**NAME:** ______________________________

# Mathematics 205B, Winter 2012, Final Examination

*Work all questions, and unless indicated otherwise give reasons for your answers. The point values for individual problems are indicated in brackets.*

Unless explicitly stated otherwise, all spaces are assumed to be **Hausdorff** and **locally arcwise connected**.

## Answer Key

1. [*25 points*]

Suppose that $X$ is an arcwise connected space whose fundamental group is finite. Prove that every continuous mapping $f$ from $X$ to $S^1$ is homotopic to a constant. [*Hint:* Show that $f$ lifts to the universal covering space of $S^1$.]

**SOLUTION:**

The map $f$ lifts to the universal covering space $\mathbb{R}$ if and only if the induced map $f_*$ of fundamental groups is trivial. This is true in our situation because $\pi_1(X)$ is finite, so that every homomorphism from $\pi_1(X)$ to the infinite cyclic group $\pi_1(S^1)$ is constant (elements of finite order go to elements of finite order, and the only such element of $\pi_1(S^1)$ is the identity). Therefore, if $p : \mathbb{R} \to S^1$ is the universal covering space projection we have a lifting $g$ of $f$ to a map $X \to \mathbb{R}$ such that $p \circ g$ is homotopic to $f$. Since $\mathbb{R}$ is contractible it follows that $g$ is homotopic to a constant map, and therefore the same is true of $f = p \circ h$.

2. [*20 points*]

Suppose that we are given a decomposition of a space $X$ as a union of two (nonempty) arcwise connected subspaces such that their intersection is also nonempty and arcwise connected. Suppose also that the maps of fundamental groups from $\pi_1(U \cap V)$ to $\pi_1(U)$ and $\pi_1(V)$ (induced by inclusions) are both onto. Prove that the the map of fundamental groups from $\pi_1(U \cap V)$ to $\pi_1(X)$ (induced by inclusion) is also onto.

**SOLUTION:**

The Seifert-van Kampen Theorem implies that $\pi_1(X)$ is generated by the images of $\pi_1(U)$ and $\pi_1(V)$. By our hypotheses the latter are generated by the images of $\pi_1(U \cap V)$, and therefore the set of generators for $\pi_1(X)$ coming from $\pi_1(U)$ and $\pi_1(V)$ must lie in the image of $\pi_1(U \cap V)$. Therefore the map from $\pi_1(U \cap V)$ to $\pi_1(X)$ maps onto a set of generators, which implies that every element of $\pi_1(X)$ lies in the image of $\pi_1(U \cap V)$.

3. [*25 points*]

Suppose that $X$ is the graph given by the Star of David figure in the drawing handed out with this examination, and take the decomposition with 12 vertices suggested by this drawing. Compute the fundamental group of $X$ and the fundamental group of a connected 3-sheeted covering of $X$.

**SOLUTION:**

See `star-graph.pcf` for a drawing.

The Star of David graph has 18 edges, so its Euler characteristic $\chi$ is equal to $12-18 = -6$. Therefore the fundamental group of $X$ is a free group on $1-\chi = 7$ generators. Now the Euler characteristic $\chi'$ of the 3-fold covering is $3 \cdot \chi = -18$, and therefore the fundamental group of the covering space is free on $1 - \chi' = 19$ generators.

4. [*25 points*]

Let $(P, \mathbf{K})$ be a simplicial complex, and let $P_1$ be the graph whose edges and vertices are the 1-simplices and vertices of $\mathbf{K}$. Prove that $P$ is connected if and only if every pair of vertices can be joined by an edge path in $P_1$.

**SOLUTION:**

The first step is to show that every point of $P$ is in the same component as some vertex; this is true because each point lies in a simplex $A$, and each simplex is connected so that $A$ lies in the connected component defined by one of its vertices.

If every pair of vertices can be joined by an edge path in $P_1$, then all vertices lie in the same component, and by the previous paragraph this means that every point of $P$ lies in the same component, which means that $P$ must be connected (in fact, it is arcwise connected).

Conversely, given an arbitrary simplicial complex $(P, \mathbf{K})$ consider the equivalence relation on the set of vertices given by $v \sim v'$ if and only if $v$ and $v'$ can be joined by an edge path in $P_1$. Given an equivalence class $C$ of vertices, define the *hull* of $C$ to be the union of all simplices in $\mathbf{K}$ which contain a vertex in $C$ among its vertices. It follows that if a simplex $A$ has one vertex in $C$ among its vertices, then all the vertices in $A$ lie in $C$. The sets $\mathbf{Hull}(C)$ form a finite, pairwise disjoint family of closed subspaces whose union is $P$. If $P$ is connected then there can only be one such subset and hence $C$ must consist of all vertices of $\mathbf{K}$. But this means that every pair of vertices can be joined by an edge path in $P_1$.

5. [*25 points*]

Suppose that the simplicial complex $\mathbf{K}$ is the union of two subcomplexes $\mathbf{K}_1$ and $\mathbf{K}_2$; choose a linear ordering $\omega$ for the vertices of $\mathbf{K}$. Prove that there are isomorphisms of chain complexes from $C_*(\mathbf{K_1}, \mathbf{K}_1 \cap \mathbf{K}_2) \oplus C_*(\mathbf{K_2}, \mathbf{K}_1 \cap \mathbf{K}_2)$ to $C_*(\mathbf{K}, \mathbf{K}_1 \cap \mathbf{K}_2)$ (defined using $\omega$) and corresponding isomorphisms of simplicial homology groups. [*Hint:* Look at the standard free generators for these relative chain complexes. Recall that $C_q(\mathbf{L}, \mathbf{M})$ is a free abelian group on the set of $q$-simplices in $\mathbf{L}$ but not in $\mathbf{M}$.]

