0,1)$.

12. The purpose of this exercise is to prove a generalization of the preceding result to certain subspaces of $\mathbb{R}^2$ which play a central role in the treatment of double integrals in multivariable calculus. — Let $a < b$ in $\mathbb{R}$, let f and g be continuous real valued functions on $[a,b]$ such that $g(x) < f(x)$ for all $x \in [a,b]$. Define $V, A \subset \mathbb{R}^2$ as follows:

$$V \;=\; \{(x,y) \mid a \;<\; x \;<\; b\,, \quad g(x) \;<\; y \;<\; f(x)\}$$
$$A \;=\; \{(x,y) \mid a \;\leq\; x \;\leq\; b\,, \quad g(x) \;\leq\; y \;\leq\; f(x)\}$$

Prove that V is open, A is closed, the closure of V (in $\mathbb{R}^2$) is equal to A, the interior of A (in $\mathbb{R}^2$) is equal to V, and the boundaries of both V and A are equal to $A - V$. [*Hints:* Why will this follow if we can find a homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that H sends $(0,1) \times (0,1)$ to V and $[0,1] \times [0,1]$ to A? To construct such a homeomorphism, first extend f and g to all points in $\mathbb{R}$ by setting $g(x) = g(a)$ and $f(x) = f(a)$ if $x \leq a$, and similarly setting $g(x) = g(b)$ and $f(x) = f(b)$ if $x \geq b$. Consider the mapping $H : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $H(s,t) = (x,y)$ where $x = a + s(b-a)$ and $y = g(x) + t(f(x) - g(x))$. Prove that H is a homeomorphism by solving these equations for s and t uniquely in terms of x and y, and complete the argument by verifying that V and A are the images of the subsets described in the first sentence of these hints. Note that homeomorphisms automatically preserve preserve closures, interiors and boundaries; you may use this fact without proving it here.]

Drawings for this problem are contained in the file `math205Asolutions2a.pdf`.

Generalizations. There are analogous results for the standard types of domains in $\mathbb{R}^3$ over which one takes triple integrals, which are defined by the following inequalities such that all functions in the displayed lines are continuous:

$$\begin{array}{rcl} a \quad \leq \quad x \quad \leq \quad b\ , & \text{where} & a \quad < \quad b \\ u(x) \quad \leq \quad y \quad \leq \quad v(x)\ , & \text{where} & u(x) \quad < \quad v(x) \quad \text{for all} \quad x \\ f(x,y) \quad \leq z \quad \leq \quad g(x,y)\ , & \text{where} & f(x,y) \quad < \quad g(x,y) \quad \text{for all} \quad (x,y) \end{array}$$

In multivariable calculus, one also sees 5 analogs of these inequalities in which the roles of the x, y and z variables are permuted. There are also similar generalizations to suitably defined subsets of $\mathbb{R}^n$ when $n \geq 4$, but we shall not attempt to describe them here.

III. Spaces with special properties

III.1 : Compact spaces – I

(Munkres, §§ 26, 27)

Munkres, § 26, pp. 170–172: 3, 7*, 8*
Munkres, § 27, pp. 177–178: 2*abcd*, 2*e**

Additional exercises

1. Let X be a compact Hausdorff space, and let $f : X \to X$ be continuous. Define $X_1 = X$ and $X_{n+1} = f[\,X_n\,]$, and set $A = \bigcap_n X_n$. Prove that A is a nonempty subset of X and $f[A] = A$.

2.** A topological space X is said to be a k-**space** if it satisfies the following condition: *A subset $A \subset X$ is closed if and only if for all compact subsets $K \subset X$, the intersection $A \cap K$ is closed.* It turns out that a large number of the topological spaces one encounters in topology, geometry and analysis are k-spaces (including all metric spaces and compact Hausdorff spaces), and the textbooks by Kelley and Dugundji contain a great deal of information about these k-spaces (another important reference is the following paper by N. E. Steenrod: *A convenient category of topological spaces*, Michigan Mathematical Journal **14** (1967), pp. 133–152).

(a) Prove that if $(X, \mathbf{T})$ is a Hausdorff topological space then there is a unique minimal topology $\mathbf{T}^\kappa$ containing $\mathbf{T}$ such that $\mathcal{K}(X) = (X, \mathbf{T}^\kappa)$ is a Hausdorff k-space.

(b) Prove that if $f : X \to Y$ is a continuous map of Hausdorff topological spaces, then f is also continuous when viewed as a map from $\mathcal{K}(X) \to \mathcal{K}(Y)$.

3.** *Non-Hausdorff topology revisited.* A topological space X is *noetherian* if every nonempty family of open subsets has a maximal element. This class of spaces is also of interest in algebraic geometry.

(a) (*Ascending Chain Condition*) Show that a space X is noetherian if and only if every increasing sequence of open subsets

$$U_1 \subset U_2 \subset \ldots$$

stabilizes; *i.e.*, there is some positive integer N such that $n \geq N$ implies $U_n = U_N$.

(b) Show that a space X is noetherian if and only if every open subset is compact.

(c) Show that a noetherian Hausdorff space is finite (with the discrete topology). [*Hint:* Show that every open subset is closed.]

(d) Show that a subspace of a noetherian space is noetherian.

4. Suppose that X is a Hausdorff space and $A \subset X$ is a subspace whose closure in X is compact. Prove that the set $L(A)$ of limit points for A is also compact. [*Hint:* Why do we know that $L(A)$ is closed in X?]

5. (i) Let $(X, \mathbf{T})$ be a compact topological space, and let $f : (X, \mathbf{T}) \to (\mathbb{R}, \mathbf{U})$ be a (lower semi-)continuous mapping. Prove that $f[X]$ has a lower bound. [*Hint:* First explain why $(\mathbb{R}, \mathbf{U})$ is not compact, then find a finite family of proper open subsets such that $f[X]$ is contained in their union, and finally explain why this yields a lower bound for $f[X]$.]

(ii) In the setting of (i), prove that f takes a minimum value. [*Hint:* By (i) we know that $f[X]$ has a greatest lower bound, say m. Why do the sets $f^{-1}\left[\,(-\infty, m + \frac{1}{n}\,\right]$ form a nested collection of nonempty closed subsets, and why is their intersection nonempty?]

III.2 : Complete metric spaces

(Munkres, §§ 43, 45)

Munkres, § 43, pp. 270–271: 1, 3*b*, 6*a*, 6*c**

Additional exercises

1. Show that the Nested Intersection Property for complete metric spaces does not necessarily hold for nested sequences of closed subsets $\{ A_n \}$ if $\lim_{n \to \infty} \mathrm{diam}(A_n) \neq 0$; *i.e.*, in such cases one might have $\cap_n A_n = \emptyset$. [*Hint:* Consider the set A_n of all continuous functions f from $[0,1]$ to itself that are zero on $\left[\frac{1}{n}, 1\right]$ and also satisfy $f(0) = 1$.]

2.** Let ℓ^2 be the set of all real sequences $\mathbf{x} = \{ x_n \}$ such that $\sum_n |x_n|^2$ converges. The results of Exercise 10 on page 128 of Munkres show that ℓ^2 is a normed vector space with the norm

$$|\,\mathbf{x}\,| = \left(\sum_n |x_n|^2 \right)^{1/2} .$$

Prove that ℓ^2 is complete with respect to the associated metric. [*Hint:* If $\mathbf{p}_i$ gives the i^{th} term of an element in ℓ^2, show that $\mathbf{p}_i$ takes Cauchy sequences to Cauchy sequences. This gives a candidate for the limit of a Cauchy sequence in ℓ^2. Show that this limit candidate actually lies in ℓ^2 and that the Cauchy sequence converges to it. See also Royden, *Real Analysis*, Section 6.3, pages 123–127, and also Rudin, *Principles of Mathematical Analysis*, Theorem 11.42 on page 329–330 together with the discussion on the following two pages.]

III.3 : Implications of completeness

(Munkres, § 48)

Munkres, § 27, pp. 177–178: 6*
Munkres, § 48, pp. 298–300: 1, 2*, 4*

Additional exercises

1. Let A and B be subspaces of X and Y respectively such that $A \times B$ is nowhere dense in $X \times Y$ (with respect to the product topology). Prove that either A is nowhere dense in X or B is nowhere dense in Y, and give an example to show that "or" cannot be replaced by "and."

