Mathematics 205B, Spring 2018, Examination 1

Answer Key

1. [35 *points*] These questions involve the pentagram (5–pointed star) graph on the next page.

(a) The fundamental group of this graph is a free group on finitely many generators. Determine the number of such generators.

(b) Sketch a maximal tree in this graph.

(c) Describe an Euler path for this graph.

There is space for drawings on this page and the next one.

SOLUTION

(a) The fundamental group is free on $1 - v + e$ generators, where e and f denote the numbers of edges and vertices. There are 15 edges and 10 vertices, so the number of generators for the fundamental group is $1 - 10 + 15 = 6$.$\blacksquare$

(b) There are many answers. One easy way of recognizing maximal trees is that sufficient conditions are connectedness, all the vertices appear on the edges in the subcomplex, and the number of edges is 1 more than the number of vertices, so that there are 9 edges in a maximal tree.$\blacksquare$

(c) Every edge must appear in the reduced edge path, and the pieces must be suitably oriented. The simplest Euler path is probably the usual method for drawing a five pointed star without lifting the pen or pencil for the paper, but there are also others (see the accompanying figures).$\blacksquare$

2. [25 *points*] Let $p : (E, e) \to (S^1, 1)$ be a connected covering space and assume E is compact Hausdorff. Prove that E is homeomorphic to S^1. [*Hint:* For each n find an n-sheeted covering space projection onto S^1.]

SOLUTION

The induced map from $\pi_1(E, e_0)$ to $\pi_1(B, b_0)$ is 1–1, and the inverse image of a point is in 1–1 correspondence with the set of cosets $\pi_1(B, b_0)/\text{Image}\,\pi_1(E, e_0)$. If $B = S^1$, then $\pi_1(B) \cong \mathbb{Z}$ and hence $\pi_1(E, e_0)$ must be either $\{0\}$ or $n\mathbb{Z}$ for some positive integer n. We claim that the latter happens under the conditions in this problem.

The inverse image of a base point in E is a closed discrete subset of E. Since E is compact, it follows that this inverse image must be finite. Therefore the image of $\pi_1(E)$ in $\pi_1(S^1) \cong \mathbb{Z}$ must be trivial, and hence it must be $n\,\mathbb{Z}$. Since connected coverings are classified by their fundamental groups, viewed as subsets of $\pi_1(S^1)$, it follows that E is homeomorphic to $R/n\mathbb{Z}$, and hence we need only show that the latter is homeomorphic to S^1.

Now $R/n\mathbb{Z}$ is homeomorphic to $[0, n]$ with the endpoints identified, and the latter quotient space is homeomorphic to S^1. Hence by the classification of coverings E is homeomorphic to $\mathbb{R}/n\mathbb{Z} \cong S^1$.■

3. [*20 points*] Suppose that X and X' are subspaces in $\mathbb{R}^n$ such that there are graph structures $\mathcal{E}$ and $\mathcal{E}'$ on X and X' respectively such that $X \cap X'$ is a single point which is a vertex of both. Explain why $X \cup X'$ has a graph complex structure.

SOLUTION

We can take the edges in $\mathcal{E}$ and $\mathcal{E}'$ to be the set of edges for the graph complex structure. By construction $X \cup X'$ is the union of the edges in $\mathcal{E} \cup \mathcal{E}'$. Suppose that two edges meet. If both edges are in $\mathcal{E}$ or $\mathcal{E}'$, then they have at most one vertex in common. If one edge is in $\mathcal{E}$ and the other is in $\mathcal{E}'$, then the intersection of the two edges lies in $X \cap X'$, which is a vertex of both. Thus in this case if two edges have a nonempty intersection, it must be the common vertex of the two graphs.$\blacksquare$

4. [20 *points*] Suppose that X is the union of the open arcwise connected subspaces U and V such that $U \cap V \neq \emptyset$ is arcwise connected and both U and X are simply connected; take a base point for all of these (sub)spaces which lies in $U \cap V$. Prove that if H is the normal subgroup $\pi_1(V)$ which is (normally) generated by the image $\pi_1(U \cap V)$, then $H = \pi_1(V)$.

SOLUTION

Since the fundamental group of U is trivial, by van Kampen's Theorem the fundamental group of X is the quotient of $\pi_1(V)$ by the normal subgroup H (normally) generated by the image of $\pi_1(U \cap V)$ in $\pi_1(V)$. Since X is simply connected, this quotient is trivial, and therefore H must be all of $\pi_1(V)$.■