**SOLUTION:**

We shall show that there is a 1–1 correspondence between sets of generators for the $q$-chain groups of both chain complexes such that the map induced by inclusion sends one set of generaors to the other. This will imply that we have an isomorphism on the level of chain groups and chain complexes, and by functoriality we will get an associated isomorphism of homology groups.

By definition the groups $C_q(\mathbf{K_1}, \mathbf{K}_1 \cap \mathbf{K}_2)$ and $C_q(\mathbf{K_2}, \mathbf{K}_1 \cap \mathbf{K}_2)$ $C_q(\mathbf{K_2}, \mathbf{K}_1 \cap \mathbf{K}_2)$ are free abelian on the $q$-simplices of $\mathbf{K_1}$ or $\mathbf{K}_2$ which are not in $\mathbf{K}_1 \cap \mathbf{K}_2$. Each of these simplices is a $q$-simplex of $\mathbf{K}$ which does not lie in $\mathbf{K}_1 \cap \mathbf{K}_2$, and the map from free generators of $C_q(\mathbf{K_1}, \mathbf{K}_1 \cap \mathbf{K}_2) \oplus C_q(\mathbf{K_2}, \mathbf{K}_1 \cap \mathbf{K}_2)$ to free generators of $C_q(\mathbf{K}, \mathbf{K_1} \cap \mathbf{K}_2)$ is 1–1 because the $q$-simplices in $\mathbf{K_1}$ but not in $\mathbf{K}_1 \cap \mathbf{K}_2$ are disjoint from the $q$-simplices in $\mathbf{K_2}$ but not in $\mathbf{K}_1 \cap \mathbf{K}_2$. But this map is also onto, because a simplex which is in $\mathbf{K} = \mathbf{K}_1 \cap \mathbf{K}_2$ but not in $\mathbf{K}_1 \cap \mathbf{K}_2$ either comes from $\mathbf{K}_1$ or $\mathbf{K}_2$. This shows that the chain level maps send a canonical set of free generators for the domain to a canonical set of free generators for the codomain, so that the induced map of chain complexes is an isomorphism (and hence the induced homology map is also an isomorphism).

6. [*30 points*]

Prove that every finitely generated abelian group is isomorphic to the fundamental group of a compact topological $n$-manifold for some choice of $n$. [*Hint:* If $X$ and $Y$ are topological manifolds of dimensions $p$ and $q$, then their product is a topological $(p+q)$-manifold.]

**SOLUTION:**

Write the finitely generated abelian group $\pi$ as a product of $B$ infinite cyclic groups and finite cyclic groups of orders $k_1, \cdots, k_r$. Consider the topological manifold given by a product of $B$ copies of $S^1$ and the lens spaces $M(k_j) = S^3/\mathbb{Z}_{k_j}$. Since the fundamental group of a product is a product of the fundamental groups of the factors, it follows that the fundamental group of the product space (which is a compact topological manifold) will be isomorphic to $\pi$.

7. [*25 points*]

One can use Mayer-Vietoris sequences and homotopy invariance in singular homology to prove the following formula in which $n$ is a positive integer and $X$ is an arbitrary nonempty space:

$$H_q(S^n \times X) \;\cong\; H_q(X) \,\oplus\, H_{q-n}(X)$$

Use this formula to compute the homology groups of the compact 6-manifolds $S^3 \times S^3$ and $S^2 \times S^4$, and explain why the computations show that these spaces are not homeomorphic (or even homotopy equivalent).

**SOLUTION:**

Use the rule to compute the homology of $S^2 \times S^4$ and $S^3 \times S^3$.

We know that

$$H_q(S^2 \times S^4) \;\cong\; H_q(S^4) \,\oplus\, H_{q-2}(S^4)\;, \quad H_q(S^3 \times S^3) \;\cong\; H_q(S^3) \,\oplus\, H_{q-3}(S^3)$$

and these yield the following computations of homology groups:

$$H_3(S^3 \times S^3) \;\cong\; H_3(S^3) \,\oplus\, H_0(S^3) \;=\; \mathbb{Z} \oplus \mathbb{Z}$$

$$H_3(S^2 \times S^4) \;\cong\; H_3(S^4) \,\oplus\, H_1(S^4) \;=\; 0$$

Since the 3-dimensional homology groups of $S^3 \times S^3$ and $S^2 \times S^4$ are not isomorphic, the spaces cannot be homeomorphic or even homotopy equivalent.

8. [*25 points*]

Suppose that $\Gamma \subset S^2$ is a simple closed curve, and let $A \subset S^2 - \Gamma$ be a set of $k \geq 2$ points. For which values of $k$ is it always possible to find two points in $A$ which can be joined by a continuous curve in $S^2 - \Gamma$, and for which valued is this not always possible? Prove that your answer is correct. [*Hint:* What can we say when there are $p$ objects which lie in $q < p$ pairwise disjoint subsets?]

**SOLUTION:**

It is always possible to find two points in $A$ which can be joined by a continuous curve in $S^2 - \Gamma$ if $k \geq 3$, for there are 2 components in $S^2 - \Gamma$, and since $3 > 2$ there must be two points which lie in the same component (equivalently, arc component), and these points can be joined by a continuous curve which lies in the component and hence in the complement of $\Gamma$.

On the other hand, this is not always possible if $k = 2$, for $S^2 - \Gamma$ has two (arc) components, and if we take one point from each and join them by a continuous curve in $S^2$, then this curve cannot be contained in the complement (if it did, both points would lie in the same arc component of the complement and we know this is not true).