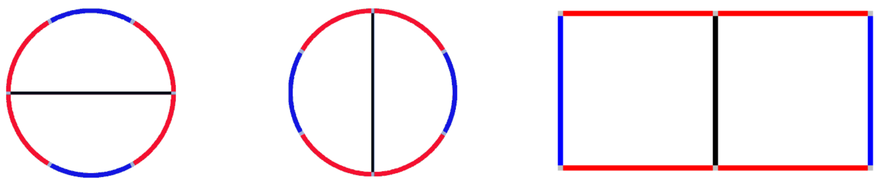
Details for Example 3 on page 362 of Munkres

In the cited example there is a statement that the Figure Theta space, which is defined by $S^1 \cup \{0\} \times [-1,1]$, is a strong deformation retract of the complement of two points in the plane, say with coordinates $(\pm\frac{1}{2}, 0)$. The goal here is to provide details for the proof for a statement of this type, the main difference being that the subspace in Munkres is replaced with

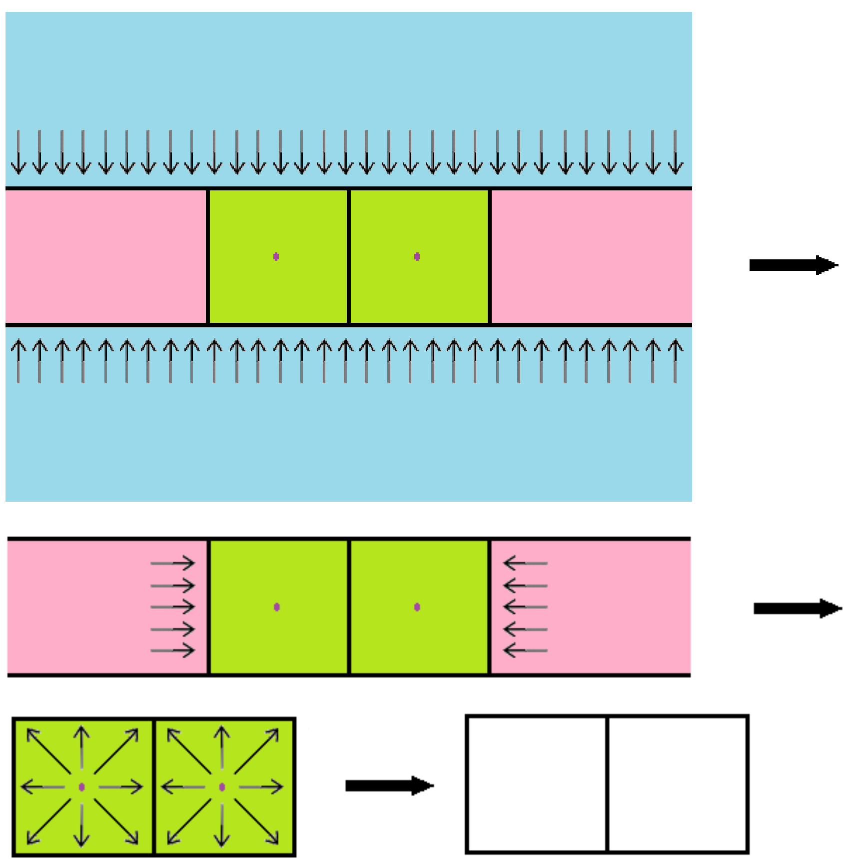
$$\{-1, 0, 1\} \times [-\tfrac{1}{2}, \tfrac{1}{2}] \;\cup\; [-1,1] \times \{-\tfrac{1}{2}, \tfrac{1}{2}\}.$$

The existence of such a homeomorphism is motivated by the sequence of drawings



and explicit formulas for such a mapping are given at the end of this document.

We begin with sketches of a deformation retract for the modified Example 3 which is similar to the construction illustrated in Figure 58.2 on page 362 of Munkres.