2. Is there an uncountable topological space of the first category?

3.* Let X be a metric space. A map $f : X \to X$ is said to be an *expanding similarity* of X if f is onto and there is a constant $C > 1$ such that

$$\mathbf{d}\,(\,f(u),\ f(v)\,) \;=\; C \cdot \mathbf{d}(u,v)$$

for all u, $v \in X$ (hence f is 1–1 and uniformly continuous). Prove that every expanding similarity of a complete metric space has a unique fixed point. [*Hint and comment:* Why does f have an inverse that is uniformly continuous, and why does $f(x) = x$ hold if and only if $f^{-1}(x) = x$? If $X = \mathbb{R}^n$ and a similarity is given by $f(x) = cAx + b$ where A comes from an orthogonal matrix and either $0 < c < 1$ or $c > 1$, then one can prove the existence of a unique fixed point directly using linear algebra.]

4. Consider the sequence $\{x_n\}$ defined recursively by the formula

$$x_{n+1} \quad = \quad \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \; .$$

If $x_0 > 0$ and $x_0^2 > a$, show that the sequence $\{\, x_n \,\}$ converges by proving that

$$\varphi(x) \quad = \quad \frac{1}{2}\left(x + \frac{a}{x}\right)$$

is a contraction mapping on $[\sqrt{a}, x_0]$.

5. (i) A *hyperplane* in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ defined by a nontrivial first degree polynomial equation

$$0 \quad = \quad F(x_1, \ \cdots \ , x_n) \quad = \quad \left(\sum_{i=0}^{n} \ a_i x_i\right) \ - \ b$$

where a_i and b are real numbers and at least one of the coefficients a_i is nonzero. If H is such a hyperplane, prove that H is closed in $\mathbb{R}^n$, its interior in $\mathbb{R}^n$ is empty and $\mathbb{R}^n - H$ is dense in $\mathbb{R}^n$. [*Hint:* If N is the normal vector to H with coordinates $N = (a_1, \ \cdots \ , a_n)$ prove that the function $F(y + tN)$ is not constant. Why does this imply that if $y \in H$ then $y + tN \notin H$ if $t \neq 0$?]

(ii) If $H_1, ..., H_k$ is a finite set of hyperplanes in $\mathbb{R}^n$, prove that $\mathbb{R}^n - (\cup_j \ H_j)$ is dense in $\mathbb{R}^n$.

Notes. The file `zariski-topology.pdf` establishes a similar result when hyperplanes are replaced by subsets of $\mathbb{R}^n$ which are the zero sets of nontrivial polynomials in n variables with real coefficients. Also, Baire's Theorem implies that this exercise and the generalization in previous sentence remain valid for countably infinite unions of hyperplanes (as in the exercise) or zero sets of polynomials (as in the generalization).

6.* Let $F(x_1, \ \cdots \ , x_n)$ be a nontrivial first degree polynomial function as in the preceding exercise, and define two subsets $H_-(F)$ and $H_+(F)$ to be the sets of points where $F(x_1, \ \cdots \ , x_n) < 0$ and $F(x_1, \ \cdots \ , x_n) > 0$ respectively.

(i) Prove that for each F the sets $H_-(F)$ and $H_+(F)$ are nonempty, open, and convex (if x and y belong to the set, then so does the closed segment joining them, which is all points of the form $(1-t)x + ty$ where $t \in [0, 1]$. [*Hint:* Use the methods of the preceding exercise to find points in each subset. Prove openness using the continuity of F, and prove convexity by first verifying that

$$F(ty \ + \ (1-t)x) \quad = \quad t\,F(y) \ + \ (1-t)\,F(x)$$

for all $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$.] — The subspaces $H_\pm(F)$ are called the *open half-spaces* determined by F.

(ii) Prove that the hyperplane H defined by $F(x_1, \ \cdots \ , x_n) = 0$ **separates** $H_+(F)$ **from** $H_-(F)$ in the following sense: If $x \in H_+(F)$ and $y \in H_-(F)$, then there is some t^* such that $0 < t^* < 1$ and $t^*x + (1-t^*)y \in H$. [*Hint:* In other words, find a solution to $F(ty \ + \ (1-t)x) \ = \ 0$ such that $0 < t < 1$.]

(iii) Prove that the family of all open half-spaces $H_\pm(F)$ — where F runs through all nontrivial first degree polynomials in the variables $x_1, \ \cdots \ , x_n$ and $\pm = +$ or $-$ as usual — forms a subbase for the usual topology on $\mathbb{R}^n$.

Notes. The conclusion of this exercise can be used to define topologies on mathematical systems which arise from various sets of axioms for classical synthetic geometry (*i.e.*, modernized and logically debugged axiomatic settings for doing geometry synthetically as in Euclid's *Elements*). See the file `synthetic-geom.pdf` for background information and some further discussion of this point.

III.4 : Connected spaces

(Munkres, §§ 23, 24, 25)

Munkres, § 23, p. 152: 2 – 5, 9, 12* assuming Y is closed
Munkres, § 24, pp. 157–159: 1b

Additional exercises

1. Prove that a topological space X is connected if and only if every open covering $\mathcal{U} = \{U_\alpha\}$ has the following property: For each pair of sets U, V in $\mathcal{U}$ there is a sequence of open sets $\{U_0, U_1, ...U_n\}$ in $\mathcal{U}$ such that $U = U_0$, $V = U_n$, and $U_i \cap U_{i+1} \neq \emptyset$ for all i.

2. Let X be a connected space, and let $\mathcal{R}$ be an equivalence relation that is locally constant (for each point x all points in some neighborhood of x lie in the $\mathcal{R}$-equivalence class of x). Prove that $\mathcal{R}$ has exactly one equivalence class.
Note. See `zerogradient.pdf` for a simple but important consequence of this exercise.

3. Prove that an open subset in $\mathbb{R}^n$ can have at most countably many components. Give an example to show this is not necessarily true for closed sets.

4. For each of the following statements, either prove that the statement is true or construct a counterexample.

(i) Suppose that X and Y are topological spaces with $A \subset X$ and $B \subset Y$. If neither $X - A$ nor $Y - B$ is connected, then $(X \times Y) - (A \times B)$ is not connected.

(ii) Suppose that A and B are subsets of a topological space X such that $A \cap B$ and $A \cup B$ are connected. Then A and B are connected.

(iii) Same statement as in (ii) with the added assumption that A and B are closed in X.

5. Suppose that A and B are subsets of a topological space X and B is connected. If $B \cap A$ and $B \cap (X - A)$ are both nonempty, prove that $B \cap \mathrm{Bdy}\,(A)$ is also nonempty.

6. (i) A topological space is said to be *totally disconnected* if it has a base consisting of subsets which are both open and closed. Explain why a discrete space is totally disconnected, and show that the set of all points x in $\mathbb{R}$ such that $x = 0$ or $x = 1/n$ for some positive integer n is a totally disconnected space which is not (homeomorphic to) a discrete space. [*Hint:* A subset of a discrete space has no limit points.]

(ii) Prove that the rational numbers (with the subspace topology inherited from $\mathbb{R}$) is a totally disconnected space in which every point is a limit point. [*Hint:* If q is a rational number, explain why for each positive integer n the sets

$$\left(q - \tfrac{1}{n}\sqrt{2},\ \ q + \tfrac{1}{n}\sqrt{2}\right) \cap \mathbb{Q} \qquad \text{and} \qquad \left[q - \tfrac{1}{n}\sqrt{2},\ \ q + \tfrac{1}{n}\sqrt{2}\right] \cap \mathbb{Q}$$

are equal, and hence these subsets are open and closed in $\mathbb{Q}$.]

(iii) Prove that a product of two totally disconnected topological spaces is also totally disconnected.

7. Let C_n denote the set of connected subsets of $\mathbb{R}^n$, where $n \geq 0$. Explain why the cardinalities of these sets satisfy $|C_1| = 2^{\aleph_0}$ and $|C_n| > 2^{\aleph_0}$ if $n \geq 2$. [*Hint:* If $n \geq 2$, look at connected subsets which lie between the open disk $x^2 + y^2 < 1$ and the closed disk $x^2 + y^2 = 1$.]

