UCR Math Dept Topology Qualifying Exam

September 2020

- There are three parts in this exam, and each part has three problems. You should complete two (and only two) problems of your choice in each part. Each problem is worth a maximum of 10 points.
- Support each answer with a complete argument. State completely any definitions and basic theorems you use.
- This is a closed-book test. You may only use the test and writing material. All other material or internet resource is prohibited. Write each of your solutions on separate sheets.
- The time for the exam is three hours.

Part A

(1) Let X, Y be topological spaces with Y Hausdorff, and let $f : X \to Y$ be a function.
- Show that if f is continuous, the *graph* Γ_f defined as
$$\Gamma_f = \{(x,y)|y = f(x)\} \subset X \times Y$$
is closed;
- Show that the Hausdorff assumption on Y cannot be removed;
- Show that if Y is compact, the converse holds true. (Hint: the projection map $\pi\colon X \times Y \to X$ is closed.)

(2) Let (X, d) be a metric space, and let $\mathcal{T}$ be a topology on X. Endow $X \times X$ with the product topology inherited from $(X, \mathcal{T})$ and $\mathbb{R}$ with the standard topology. Show that if the distance function $d\colon X \times X \to \mathbb{R}$ is continuous w.r.t. these topologies, then $\mathcal{T}$ is finer than the metric topology of X.

(3) Let X be a nonempty compact Hausdorff space. Assume X has no isolated points (i.e. points $x \in X$ such that $\{x\}$ is open).
- (a) Show that if $U \subset X$ is open, for any $x \in X$ there exists a nonempty open set $V \subset U$ such that $x \notin \overline{V}$.
- (b) Show that there is no surjective map $f\colon \mathbb{N} \to X$, where $\mathbb{N}$ is the set of natural numbers. (Hint: use the previous result and the Hausdorff condition to define a nonempty collection of open sets $\{V_n, n \in \mathbb{N}\}$, where "$V_n$ avoids $f(n)$" whose closures have nonempty intersection.)

Part B

(4) Consider the figure-8, $E = S^1 \vee S^1$.
- Give a 2-sheeted cover $\tilde{E} \to E$ and compute $\pi_1(\tilde{E})$.
- What does it mean for a cover to be normal? Is the cover $\tilde{E} \to E$ normal?

(5) Let E be a figure-8 embedded in $\mathbb{R}^2$. For example, E could be the union of the unit circles centered at $(-1, 0)$ and $(1, 0)$. Compute the relative homology groups $H_*(\mathbb{R}^2, E; \mathbb{Z})$.

(6) Consider the genus two surface, S_2, and a genus 1 surface with boundary component, $S_{1,1}$ (see Figure 1). Let C be the space obtained by gluing the "middle curve" α of S_2 to the boundary curve of $S_{1,1}$. Compute the homology groups $H_*(C, \mathbb{Z})$.

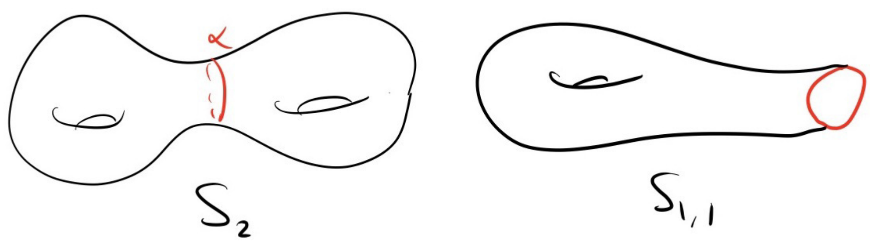

Figure 1. The surfaces in problem 6.

Part C

(7) Let

$$E := \left\{ (z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \;\middle|\; z_0^2 + \cdots + z_n^2 = 1 \right\}.$$

- Show that E is a smooth submanifold of $\mathbb{C}^{n+1}$.
- Let X and Y be smooth vector fields on $\mathbb{C}^{n+1}$ whose restriction to E is tangent to E. Show that the restriction of $[X, Y]$ to E is tangent to E.