Details of the construction(s)

At the first step we shrink the closed region $|y| \geq \frac{1}{2}$ to its frontier $|y| = \frac{1}{2}$ by a vertical straight line homotopy. Formally, the retraction takes (x, t) to itself if $|y| \leq \frac{1}{2}$ and to $(x, \mathrm{sgn}(t) \cdot \frac{1}{2})$ if $|t| \geq \frac{1}{2}$, where $\mathrm{sgn}(t)$ is 1 if $t > 0$ and -1 if $t < 0$. These definitions agree on the overlapping set where $|t| = \frac{1}{2}$, and therefore one has a well-defined continuous retraction r_1 from $\mathbb{R}^2 - \{\mathbf{p}, \mathbf{q}\}$ to

$$E \quad = \quad \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \; - \; \{\mathbf{p}, \mathbf{q}\} \; .$$

If i_1 denotes the inclusion of A in $\mathbb{R}^2 - \{\mathbf{p}, \mathbf{q}\}$, then the vertical straight line segment joining (x, t) to $r_1(x, t)$ defines a relative homotopy from the identity on $\mathbb{R}^2 - \{\mathbf{p}, \mathbf{q}\}$ to $i_1 \circ r_1$ which is fixed on A.

We can now define the second retraction from A to

$$B \quad = \quad [-1, 1] \times \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \; - \; \{\mathbf{p}, \mathbf{q}\}$$

similarly: The map sends (x, t) to itself if $|x| \geq q$ and to $(x, 1)$ if $|x| \leq 1$. These definitions agree on the overlapping subset where $|x| = 1$, and therefore one has a well-defined continuous retraction r_2 from A to B. In analogy with the preceding discussion, if $i_2 : B \to A$ denotes the inclusion mapping, then the straight line homotopy from (x, t) to $r_2(x, t)$ defines a relative homotopy from the identity on B to $i_2 \circ r_2$ which is fixed on A.

The drawings for the next step indicate that we want to push out radially from the centers of the two squares and send everything into the latter in this way. This can be disposed of most easily by the following digression:

LEMMA. *Let $a < b$ be real numbers, and also let $C \in \mathbb{R}$. If $[a, b] \times [a + C, b + C]$ is a solid square in $\mathbb{R}^2$ and $\mathbf{z}$ is the center point whose coordinates are equal to $\frac{1}{2}(a + b)$ and $\frac{1}{2}(a + b) + C$ respectively, Then*

$$\text{Boundary } [a, b] \times [a + C, b + C] \quad = \quad [a, b] \times \{a + C, b + C\} \; \cup \; \{a, b\} \times [a + C, b + C]$$

is a strong deformation retract of $[a, b] \times [a + C, b + C] - \{\mathbf{z}\}$.

Proof of Lemma. If $[a, b] = [-1, 1]$ and $C = 0$ this can be shown simply and explicitly: The center of the square is then $\mathbf{0}$, and if $|\mathbf{v}|_0$ denotes the norm $\max\{|x|, |y|\}$ on $\mathbb{R}^2$, then the retraction sends $\mathbf{v} \in [-1, 1] \times [-1, 1] - \{\mathbf{z}\}$ to $(|\mathbf{v}|_0)^{-1} \cdot \mathbf{v}$, and the homotopy is once again a straight line homotopy from $\mathbf{v}$ to $(|\mathbf{v}|_0)^{-1} \cdot \mathbf{v}$ which lie inside $[-1, 1] \times [-1, 1] - \{\mathbf{z}\}$.

To prove the general case, let ρ and H be the deformation retraction data for the special case, let $\lambda : [a, b] \to [-1, 1]$ be a strictly increasing linear homeomorphism from the domain to the codomain, let $\lambda_C(t) = \lambda(t - C)$, define $\Lambda = \lambda_C$, and take the data given by the retraction $\rho'(u, v) = \Lambda^{-1}(\rho(\Lambda(u, v))$ and the homotopy H' given by $H'(u, v; t) = \Lambda^{-1}(H(\Lambda(u, v), t)$.■

We now return to the third step of our construction. Since $B \cup \{\mathbf{p}, \mathbf{q}\}$ is the union of two squares

$$B_- \quad = \quad [-1, 0] \times \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \; , \qquad B_+ \quad = \quad [0, 1] \times \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]$$

which only meet along a common edge, we can define strong deformation retract data for B separately on the pieces $B_- - \{\mathbf{q}\}$ and $B_+ - \{\mathbf{p}\}$, and the Lemma gives us the data needed to show that the boundaries of B_- and B_+ are strong deformation retracts of $B_- - \{\mathbf{q}\}$ and $B_+ - \{\mathbf{p}\}$ respectively. By the preceding sentence, we can piece these together to obtain deformation retract data for the inclusion of the Theta Space X in B.