8.** The goal of this exercise is to prove the following result: *If $n \geq 2$ and $D \subset \mathbb{R}^n$ is countable, then $\mathbb{R}^n - D$ is arcwise connected.*

(i) Let D be a countable subset of $\mathbb{R}^n$ where $n \geq 2$, suppose $p \notin D$, and define the set $\mathrm{Lines}\,(p, \mathbb{R}^n - D)$ to be the set of all points which lie on a line px such that $px \subset \mathbb{R}^n - D$. Explain why $\mathrm{Lines}\,(p, \mathbb{R}^n - D)$ is arcwise connected and its complement is the union of countably many sets of the form $pz - \{p\}$, where z runs through the points in D.

(ii) Suppose that $p, q \in \mathbb{R}^n - D$. Prove that

$$\mathrm{Lines}\,(p, \mathbb{R}^n - D) \;\cap\; \mathrm{Lines}\,(q, \mathbb{R}^n - D) \quad \neq \quad \emptyset\ .$$

[*Hint:* If x is not on the line joining p and q, why are there only countably many points $y \in px$ such that qy contains a point of D? — Given two points a and b, the line ab is defined to be all points z such that $z = ta + (1-t)b$ for some $t \in \mathbb{R}$. You may use the fact that two distinct lines have at most one point in common; proofs of this fact are discussed in the solutions file.]

(iii) Explain why the preceding observation implies that $\mathbb{R}^n - D$ is arcwise connected.

III.5 : Variants of connectedness

(Munkres, §§ 23, 24, 25)

Munkres, § 24, pp. 157–159: 8b
Munkres, § 25, pp. 162–163: 10ab, 10c^* [only Examples A and B]

Additional exercises

1. Prove that a compact locally connected space has only finitely many components.

2.* Give and example to show that if X and Y are locally connected metric spaces and $f : X \to Y$ is continuous then $f[X]$ is not necessarily locally connected.

3. Prove that the closed annulus (ring shaped region) in $\mathbb{R}^2$ defined by $1 \leq x^2 + y^2 \leq 2$ is arcwise connected.

4. For each $n \geq 2$ prove that the unit sphere

$$S^{n-1} \quad = \quad \{\ v \in \mathbb{R}^n \mid |v|^2 \quad = \quad 1\ \}$$

and $\mathbb{R}^n - \{\mathbf{0}\}$ are arcwise connected. [*Hint:* Do the second one first.]

5. (i) Let X be a topological space, and let $p \in X$. Prove that the quasicomponent of X (as defined in Munkres, Exercise 25.10) is the intersection of all clopen (closed + open) subsets of X which contain p.

(ii) Let X be a space with finitely many components. Prove that each component is also a quasicomponent.

IV. Function spaces

IV.1 : General properties

(Munkres, §§ 45–47)

Additional exercises

1. Suppose that X and Y are metric spaces such that X is compact. Let Y^X denote the cartesian product of the spaces $Y \times \{x\} \cong Y$ with the product topology, and for each $x \in X$ let $p_x : Y^X \to Y$ denote projection onto the factor corresponding to x. Let $q : \mathbf{C}(X,Y) \to Y^X$ be the map such that for each x the composite $p_x \circ q$ sends f to $f(x)$. Prove that q is a continuous 1–1 mapping.

2. Suppose that X, Y and Z are metric spaces such that X is compact, and let p_Y, p_Z denote projections from $Y \times Z$ to Y and Z respectively. Assume that one takes the $\mathbf{d}_\infty$ or maximum metric on the product (*i.e.*, the distance between (x_1, y_1) and (x_2, y_2) is the larger of $\mathbf{d}(x_1, y_1)$ and $\mathbf{d}(x_2, y_2)$). Prove that the map from $\mathbf{C}(X, Y \times Z)$ to $\mathbf{C}(X,Y) \times \mathbf{C}(X,Z)$ sending f to $(p_y \circ f, p_Z \circ f)$ is an isometry (where the codomain also has the corresponding $\mathbf{d}_\infty$ metric).

3. Suppose that X and Y are metric spaces such that X is compact, and assume further that X is a union of two disjoint open and closed subsets A and B. Prove that the map from $\mathbf{C}(X,Y)$ to $\mathbf{C}(A,Y) \times \mathbf{C}(B,Y)$ sending f to $(f|A, f|B)$ is also an isometry, where as before the product has the $\mathbf{d}_\infty$ metric.

4. Suppose that X, Y, Z, W are metric spaces such that X and Z are compact. Let $P : \mathbf{C}(X,Y) \times \mathbf{C}(Z,W) \to \mathbf{C}(X \times Z, Y \times W)$ be the map sending (f, g) to the product map $f \times g$ (recall that $f \times g(x,y) = (f(x), g(y))$). Prove that P is continuous, where again one uses the associated $\mathbf{d}_\infty$ metrics on all products.

5. Suppose that X and Y are metric spaces and $f : X \to Y$ is a homeomorphism.

(*i*) If A is a compact metric space, show that the map $V(f)$ defines a homeomorphism from $\mathbf{C}(A,X)$ to $\mathbf{C}(A,Y)$. [*Hint:* Consider $V(h)$, where $h = f^{-1}$.]

(*ii*) If X and Y are compact and B is a metric space, show that the map $U(f)$ defines a homeomorphism from $\mathbf{C}(Y,B)$ to $\mathbf{C}(X,B)$. [*Hint:* Consider $U(h)$, where $h = f^{-1}$.]

(*iii*) Suppose that A and A' are homeomorphic compact metric spaces and B and B' are homeomorphic metric spaces. Prove that $\mathbf{C}(A,B)$ is homeomorphic to $\mathbf{C}(A', B')$.

6. Given a metric space X and $a, b \in X$, let $\mathbf{P}(X; a, b)$ be the space of all continuous functions or curves γ from $[0,1]$ to X such that $\gamma(0) = a$ and $\gamma(1) = b$. Given $a, b, c \in X$ define a concatenation map

$$\alpha : \mathbf{P}(X; a, b) \times \mathbf{P}(X; b, c) \longrightarrow \mathbf{P}(X; a, c)$$

such that $\alpha(\gamma, \gamma')(t) = \gamma(2t)$ if $t \leq \frac{1}{2}$ and $\gamma'(2t-1)$ if $t \geq \frac{1}{2}$. Informally, this is a reparametrization of the curve formed by first going from a to b by γ, and then going from b to c by γ'. **Prove** that this concatenation map is uniformly continuous (again take the $\mathbf{d}_\infty$ metric on the product).

7. Let X and Y be topological spaces, take the compact-open topology on $\mathbf{C}(X,Y)$, and let $k : Y \to \mathbf{C}(X,Y)$ be the map sending a point $y \in Y$ to the constant function $k(y)$ whose value at every point is equal to y. Prove that k maps Y homeomorphically to $k(Y)$. [*Hints:* First verify that k is 1–1. If $\mathcal{W}(K,U)$ is a basic open set for the compact-open topology with $K \subset X$ compact and $U \subset Y$ open, prove that $k^{-1}(\mathcal{W}(K,U)) = U$ and $k(U) = \mathcal{W}(K,U) \cap \mathrm{Image}(k)$. Why do the latter identities imply the conclusion of the exercise?]

8. Let $\mathcal{C}[0,1]$ be the Banach space of continuous real valued functions on $[0,1]$ with the usual norm ($|f|$ = maximum value of $|f(t)|$ for $t \in [0,1]$), let $A, B > 0$, and define $\mathbf{Diff}(A,B)$ to be the subset of all continuously differentiable functions f such that $|f(t)| \leq A$ and $|f'(t)| \leq B$ for all $t \in [0,1]$. Prove that $\mathbf{Diff}(A,B)$ is equicontinuous, and hence it has a sequentially compact closure by the Arzelà-Ascoli Theorem. [*Hint:* Use the Mean Value Theorem to show that $|f(s) - f(t)| \leq B \cdot |s - t|$ for all $s, t \in [0,1]$; without loss of generality, we might as well assume that $s < t$.]

IV.2 : Adjoint equivalences

(Munkres, §§ 45–46)

Additional exercises

1. State and prove an analog of the main result of this section in which X, Y, Z are untopologized sets (with no cardinality restrictions!) and spaces of continuous functions are replaced by sets of set-theoretic functions.

