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Mathematics 205B, Winter 2019, Examination 1

Answer Key

1. [*25 points*] Given two chain complexes (A_*, d^A) and (B_*, d^B), explain why the direct sum $(A_* \oplus B_*, d^A \oplus d^B)$ is also a chain complex and for each integer q we have an isomorphism $H_q(A \oplus B) \cong H_q(A) \oplus H_q(B)$.

SOLUTION

By the construction of a direct sum homomorphism we know that

$$\mathrm{Ker}(d^A \oplus d^B) \ = \ \mathrm{Ker}(d^A) \oplus \mathrm{Ker}(d^B) \ , \quad \mathrm{Image}(d^A \oplus d^B) \ = \ \mathrm{Image}(d^A) \oplus \mathrm{Image}(d^B)$$

and we also know that if we have abelian group inclusions $S \subset U$ and $T \subset V$ then $(U \oplus V)/(S \oplus T) \ \cong \ (U/S) \oplus (V/T)$. If we combine these observations with the definition of homology

$$H_q \ = \ \mathrm{Ker}\, d_q / \mathrm{Image}\, d_{q+1}$$

we see that $H_q(A \oplus B) \cong H_q(A) \oplus H_q(B)$.$\blacksquare$

2. [*25 points*] Suppose that Q is a solid square in $\mathbb{R}^2$ with vertices A, B, C, D, and take the simplicial decomposition of Q whose vertices are the latter plus the center point E; assume the ordering of the vertices is the usual alphabetical order. Find a simplicial 2-chain for the associated simplicial complex such that the boundary chain is a linear combination of AB, BC, CD, AD such that the coefficient of each edge is either $+1$ or -1.

SOLUTION

See the last page for a drawing. The first step is to compute the boundaries of the four 2-simplices:

$$d(XYE) \;=\; XY \;+\; YE \;-\; XE\;, \qquad (X,Y) \;=\; (A,B),(B,C),(C,D),(A,D)$$

Direct calculation then yields the following identity:

$$d(ABE + BCE + CDE - ADE) \;=\; AB + BC + CD - AD$$

3. [*25 points*] Find a tree which is not homeomorphic to one of the examples H, X, Y, Z, where in each case the font is identical to the one which is used for the letters. Give valid mathematical reasons for your answer.

SOLUTION

There are many correct possibilities. Here is one:

$$\neq$$

This tree has two vertices which lie on exactly four edges. However, only one of the given letters (namely, X) has a vertex which lies on four edges, and in this case there is only one such vertex. Now the number of vertices which lie on exactly four edges is invariant under homeomorphism by an analysis of local homology groups at points, and therefore the new example cannot be homeomorphic to the four which are listed in the problem.∎

4. [*25 points*] Prove that no two of the spaces $S^1 \times \mathbb{R}^2$, $S^2 \times \mathbb{R}^2$ and $S^2 \times \mathbb{R}$ are homeomorphic to each other. [*Hint:* The set $S^{n-1} \times \mathbb{R}$ is homeomorphic to the open subset $\mathbb{R}^n - \{\mathbf{0}\}$ in $\mathbb{R}^n$. What happens if we take Cartesian products with some $\mathbb{R}^k$?]

SOLUTION

The first is not homeomorphic to the other two because it has fundamental group isomorphic to $\mathbb{Z}$ and the other two are simply connected; since homeomorphic arcwise connected spaces have isomorphic fundamental groups, we know that $S^1 \times \mathbb{R}^2$ cannot be homeomorphic to the other two spaces. The second and third spaces cannot be homeomorphic, for the hint implies that $S^2 \times \mathbb{R}$ is homeomorphic to a nonempty open subset of $\mathbb{R}^2$ and $S^2 \times \mathbb{R}^2 \cong (\mathbb{R}^3 - \{0\}) \times \mathbb{R}$ is homeomorphic to a nonempty open subset of $\mathbb{R}^4$. By the Invariance of Dimension Theorem, nonempty open subsets in $\mathbb{R}^3$ and $\mathbb{R}^4$ are never homeomorphic, and therefore $S^2 \times \mathbb{R}^2$ and $S^2 \times \mathbb{R}$ are not homeomorphic.$\blacksquare$