To summarize, we have shown that the subspace X is a strond deformation retract of B, the subspace X is a strong deformation retract of A, and the subspace A is a strong deformation retract of $\mathbb{R}^2 - \{\mathbf{p}, \mathbf{q}\}$. These combine to show that the Theta Space X is a strong deformation retract of $\mathbb{R}^2 - \{\mathbf{p}, \mathbf{q}\}$.$\blacksquare$

A HOMEOMORPHISM OF THETA SPACES. For the sake of completeness, here is a proof that the model for a Figure Theta Space used in this document is homeomorphic to the model described in Munkres. The key features of a Figure Theta space are (1) it is a union of three closed subsets A, B, C, each of which is homeomorphic to $[-1, 1]$, (2) for each pair of these subspaces, the intersection corresponds to the two endpoints. Recall that we can characterize the endpoints p of a space J homeomorphic to $[-1, 1]$ in terms of the topology on J — they are precisely those points for which the complement $J - \{p\}$ is connected. It is a straightforward exercise to prove that if we have two spaces $X = A \cup B \cup C$ and $X' = A' \cup B' \cup C'$ which satisfy these properties, then X and X' are homeomorphic such that A corresponds to A', B corresponds to B' and C corresponds to C'. [PROOF: Let p and q be the common points in X, and let p' and q' be the common points in X'. Then there are homeomorphisms $A \to A'$, $B \to B'$ and $C \to C'$ which send p to p' and q to q'. For example, we are given homeomorphisms $A \cong [-1, 1] \cong A'$, and this map either sends p to p' and q to q' or vice versa; in the second case, if we replace our homeomorphism with $A \cong [0, 1] \to [0, 1] \cong A'$, where the map in the middle sends t to $-t$, the new map will send p to p' and q to q'. For these choices, we can assemble the homeomorphisms $A \to A'$, $B \to B'$ and $C \to C'$ into a homeomorphism from X to X'.]

For the model in Munkres, we can set

> A equal to the semicircular arc defined by $|x| \leq 1$ and $x = -\sqrt{1 - y^2}$, in which case the homeomorphism sends $y \in [-1, 1]$ to $(-\sqrt{1 - y^2}, y)$,
>
> B equal to the closed segment $\{0\} \times [-1, 1]$, in which case the homeomorphism is the obvious slice inclusion, and
>
> C equal to the semicircular arc defined by $|x| \leq 1$ and $x = +\sqrt{1 - y^2}$, in which case the homeomorphism sends $y \in [-1, 1]$ to $(+\sqrt{1 - y^2}, y)$.

For our model, we can set

> A equal to the broken line curve consisting of closed straight line segments, first horizontally from $\left(0, -\frac{1}{2}\right)$ to $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, then vertically from $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ to $\left(-\frac{1}{2}, +\frac{1}{2}\right)$, and finally horizontally from $\left(-\frac{1}{2}, +\frac{1}{2}\right)$ to $\left(0, +\frac{1}{2}\right)$, in which case the homeomorphism is the standard piecewise linear parametrization of the broken line which maps each third of $[-1, 1]$ to a closed segment in A,
>
> B equal to the closed segment $\{0\} \times \left[-\frac{1}{2}, +\frac{1}{2}\right]$, in which case the homeomorphism is $\frac{1}{2}$ times the obvious slice inclusion, and
>
> C equal to the broken line curve consisting of closed straight line segments, first horizontally from $\left(0, -\frac{1}{2}\right)$ to $\left(+\frac{1}{2}, -\frac{1}{2}\right)$, then vertically from $\left(+\frac{1}{2}, -\frac{1}{2}\right)$ to $\left(+\frac{1}{2}, +\frac{1}{2}\right)$, and finally horizontally from $\left(+\frac{1}{2}, +\frac{1}{2}\right)$ to $\left(0, +\frac{1}{2}\right)$, in which case the homeomorphism is the standard piecewise linear parametrization of the broken line which maps each third of $[-1, 1]$ to a closed segment in C.

Since these two decompositions satisfy the abstract characterization of Theta Spaces, it follows that the decomposed spaces are homeomorphic to each other.$\blacksquare$