2. Suppose that X is a compact metric space and Y is a convex subset of $\mathbb{R}^n$. Prove that $\mathbf{C}(X,Y)$ is arcwise connected. [*Hint:* Let $f : X \to Y$ be a continuous function. Pick some $y_0 \in Y$ and consider the map $H : X \times [0,1] \to Y$ sending (x,t) to the point $(1-t)f(x) + ty_0$ on the line segment joining $f(x)$ to y_0.]

3. Suppose that X, Y, Z are metric spaces such that X and Y are compact. Prove that there is a 1–1 correspondence between continuous functions from X to $\mathbf{C}(Y,Z)$ and continuous functions from Y to $\mathbf{C}(X,Z)$. As usual, assume the function spaces have the topologies determined by the uniform metric.

V. Constructions on spaces

V.1 : Quotient spaces

(Munkres, § 22)

Munkres, § 22, pp. 144–145: 4

Additional exercises

0. Suppose that X is a space with the discrete topology and $\mathcal{R}$ is an equivalence relation on X. Prove that the quotient topology on $X/\mathcal{R}$ is discrete.

1.* If A is a subspace of X, a continuous map $r : X \to A$ is called a *retraction* if the restriction of r to A is the identity. Show that a retraction is a quotient map.

2. Let $\mathcal{R}$ be an equivalence relation on a space X, and assume that $A \subset X$ contains points from every equivalence class of $\mathcal{R}$. Let $\mathcal{R}_0$ be the induced equivalence relation on A, and let

$$j : A/\mathcal{R}_0 \to X/\mathcal{R}$$

be the associated $1-1$ correspondence of equivalence classes. Prove that j is a homeomorphism if there is a retraction $r : X \to A$ such that each set $r^{-1}[\,\{a\}\,]$ is contained in an $\mathcal{R}$-equivalence class.

3. (a) Let 0 denote the origin in $\mathbb{R}^3$. In $\mathbb{R}^3 - \{0\}$ define $x\mathcal{R}y$ if y is a nonzero multiple of x (geometrically, if x and y lie on a line through the origin). Show that $\mathcal{R}$ is an equivalence relation; the quotient space is called the *real projective plane* and denoted by $\mathbb{R}\mathbf{P}^2$.

(b) Using the previous exercise show that $\mathbb{RP}^2$ can also be viewed as the quotient of S^2 modulo the equivalence relation $x \sim y \Longleftrightarrow y = \pm\, x$. In particular, this shows that $\mathbb{RP}^2$ is compact. [*Hint:* Let r be the radial compression map that sends v to $|v|^{-1}v$.]

NOTE. One can also prove that $\mathbb{RP}^2$ is Hausdorff, but this property is more difficult to verify. The file `rpn-in-rk.pdf` proves a stronger result; namely, $\mathbb{RP}^2$ is homeomorphic to a subset of $\mathbb{R}^M$ for some positive integer M. A more geometrical proof of the Hausdorff property for $\mathbb{RP}^2$ is contained in the file

`http://math.ucr.edu/∼res/progeom/quadrics0.pdf`

and in fact both of these documents state and prove more general results.

4.** In $D^2 = \{x \in \mathbb{R}^2 |\ |x| \leq 1\}$, consider the equivalence relation generated by the condition $x\mathcal{R}'y$ if $|x| = |y| = 1$ and $y = -x$. Show that this quotient space is homeomorphic to $\mathbb{R}\mathbf{P}^2$.

[*Hints:* Use the description of $\mathbb{RP}^2$ as a quotient space of S^2 from the previous exercise, and let $h : D^2 \to S^2$ be defined by

$$h(x,y) = (x, y, \sqrt{1 - x^2 - y^2}\,).$$

Verify that h preserves equivalence classes and therefore induces a continuous map $\overline{h}$ on quotient spaces. Why is $\overline{h}$ a $1-1$ and onto mapping? Finally, prove that $\mathbb{RP}^2$ is Hausdorff and $\overline{h}$ is a closed mapping.]

5. Suppose that X is a topological space with topology $\mathbf{T}$, and suppose also that Y and Z are sets with set-theoretic maps $f : X \to Y$ and $g : Y \to Z$. Prove that the quotient topologies satisfy the condition

$$(g^\circ f)_*\mathbf{T} = g_*\,(f_*\mathbf{T})\ \ .$$

(Informally, a quotient of a quotient is a quotient.)

6.* If Y is a topological space with a topology $\mathbf{T}$ and $f; X \to Y$ is a set-theoretic map, then the *induced topology* $f^*\mathbf{T}$ on X is defined to be the set of all subsets $W \subset X$ having the form $f^{-1}(U)$ for some open set $U \in \mathbf{T}$. Prove that $f^*\mathbf{T}$ defines a topology on X, that it is the unique smallest topology on X for which f is continuous, and that if $h : Z \to X$ is another set-theoretic map then

$$(f \circ h)^*\mathbf{T} = h^* \left(f^*\mathbf{T} \right) \ .$$

7. Let X and Y be topological spaces, and define an equivalence relation $\mathcal{R}$ on $X \times Y$ by $(x, y) \sim (x', y')$ if and only if $x = x'$. Show that $X \times Y/\mathcal{R}$ is homeomorphic to X.

8. Let $\mathcal{R}$ be an equivalence relation on a topological space X, let $\Gamma_{\mathcal{R}}$ be the graph of $\mathcal{R}$, and let $\pi : X \to X/\mathcal{R}$ be the quotient projection. Prove the following statements:

(a) If $X/\mathcal{R}$ satisfies the Hausdorff Separation Property then $\Gamma_{\mathcal{R}}$ is closed in $X \times X$.
(b) If $\Gamma_{\mathcal{R}}$ is closed and π is open, then $X/\mathcal{R}$ is Hausdorff.
(c) If $\Gamma_{\mathcal{R}}$ is open then $X/\mathcal{R}$ is discrete.

9. Let $D^2 \subset \mathbb{C} \cong \mathbb{R}^2$ be the set of all points z such that $|z| \leq 1$, let $d \geq 2$ be an integer, and let $\mathcal{E}$ be the equivalence relation on D^2 generated by the binary relation $z \ \mathcal{R} \ \alpha \, z$ where $\alpha^d = 1$ and $|z| = 1$ (it follows that $|\alpha| = 1$ and there are d possible choices for α).

(i) Explain why the equivalence classes of $\mathcal{E}$ are one point subsets of the form $\{w\}$ where $|w| < 1$ and all d point sets $\{\alpha^k \, w\}$ where $|w| = 1$ and $\alpha = \exp(2\pi \, i/d)$ is a primitive d^{th} root of unity.

(ii) Let $X = D^2/\mathcal{E}$ be the quotient space where $\mathcal{E}$ is given as above, and let $p : D^2 \to X$ denote the quotient space projection. Prove that the map $h : D^2 \to \mathbb{C}^2$ defined by $h(w) = (\,(1 - |w|)z, z^d)$ passes to a map $h^* : X \to \mathbb{C}^2$ such that $h = h^* \circ p$ and h^* maps X homeomorphically onto its image. [*Hint:* Why does it suffice to show that h^* is 1–1?]

(iii) Suppose instead that we take the equivalence relation on D^2 defined by $u \ \mathcal{F} \ v$ if and only if $u^d = v^d$, where d is again some integer ≥ 2. Prove that $D^2/\mathcal{F}$ is homeomorphic to D^2. [*Hint:* Show that the complex d^{th} power map $z \to z^d$ maps D^2 to itself, and it passes to a map from $D^2/\mathcal{F}$ to D^2 which is continuous, 1–1 and onto.

V.2 : Sums and cutting and pasting

(not covered in the texts)

Additional exercises

1. Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of topological spaces, and let $X = \coprod_\alpha A_\alpha$. Prove that X is locally connected if and only if each A_α is locally connected.

2. In the preceding exercise, formulate and prove necessary and sufficient conditions on $\mathcal{A}$ and the sets A_α for the space X to be compact.

