Mathematics 205B, Winter 2014, Examination 1

Answer Key

1. [*30 points*] (*a*) Suppose that $\pi_1(B, b_0) \cong \mathbb{Z}_6$. Prove that, up to equivalence, there are exactly two connected coverings $p : (E, e_0) \to (B, b_0)$ such that p is not a homeomorphism and E is not simply connected.

(*b*) Suppose that $p : E \to B$ is a connected covering where E is simply connected, let Y be a simply connected space, and let $q : X \to B \times Y$ be a covering where X is also simply connected. Prove that X is homeomorphic to $E \times Y$. [*Hint:* What can we say about the product mapping $p \times \mathrm{id}_Y$?]

SOLUTION

(*a*) The connected coverings correspond to subgroups of $\pi_1(B, b_0) \cong \mathbb{Z}_6$, and the subgroups are $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_6$, and the correspondence is that the subgroup is equal to $\pi_1(E, e_0)$. If this subgroup is the trivial group then E is simply connected, and if this subgroup is the group $\mathbb{Z}_6$ itself, then the covering must be topologically equivalent to the identity map and hence $E \to B$ must be a homeomorphim. The only other possibilities are that the subroup in question is $\mathbb{Z}_2$ or $\mathbb{Z}_3$, and by the classification of connected coverings every covering of the prescribed type must be equivalent to an example from one of these two classes.■

(*b*) A product of two covering maps is also a covering map, so the product mapping $p \times \mathrm{id}_Y$ is a covering map from $E \times Y$ to $B \times Y$. The total space $E \times Y$ is a product of simply connected spaces and hence it is also simply connected. By the classification theorem for covering spaces, we know that two simply connected covering spaces of X are homeomorphic. In particular, this means that X must be homeomorphic to $E \times Y$.■

2. [*20 points*] Let $n \geq 2$, and let $\mathbb{RP}^n$ denote real projective n-space. Prove that every continuous mapping from $\mathbb{RP}^n$ to S^1 is homotopic to a constant. [*Hint:* Why is the same conclusion true if S^1 is replaced by $\mathbb{R}$?]

SOLUTION

Following the hint, we start by proving that every continuous mapping $f : \mathbb{RP}^n \to S^1$ can be lifted to a continuous mapping $F : \mathbb{RP}^n \to \mathbb{R}$ such that $p \circ F = f$, where $p : \mathbb{R} \to S^1$ is the usual covering space projection. The Lifting Criterion implies that f exists if and only if the image of the map $f_* : \pi_1(\mathbb{RP}^n) \to \pi_1(S^1)$ is contained in the image of $p_* : \pi_1(\mathbb{R}) \to \pi_1(S^1)$. The image of the latter map is the trivial subgroup. We claim the image of f_* is also trivial. This is true because $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$, $\pi_1(S^1) \cong \mathbb{Z}$, and all homomorphisms from $\mathbb{Z}_2$ to $\mathbb{Z}$ are trivial (a homomorphism sends elements of finite order to elements of finite order, and in $\mathbb{Z}$ only the trivial element has finite order). Since the images of f_* and p_* are both trivial, we can use the Lifting Criterion to conclude that f lifts to a continuous mapping $f : \mathbb{RP}^n \to S^1$, as claimed in the first sentence of this paragraph.

Since $\mathbb{R}$ is contractible, the map F is homotopic to a constant map c. Therefore $f = p \circ F$ is homotopic to $p \circ c$, which is also a constant map.∎

3. [*20 points*] Let $(X, \mathcal{E})$ be a finite graph (recall this means that it has finitely many edges). Prove that X is connected if and only if each pair of vertices can be joined by an edge path.

SOLUTION

Suppose first that each pair of vertices can be joined by an edge path. Since each edge is a connected set, each edge is contained in some component of X, and hence X has at most finitely many components (since X is the union of finitely many edges). Note also that two edges must lie in the same component if they have a common vertex. This implies that if we are given an edge path in $(X, \mathcal{E})$, then all of its edges and vertices lie in a single component. By the assumption in the first sentence, this means that all vertices lie in the same component. Since v and E are contained in the same component if v is a vertex of E, this means that every edge must also be contained in the component containing all the vertices. Therefore X has only one component and hence is connected.

Conversely, suppose that X is connected, and define an equivalence relation on vertices with $v \sim w$ if and only if $v = w$ or there is an edge path joining v to w. As in the previous paragraph, the space X has only finitely many components, and equivalent vertices lie in the same component of X. For each equivalence class $\mathcal{A}$ of the equivalence relation, let $X(\mathcal{A})$ denote the subgraph of X consisting of the vertices in $\mathcal{A}$ and all edges in the graph which join these edges.

We claim that distinct equivalence classes define disjoint subgraphs. To see this, suppose that $X(\mathcal{A})$ and $X(\mathcal{B})$ have a point in common. This point lies on some edge E, and by construction the vertices of this edge must lie in both $\mathcal{A}$ and $\mathcal{B}$; since $\mathcal{A}$ and $\mathcal{B}$ are equivalence classes, they are disjoint or identical, and consequently we must have $\mathcal{A} = \mathcal{B}$.

Next, we claim that every edge belongs to one of the sets $X(\mathcal{A})$. Let E be an edge and v_0, v_1 are its vertices; then $v_0 \in \mathcal{A}$ for some $\mathcal{A}$, and since v_1 is another vertex of E it follows that v_1 also belongs to $\mathcal{A}$, which means that E is contained in $X(\mathcal{A})$.

The preceding discussion shows that a finite graph has a decomposition into a finite collection of pairwise disjoint subgraphs, each of which is connected by the first part of the problem. Therefore, if X is connected there can only be one such class, which means that every pair of vertices can be joined by an edge path if X is connected.■

4. [*30 points*] Let $(X, \mathcal{E})$ be the connected graph in $\mathbb{R}^2$ whose vertices are given by the set $L = \{0,1,2,3\} \times \{0,1,2,3\}$ and whose edges are the vertical and horizontal segments whose endpoints are have the form (a,b) and $(a,b+1)$ or (a,b) and $(a+1,b)$ for suitable points in L (so X is the framework for a 3×3 grid in the plane).

(a) Find a maximal tree in $(X, \mathcal{E})$ and compute the fundamental group of X. For the first part, it will suffice to give an accurate drawing of a maximal tree.

(b) If $Y \to X$ is a connected n-sheeted covering space, what is the fundamental group of Y?

SOLUTION

(a) There are 16 vertices and 24 edges in the graph (12 horizontal and 12 vertical), so $\chi(X, \mathcal{E}) = -8$ and the fundamental group is a free group on $1 - \chi(X, \mathcal{E}) = 9$ generators. A maximal tree will contain all of the vertices, and it will have 15 edges because its Euler characteristic is 1. Drawings for 9 examples appear on the next page; of course, there are also many other possibilities.■

(b) Since Y is an n-sheeted covering we have an associated graph complex structure $\mathcal{E}'$ on Y such that $\chi(Y, \mathcal{E}') = n \cdot \chi(X, \mathcal{E}) = -8n$, and thererfore the fundamental group of Y is a free group on $8n + 1$ generators.■