3.** Prove that $\mathbb{RP}^2$ can be constructed by identifying the edge of a Möbius strip with the edge circle on a closed 2-dimensional disk by filling in the details of the following argument: Let $A \subset S^2$ be the set of all points $(x, y, z) \in S^2$ such that $|z| \leq \frac{1}{2}$, and let B be the set of all points where $|z| \geq \frac{1}{2}$. If $T(x) = -x$, then $T(A) = A$ and $T(B) = B$ so that each of A and B (as well as their intersection) can be viewed as a union of equivalence classes for the equivalence relation that produces $\mathbb{RP}^2$. By construction B is a disjoint union of two pieces $B_\pm$ consisting of all points where $\mathrm{sign}(z) = \pm 1$, and thus it follows that the image of B in the quotient space is homeomorphic to $B_+ \cong D^2$. Now consider A. There is a homeomorphism h from $S^1 \times [-1,1]$ to A sending (x,y,t) to $\left(\alpha(t)x, \alpha(t)y, \frac{1}{2}t\right)$ where

$$\alpha(t) = \sqrt{1 - \frac{t^2}{4}}$$

and by construction $h(-v) = -h(v)$. The image of A in the quotient space is thus the quotient of $S^1 \times [-1,1]$ modulo the equivalence relation $u \sim v \iff u = \pm v$. This quotient space is in turn homeomorphic to the quotient space of the upper semicircular arc S^1_+ (all points with nonnegative y-coordinate) modulo the equivalence relation generated by setting $(-1,0,t)$ equivalent to $(1,0,-t)$, which yields the Möbius strip. The intersection of this subset in the quotient with the image of B is just the image of the closed curve on the edge of B_+, which also represents the edge curve on the Möbius strip.

4. Suppose that the topological space X is a union of two closed subspaces A and B, let $C = A \cap B$, let $h : C \to C$ be a homeomorphism, and let $A \cup_h B$ be the space formed from $A \sqcup B$ by identifying $x \in C \subset A$ with $h(x) \in C \subset B$. Prove that $A \cup_h B$ is homeomorphic to X if h extends to a homeomorphism $H : A \to A$, and give an example for which X is not homeomorphic to $A \cup_h B$. [*Hint:* Construct the homeomorphism using H in the first case, and consider also the case where $X = S^1 \sqcup S^1$, with $A_\pm == S^1_\pm \sqcup S^1_\pm$; then $C = \{\pm 1\} \times \{1,2\}$, and there is a homeomorphism from h to itself such that $A_+ \cup_h A_-$ is connected.]

5.** *One-point unions.* One conceptual problem with the disjoint union of topological spaces is that it is never connected except for the trivial case of one summand. In many geometrical and topological contexts it is extremely useful to construct a modified version of disjoint unions that is connected if all the pieces are. Usually some additional structure is needed in order to make such constructions.

In this exercise we shall describe such a construction for objects known as *pointed spaces* that are indispensable for many purposes (*e.g.*, the definition of fundamental groups as in Munkres). A pointed space is a pair (X,x) consisting of a topological space X and a point $x \in X$; we often call x the *base point*, and unless stated otherwise *the one point set consisting of the base point is assumed to be closed.* If (Y,y) is another pointed space and $f : X \to Y$ is continuous, we shall say that *f is a base point preserving continuous map from (X,x) to (Y,y)* if $f(x) = y$, In this case we shall often write $f : (X,x) \to (Y,y)$. Identity maps are base point preserving, and composites of base point preserving maps are also base point preserving.

(a) Given a finite collection of pointed spaces (X_i, x_i), define an equivalence relation on $\coprod_i X_i$ whose equivalence classes consist of $\coprod_j \{x_j\}$ and all one point sets y such that $y \notin \coprod_j \{x_j\}$. Define the *one point union* or *wedge*

$$\bigvee_{i=1}^{n} (X_j, x_j) = (X_1, x_1) \vee \ \cdots \ \vee (X_n, x_n)$$

to be the quotient space of this equivalence relation with the quotient topology. The base point of this space is taken to be the class of $\coprod_j \{x_j\}$.

(a) Prove that the wedge is a union of closed subspaces Y_j such that each Y_j is homeomorphic to X_j and if $j \neq k$ then $Y_j \cap Y_k$ is the base point. Explain why $\vee_k\ (X_k, x_k)$ is Hausdorff if and only if each X_j is Hausdorff, why $\vee_k\ (X_k, x_k)$ is compact if and only if each X_j is compact, and why $\vee_k\ (X_k, x_k)$ is connected if and only if each X_j is connected (and the same holds for arcwise connectedness).

(b) Let $\varphi_j : (X_j, x_j) \to \vee_k\ (X_k, x_k)$ be the composite of the injection $X_j \to \coprod_k\ X_k$ with the quotient projection; by construction φ_j is base point preserving. Suppose that (Y, y) is some arbitrary pointed space and we are given a sequence of base point preserving continuous maps $F_j : (X_j, x_j) \to (Y, y)$. Prove that there is a unique base point preserving continuous mapping

$$F : \vee_k\ (X_k, x_k) \to (Y, y)$$

such that $F \circ \varphi_j = F_j$ for all j.

(c) In the infinite case one can carry out the set-theoretic construction as above but some care is needed in defining the topology. Show that if each X_j is Hausdorff and one takes the so-called *weak topology* whose closed subsets are generated by the family of subsets $\varphi_j(F)$ where F is closed in X_j for some j, then [1] a function h from the wedge into some other space Y is continuous if and only if each composite $h \circ \varphi_j$ is continuous, [2] the existence and uniqueness theorem for mappings from the wedge (in the previous portion of the exercise) generalizes to infinite wedges with the so-called weak topologies.

(d) Suppose that we are given an infinite wedge such that each summand is Hausdorff and contains at least two points. Prove that the wedge with the so-called weak topology is not compact.

Remark. If each of the summands in (d) is compact Hausdorff, then there is a natural candidate for a *strong topology* on a countably infinite wedge which makes the latter into a compact Hausdorff space. Specifically, if the wedge summands are (X_j, x_j), then we can view $\vee_k\ (X_k, x_k)$ as the subgroup of $\prod_k\ X_k$ consisting of all sequences $\{y_k \mid k \in K\}$ such that $y_k = x_k$ for all but at most one value of k, and the strong topology is given by taking the subspace topology associated to the inclusion $\vee_k\ (X_k, x_k) \subset \prod_k\ X_k$, where the latter is equipped with the subspace topology. — By construction, every open neighborhood of the basepoint, whose coordinates are given by x_k, contains all but finitely many of the subspaces X_j. This is not true for open neighborhoods in the weak topology.

In some cases the strong topology can be viewed more geometrically; for example, if each (X_j, x_j) is equal to $(S^1, 1)$ and there are countably infinitely many of them, then the space one obtains is the *Hawaiian earring* (or necklace) in $\mathbb{R}^2$ given by the union of the circles defined by the equations

$$\left(x - \frac{1}{2^k}\right)^2 + y^2 = \frac{1}{2^{2k}}\ .$$

As usual, drawing a picture may be helpful. The k^{th} circle has center $(1/2^k,\ 0)$ and passes through the origin; the y-axis is the tangent line to each circle at the origin.

VI. Spaces with additional properties

VI.1 : Second countable spaces

(Munkres, § 30)

Munkres, § 30, pp. 194–195: 9 (first part only), 10, 13*, 14*

Additional exercises

1. If $(X, \mathbf{T})$ is a second countable Hausdorff space, prove that the cardinalities of both X and $\mathbf{T}$ are less than or equal to $2^{\aleph_0}$. (Using the formulas for cardinal numbers in Section I.3 of the course notes and the separability of X one can prove a similar inequality for $\mathbf{BC}(X)$.)

2.** *Separability and subspaces.* The following example shows that a closed subspace of a separable Hausdorff space is not necessarily separable.

(a) Let X be the upper half plane $\mathbb{R} \times [0, \infty)$ and take the topology generated by the usual metric topology plus the following sets:

$$T_\varepsilon(x) = \{\,(x, 0)\,\} \quad \cup \quad N_\varepsilon\,(\,(x, \varepsilon)\,)\,, \quad \text{where} \;\; x \in \mathbb{R} \;\; \text{and} \;\; \varepsilon \; > \; 0$$

Geometrically, one takes the interior region of the circle in the upper half plane that is tangent to the x-axis at $(x, 0)$ and adds the point of tangency. — Show that the x-axis is a closed subset and has the discrete topology.

(b) Explain why the space in question is Hausdorff. [*Hint:* The topology contains the metric topology. If a topological space is Hausdorff and we take a larger topology, why is the new topology Hausdorff?]

(c) Show that the set of points (u, v) in X with $v > 0$ and $u, v \in \mathbb{Q}$ is dense. [*Hint:* Reduce this to showing that one can find such a point in every set of the form $T_\varepsilon(x)$.]

3. Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of topological spaces, and let $X = \coprod_\alpha A_\alpha$. Formulate and prove necessary and sufficient conditions on $\mathcal{A}$ and the sets A_α for the space X to be second countable, separable or Lindelöf.

4. Let X and Y be second countable topological spaces, and let $W \subset X \times Y$ be open in the product topology. Prove that W is a union of countably many rectangular open subsets of the form $U_k \times V_k$, where U_k is open in X and V_k is open in Y.

> *Note.* In `nonrectangular.pdf` there is a proof that the open unit disk W in $\mathbb{R}^2$ defined by $x^2 + y^2 < 1$ cannot be expressed as a *finite* union of rectangular open sets in $\mathbb{R}^2$. This probably seems clear intuitively, but the preceding exercise implies that W can be written as a countably infinite union of rectangular open subsets, and this conclusion seems somewhat less obvious.

VI.2 : Compact spaces – II

(Munkres, §§ 26, 27,28)

Munkres, § 28, pp. 181–182: 6

Additional exercises

1.* Let X be a compact Hausdorff space, let Y be a Hausdorff space, and let $f : X \to Y$ be a continuous map such that f is locally 1–1 (each point x has a neighborhood U_x such that $f|U_x$ is 1–1) and there is a closed subset $A \subset X$ such that $f|A$ is 1–1. Prove that there is an open neighborhood V of A such that $f|V$ is 1–1. [*Hint:* A map g is 1–1 on a subset B is and only if

$$B \times B \cap (g \times g)^{-1} [\, \Delta_Y \,] \quad = \quad \Delta_B ,$$

where Δ_S denotes the diagonal in $S \times S$. In the setting of the exercise show that

$$(f \times f)^{-1} [\, \Delta_Y \,] \quad = \quad \Delta_X \cup D' ,$$

where D' is closed and disjoint from the diagonal. Also show that the subsets D' and $A \times A$ are disjoint, and find a square neighborhood of $A \times A$ disjoint from D'.]

2.** Let U be open in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^n$ be a $\mathbf{C}^1$ map such that $Df(x)$ is invertible for all $x \in U$ and there is a compact subset $A \subset U$ such that $f|A$ is 1–1. Prove that there is an open neighborhood V of A such that $f|V$ is a homeomorphism onto its image.

3.** Let $\mathbf{d}_p$ be the metric on the integers constructed in Exercise I.1.1, and let $\widehat{\mathbb{Z}_p}$ be the completion of this metric space. Prove that $\widehat{\mathbb{Z}_p}$ is (sequentially) compact. [*Hint:* For each integer $r > 0$ show that every integer is within p^{-r} of one of the first p^{r+1} nonnegative integers. Furthermore, each open neighborhood of radius p^{-r} centered at one of these integers a is a union of p neighborhoods of radius $p^{-(r+1)}$ over all of the first p^{r+2} integers b such that $b \equiv a$ mod p^{r+1}. Now let $\{\, a_n \,\}$ be an infinite sequence of integers, and assume the sequence takes infinitely many distinct values (otherwise the sequence obviously has a convergent subsequence). Find a sequence of positive integers $\{\, b_r \,\}$ such that the open neighborhood of radius p^{-r} centered at b_r contains infinitely many points in the sequence and $b_{r+1} \equiv b_r$ mod p^{r+1}. Prove that this yields a Cauchy sequence in $\mathbb{Z}$ (with respect to the $\mathbf{d}_p$ metric). Form a subsequence of $\{\, a_n \,\}$ by choosing distinct points $a_{n(k)}$ recursively such that $n(k) > n(k-1)$ and $a_{n(k)} \in N_{p^{-k}}(b_k)$. Prove that this subsequence is a Cauchy sequence and hence converges.]

VI.3 : Separation axioms

(Munkres, §§ 31, 32, 33, 35)

Munkres, § 26, pp. 170–172: 11*
Munkres, § 33, pp. 212–214: 2a [for metric spaces], 6a, 8

Additional exercises

1. If $(X, \mathbf{T})$ is compact Hausdorff and $\mathbf{T}^*$ is strictly contained in $\mathbf{T}$, prove that $(X, \mathbf{T}^*)$ is compact but not Hausdorff.

2. (a) Prove that a topological space is $\mathbf{T_3}$ if and only if it is $\mathbf{T_1}$ and there is a basis $\mathcal{B}$ such that for every $x \in X$ and every open set $V \in \mathcal{B}$ containing x, there is an open subset $W \in \mathcal{B}$ such that $x \in W \subset \overline{W} \subset V$.

(b)** Prove that the space constructed in Exercise VI.1.2 is $\mathbf{T_3}$. [*Hint:* Remember that the "new" topology contains the usual metric topology.]

3. If X is a topological space and $A \subset X$ is nonempty then X/A (in words, "X mod A" or "X modulo A collapsed to a point") is the quotient space whose equivalence classes are A and all one point subsets $\{x\}$ such that $x \notin A$. Geometrically, one is collapsing A to a single point.

(a) Suppose that A is closed in X. Prove that X/A is Hausdorff if either X is compact Hausdorff or X is metric (in fact, if X is $\mathbf{T_4}$).

(b) Still assuming A is closed but not making any assumptions on X (except that it be nonempty), show that the quotient map $X \to X/A$ is always closed but not necessarily open. [*Note:* For reasons that we shall not discuss, it is appropriate to define $X/\emptyset$ to be the disjoint union $X \sqcup \{\emptyset\}$.]

(c) Suppose that we are given a continuous map of topological spaces $f : X \to Y$, and that $A \subset X$ and $B \subset Y$ are nonempty closed subsets satisfying $f[A] \subset B$. Prove that there is a unique continuous map $F : X/A \to Y/B$ such that for all $\mathbf{c} \in X/A$, if $\mathbf{c}$ is the equivalence class of $x \in X$, then $F(\mathbf{c})$ is the equivalence class of $f(x)$.

4. Prove that the topologies of upper and lower semicontinuity (see Additional Exercise II.1.5) are $\mathbf{T}_0$ but not $\mathbf{T}_1$.

VI.4 : Local compactness and compactifications

(Munkres, §§ 29, 37, 38)

Munkres, § 38, pp. 241–242: 2*, 3 (just give a necessary condition on the topology of the space)

Additional exercises

Definition. If $f : X \to Y$ is continuous, then f is *proper* (or *perfect*) if for each compact subset $K \subset Y$ the inverse image $f^{-1}(K)$ is a compact subset of X.

1. Suppose that $f : X \to Y$ is a continuous map of noncompact locally compact T_2 spaces. Let $f^\bullet : X^\bullet \to Y^\bullet$ be the map of one point comapactifications defined by $f^\bullet|X = f$ and $f^\bullet(\infty_X) = (\infty_Y)$. Prove that f is proper if and only if $f^\bullet$ is continuous.

2. Prove that a proper map of noncompact locally compact Hausdorff spaces is closed.

3. If $\mathbb{F}$ is the reals or complex numbers, prove that every polynomial map $p : \mathbb{F} \to \mathbb{F}$ is proper. [*Hint:* Show that

$$\lim_{|x|\to\infty} \left|p(z)\right| \quad = \quad \infty$$

and use the characterization of compact subsets as closed and bounded subsets of $\mathbb{F}$.]

4.* Let ℓ^2 be the complete metric space described above, and view $\mathbb{R}^n$ as the subspace of all sequences with $x_k = 0$ for $k > n$. Let $A_n \subset \ell^2 \times \mathbb{R}$ be the set of all ordered pairs (x, t) with $x \in \mathbb{R}^n$ and $0 < t \leq 2^{-n}$. Show that $A = \bigcup_n A_n$ is locally compact but its closure is not. Explain why this shows that the completion of a locally compact metric space is not necessarily locally compact. [*Hint:* The family $\{\, A_n \,\}$ is a locally finite family of closed locally compact subspaces in A. Use this to show that the union is locally compact, and show that the closure of A contains all of $\ell^2 \times \{0\}$. Explain why ℓ^2 is not locally compact.]

5. Let X be a compact Hausdorff space, and let $U \subset X$ be open and noncompact. Prove that the collapsing map $c : X \to U^\bullet$ such that $c|U = \mathrm{id}_U$ and $c = \infty_U$ on $X - U$ is continuous. Show also that c is not necessarily open.

6.* (a) Explain why a compact Hausdorff space has no nontrivial Hausdorff abstract closures.

(b) Prove that a Hausdorff space X has a maximal abstract Hausdorff closure that is unique up to equivalence. [*Hint:* Consider the identity map.]

7. Suppose that X is compact Hausdorff and A is a closed subset of X. Prove that X/A is homeomorphic to the one point compactification of $X - A$.

8.* Suppose that X is a metric space that is uniformly locally compact in the sense that there is some $\delta > 0$ such that for each $x \in X$ the neighborhood $N_\delta(x)$ has compact closure. Prove that X is complete. Explain why the conclusion fails if one only assumes that for each x there is some $\delta(x) > 0$ with the given property (give an example).

9. Suppose that X is a noncompact locally compact $\mathbf{T}_2$ space and A is a noncompact closed subset of X. Prove that the one point compactification $A^\bullet$ is homeomorphic to a subset of the one point compactification $X^\bullet$.

10. Let U be an open subset of $\mathbb{C}$, let $a \in U$, and let $f : U - \{a\} \to \mathbb{C}$ be a continuous function with the following properties: (i) There is an open neighborhood $V \subset U$ of a such that f is nonzero on $V - \{a\}$. (ii) We have

$$\lim_{z \to a} \frac{1}{f(z)} \quad = \quad 0 \; .$$

If $F : U \to \mathbb{C}^\bullet$ is defined by $F|U - \{a\} = f$ and $F(a) = \infty$, prove that F is continuous. [*Hint:* It is only necessary to show that F is continuous at a, which means that for each compact subset $K \subset \mathbb{C}$ there is some $\delta > 0$ such that $0 < |z - a| < \delta$ implies that $f(z) \notin K$.]

VI.5 : Metrization theorems

(Munkres, §§ 39, 40, 41, 42)

Munkres, § 40, p. 252: 2, 3

Additional exercises

1. A *pseudometric space* is a pair $(X, \mathbf{d})$ consisting of a nonempty set X and a function $\mathbf{d} : X \times X \to \mathbb{R}$ that has all the properties of a metric except possibly the property that $\mathbf{d}(u, v) = 0$ implies $u = v$.

(a) If ε-neighborhoods and open sets are defined as for metric spaces, explain why one still obtains a topology for pseudometric spaces.

(b) Given a pseudometric space, define a binary relation $x \sim y$ if and only if $\mathbf{d}(x, y) = 0$. Show that this defines an equivalence relation and that $\mathbf{d}(x, y)$ only depends upon the equivalence classes of x and y.

(c) Given a sequence of pseudometrics $\mathbf{d}_n$ on a set X, let $\mathbf{T}_\infty$ be the topology generated by the union of the sequence of topologies associated to these pseudometrics, and suppose that for each pair of distinct points $u\, v \in X$ there is some n such that $\mathbf{d}_n(u, v) > 0$. Prove that $(X, \mathbf{T}_\infty)$ is metrizable and that

$$\mathbf{d}_\infty \quad = \quad \sum_{n=1}^{\infty} \frac{\mathbf{d}_n}{2^n(1 + \mathbf{d}_n)}$$

defines a metric whose underlying topology is $\mathbf{T}_\infty$.

(d) Let X be the set of all continuous real valued functions on the real line $\mathbb{R}$. Prove that X is metrizable such that the restriction maps from X to $\mathbf{BC}(\,[-n,n]\,)$ are uniformly continuous for all n. [*Hint:* Let $\mathbf{d}_n(f,g)$ be the maximum value of $|f(x)-g(x)|$ for $|x| \leq n$.]

(e) Given X and the metric constructed in the previous part of the problem, prove that a sequence of functions $\{\,f_n\,\}$ converges to f if and only if for each compact subset $K \subset \mathbb{R}$ the sequence of restricted functions $\{\,f_n|K\,\}$ converges to $f|K$.

(f) Is X complete with respect to the metric described above? Prove this or give a counterexample.

(g) Explain how the preceding can be generalized from continuous functions on $\mathbb{R}$ to continuous functions on an arbitrary open subset $U \subset \mathbb{R}^n$.

2. (a) Let U be an open subset of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^n$ be a continuous function such that $f^{-1}(\{0\})$ is contained in an open subset V such that $V \subset \overline{V} \subset U$. Prove that there is a continuous function g from $S^n \cong (\mathbb{R}^n)^\bullet$ to itself such that $g|V = f|V$ and $f^{-1}(\{0\}) = g^{-1}(\{0\})$. [*Hint:* Note that

$$(\mathbb{R}^n)^\bullet \;-\; \{0\} \quad\cong\quad \mathbb{R}^n$$

and consider the continuous function on

$$\left(\overline{V} - V\right) \,\sqcup \{\infty\} \quad\subset\quad (\mathbb{R}^n)^\bullet \;-\; \{0\}$$

defined on the respective pieces by the restriction of f and ∞. Why can this be extended to a continuous function on $(\mathbb{R}^n)^\bullet \;-\; V$ with the same codomain? What happens if we try to piece this together with the original function f defined on U?]

(b) Suppose that we are given two continuous functions g and g' satisfying the conditions of the first part of this exercise. Prove that there is a continuous function

$$G : S^n \times [0,1] \longrightarrow S^n$$

such that $G(x,0) = g(x)$ for all $x \in S^n$ and $G(x,1) = g'(x)$ for all $x \in S^n$ (*i.e.*, the mappings g and g' are **homotopic**).

3. Suppose that X is a Hausdorff **UL** space (for example, a metrizable space). Define G_δ and F_σ sets as in Section 40 of Munkres and its exercises.

(a) Show that every open subset of X is an F_σ set. [*Hint:* Look at Example 2 on page 249.]

(b) Show that a subset A of X is a closed G_δ set if and only if there is some continuous real valued function $f : X \to [0,1]$ such that $A = f^{-1}[\,\{0\}\,]$.

(c) Show that a subset A of X is an open F_σ set if and only if there is some continuous real valued function $f : X \to [0,1]$ such that A is the set of all points x such that $f(x)$ is positive.

4. Let X be compact, and let $\mathcal{F}$ be a family of continuous real valued functions on X that is closed under multiplication and such that for each $x \in X$ there is a neighborhood U of X and a function $f \in \mathcal{F}$ that vanishes identically on U. Prove that $\mathcal{F}$ contains the zero function.

5.** Let X be a compact metric space, and let J be a nonempty subset of the ring $\mathbf{BC}(X)$ of (bounded) continuous functions on X such that J is closed under addition and subtraction, it is an *ideal* in the sense that $f \in J$ and $g \in \mathbf{BC}(X) \Longrightarrow f \cdot g \in J$, and for each $x \in X$ there is a function $f \in J$ such that $f(x) \neq 0$. Prove that $J = \mathbf{BC}(X)$. [*Hints:* This requires the existence of partitions of unity as established in Theorem 36.1 on pages 225–226 of Munkres; as noted there, the result works for arbitrary compact Hausdorff spaces, but we restrict to metric spaces because the course does not cover Urysohn's Lemma in that generality. Construct a finite open covering of X, say $\mathcal{U}$, such that for each $U_i \in \mathcal{U}$ there is a function $f_i \in J$ such that $f_i > 0$ on U_i. Let $\{\, \varphi_i \,\}$ be a partition of unity dominated by $\mathcal{U}$, and form $h = \sum_i \varphi_i \cdot f_i$. Note that $h \in J$ and $h > 0$ everywhere so that h has a reciprocal $k = 1/h$ in $\mathbf{BC}(X)$. Why does this imply that the constant function 1 lies in J, and why does the latter imply that everything lies in J?]

6.** In the notation of the preceding exercise, an ideal $\mathbf{M}$ in $\mathbf{BC}(X)$ is said to be a *maximal ideal* if it is a proper ideal and there are no ideals $\mathbf{A}$ such that $\mathbf{M}$ is properly contained in $\mathbf{A}$ and $\mathbf{A}$ is properly contained in in $\mathbf{BC}(X)$. Prove that there is a 1–1 correspondence between the maximal ideals of $\mathbf{BC}(X)$ and the points of X such that the ideal $\mathbf{M}_x$ corresponding to X is the set of all continuous functions $g : X \to \mathbb{R}$ such that $g(x) = 0$. [*Hint:* Use the preceding exercise.]

APPENDICES

Appendix A : Topological groups

(Munkres, Supplementary exercises following \$ 22; *see also* course notes, Appendix D)

Munkres, Supplementary Exercises, pp. 144–145: 5–7

Munkres, § 26, pp. 170–172: 12, 13

Munkres, § 30, pp. 194–195: 18

Munkres, § 31, pp. 199–200: 8

Munkres, § 33, pp. 212–214: 10

Additional exercises

Let $\mathbb{F}$ be the real or complex numbers. Within the matrix group $\mathbf{GL}(n, \mathbb{F})$ there are certain subgroups of particular importance. One such subgroup is the *special linear group* $\mathbf{SL}(n, \mathbb{F})$ of all matrices of determinant 1.

0. Prove that the group $\mathbf{SL}(2, \mathbb{C})$ has no nontrivial proper normal subgroups except for the subgroup $\{\pm I\}$. [*Hint:* If N is a normal subgroup, show first that if $A \in N$ then N contains all matrices that are similar to A. Therefore the proof reduces to considering normal subgroups containing a Jordan form matrix of one of the following two types:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad , \qquad \begin{pmatrix} \varepsilon & 1 \\ 0 & \varepsilon \end{pmatrix}$$

Here α is a complex number not equal to 0 or ± 1 and $\varepsilon = \pm 1$. The idea is to show that if N contains one of these Jordan forms then it contains all such forms, and this is done by computing sufficiently many matrix products. Trial and error is a good way to approach this aspect of the problem.]

Definition. The *orthogonal group* $\mathbf{O}(n)$ consists of all transformations in $\mathbf{GL}(n, \mathbb{R})$ that take each orthonormal basis for $\mathbb{R}^n$ to another orthonormal basis, or equivalently, the subgroup of all matrices whose columns form an orthonormal basis. It is an easy exercise in linear algebra to show that the determinant of all matrices in $\mathbf{O}(n)$ is ± 1. The *special orthogonal group* $\mathbf{SO}(n)$ is the subgroup of $\mathbf{O}(n)$ consisting of all matrices whose determinants are equal to $+1$. Replacing If we replace the real numbers $\mathbb{R}$ by the complex numbers $\mathbb{C}$ we get the unitary groups $\mathbf{U}(n)$ and the *special unitary groups* $\mathbf{SU}(n)$, which are the subgroups of $\mathbf{U}(n)$ given by matrices with determinant 1. The determinant of every matrix in $\mathbf{U}(n)$ is of absolute value 1 just as before, but in the complex case this means that the determinant is a complex number on the unit circle. In Appendix A the orthogonal and unitary groups were shown to be compact.

1. Show that $\mathbf{O}(1)$ is isomorphic to the cyclic group of order 2 and that $\mathbf{SO})(2)$ is isomorphic as a topological group to the circle group S^1. Conclude that $\mathbf{O}(2)$ is homeomorphic as a space to $\mathbf{SO}(2) \times \mathbf{O}(1)$, but that as a group these objects are not isomorphic to each other. [*Hint:* In Appendix D it is noted that every element of $\mathbf{O}(2)$ that is not in $\mathbf{SO}(2)$ has an orthonormal basis of eigenvectors corresponding to the eigenvalues ± 1. What does this say about the orders of such group elements?]

2. Show that $\mathbf{U}(1)$ is isomorphic to the circle group S^1 as a topological group.

3. For each positive integer n, show that $\mathbf{SO}(n)$, $\mathbf{U}(n)$ and $\mathbf{SU}(n)$ are connected spaces and that $\mathbf{U}(n)$ is homeomorphic as a space (but not necessarily isomorphic as a topological group) to $\mathbf{SU}(n) \times S^1$. [*Hints:* In the complex case use the Spectral Theorem for normal matrices to show that every unitary matrix lies in the path component of the identity, which is a normal subgroup. In the real case use the results on normal form in Appendix D.]

4. Show that $\mathbf{O}(n)$ has two connected components both homeomorphic to $\mathbf{SO}(n)$ for every n.

5. Show that the inclusions of $\mathbf{O}(n)$ in $\mathbf{GL}(n, \mathbf{R})$ and $\mathbf{U}(n)$ in $\mathbf{GL}(n, \mathbb{C})$ determine 1–1 correspondences of path components. [*Hint:* Show that the Gram-Schmidt orthonormalization process expresses an arbitrary invertible matrix over the real numbers as a product PQ where P is upper triangular and Q is orthogonal (real case) or unitary (complex case), and use this to show that every invertible matrix can be connected to an orthogonal or unitary matrix by a continuous curve.]

*Local group actions on spaces.*** In the notes we defined the notion of a *local topological group* to be a structure which behaves like a neighborhood of the identity in a topological group. There is an accompanying concept of a *local (continuous) action* on a local topological group. Specfically, a local (continuous) action of a local topological group is a pair consisting of a local group U, where U is closed under taking inverses, and a mapping Φ defined on an open neighborhood $\mathcal{N}(\Phi)$ of $\{1\} \times X$ in $U \times X$ — whose value at (u, x) is often abbreviated to $u \cdot x$ — such that the following statements are valid:

(1) For all $x \in X$ we have $1 \cdot x = x$.

(2) If $(u, x) \in \mathcal{N}(\Phi)$ and $(v, u \cdot x) \in \mathcal{N}(\Phi)$, then $(v \cdot u, x) \in$ and $(v \cdot u) \cdot x = v \cdot (u \cdot x)$.

Every topological group action in the sense of Munkres, page 199. In 205C and continuations of that course, one naturally encounters local actions of $\mathbb{R}$ on open sets in $\mathbb{R}^n$ by considering the solutions to differential equations of the form

$$\frac{d\mathbf{y}}{dt} \quad = \quad \mathbf{F}(\mathbf{y})$$

where $\mathbf{F}$ is a function on U whose coordinate functions have continuous partial derivatives. The local group action is then given by a map $\Phi : \mathcal{N} \to U$ such that, for all $\mathbf{x}$, $\gamma(t) = (t, \mathbf{x})$ is the unique solution to the differential equation with $\gamma'(t) = \mathbf{F}(\gamma(t))$ and $\gamma(0) = \mathbf{x}$. Basic results on solutions to differential equations imply that one obtains a local group action if $\mathbf{F}$ satisfies the hypotheses (see the file `math205Asolutions06a.pdf` for more details), but in many cases this local group action does not come from a global group action; what can happen is that one might only be able to define the integral curve on an open neighborhood of $0 \in \mathbb{R}$. One simple example, which only involves lower level undergraduate mathematics, is $\mathbf{F}(x, y) = (y^2, x^2)$; the resulting differential equation can be rewritten in the form $x^2\,dx - y^2\,dy \;=\; 0$, and this is an exact first order differential equation.

> We claim that some solution curves for this differential equation can only be defined on bounded intervals. The following is adapted from a posting in `math.stackexchange.com` by T. Shifrin (there is a more complete citation in `math205Asolutions06a.pdf` along with sketches of the integral curves): One can check directly that the integral curves of

this vector field are given by $y^3 = x^3 + c$. In particular, if an integral curve has initial point $(1,1)$, then the curve has the form $(x(t), x(t))$ for a suitable function $x(t)$. The data in the vector field imply that $x(t) = 1/(1-t)$ is the solution curve such that $x(0) = 1$, and we can only define this solution curve on the interval $(-\infty, 1)$.

The proof that the integral curves of a differential equation define a local group action requires more input than the contents of a lower level undergraduate course on solving certain ordinary differential equations. One reference for the necessary background material is Chapter 9 of the following book (the 205C text):

J. N. Lee. Introduction to Smooth Manifolds (Second Edition). Graduate Texts in Mathematics, Vol. 218. *Springer-Verlag, New York etc.*, 2013. ISBN: 978-1-4419-9981-8.

6. Suppose that we are given a local group action of a connected Hausdorff topological group G on a compact Hausdorff space X. Prove that $\mathcal{N}(\Phi) = G \times X$; *i.e.*, the local action arises from a global action of G on X.

When $G = \mathbb{R}$, this result generalizes some fundamental facts about integral curves in Chapter 9 of Lee's book.