(8) Let $M^k \subset \mathbb{R}^{n+1}$ be a compact submanifold of dimension k so that $n \geq 2k + 1$. Show that there is an n–dimensional vector subspace $V \subset \mathbb{R}^{n+1}$ so that orthogonal projection

$$\pi : \mathbb{R}^{n+1} \longrightarrow V$$

restricts of an embedding $M^k \hookrightarrow V$.

(9) Show that if $f : \mathbb{R}P^n \longrightarrow \mathbb{R}$ is differentiable, then there are two distinct points $p, q \in \mathbb{R}P^n$ where the differential of f is the 0 map.

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September 2019

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Part I

1. Let X be a topological space. An equivalence relation $\sim$ on X is called *open* if the associated quotient map

$$\pi : X \to X/\sim$$

is open, when $X/\sim$ is endowed with the quotient topology. Let $\sim$ be an open equivalence relation.

Show that $X/\sim$ is Hausdorff if and only if

$$R := \{(x, y) \mid x \sim y\} \subset X \times X$$

is closed in the product topology of $X \times X$.

2. Let X be a topological space, and $C_\alpha \subset X, \alpha \in A$, be a locally finite family of closed sets.

(a) Show that $\bigcup_{\alpha \in A} C_\alpha \subset X$ is closed.

(b) Show that this property may fail without the assumption of local finiteness.

3. A topological space X is *normal* if given any two disjoint closed subsets $C, D \subset X$ there exist two disjoint open sets $U \supset C, V \supset D$. Prove that a compact Hausdorff space is normal.

Part II

4. Let X be a nonempty space which is Hausdorff, connected, simply connected, and locally path connected. Prove that every continuous mapping $f : X \to S^1$ is homotopic to a constant map.

5. Suppose that Q is a solid square in $\mathbb{R}^2$ with vertices A, B, C, D, let E denote the center point of Q, and consider the graph X whose edges are the boundary of Q plus the segments joining the vertices of Q to E. If $Y \subset X$ is the subgraph given by the boundary of Q, show that Y is not a deformation retract of X. [*Hint:* What are the fundamental groups?]

6. If E is a topological space with $e \in E$, then the local homology groups of E at e are defined by $H_k(E, E - \{e\})$. Show that if $f : X \to Y$ is a homeomorphism and $x \in X$, then f induces an isomorphism from $H_k(X, X - \{x\})$ to $H_k(Y, Y - \{f(x)\})$ for every integer k. [*Hint:* What is $f[X - \{x\}]$?]

Part III

7. Give an example of a smooth map $F : M \longrightarrow N$ and a smooth vector field X on M that is not F–related to any vector field on N.

8. Suppose $M \subset \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subset T\mathbb{R}^n$ be the set of all unit vectors tangent to M :

$$UM := \{ (x, v) \in T\mathbb{R}^n \mid x \in M, v \in T_x M, \ |v| = 1 \} .$$

It is called the **unit tangent bundle** of M. Prove that UM is an embedded $(2m-1)$–dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$.

9. Show that $S^n \times \mathbb{R}$ is parallelizable for all $n \geq 1$.

UCR Math Dept Topology Qualifying Exam

September 2018

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Part I

1. Let

$$p : (E, e_0) \longrightarrow (B, b_0)$$

be a covering map of pointed spaces. Show that

$$p_* : \pi_1(E, e_0) \longrightarrow \pi_1(B, b_0)$$

is injective. Here p_* is the map that p induces $\pi_1(E, e_0) \longrightarrow \pi_1(B, b_0)$.

2. Let $h : S^1 \longrightarrow S^1$ be nullhomotopic.

(a) Show that h has a fixed point.

(b) Show that h maps some point x to its antipode $-x$.

3. Let X be a compact metric space. Let

$$C_1 \supset C_2 \supset C_3 \supset \cdots$$

be an infinite sequence of closed, nonempty subsets. Prove that

$$\bigcap_{i=1}^{\infty} C_i \neq \emptyset.$$

Part II

4. Suppose that X is a connected, locally arcwise connected, Hausdorff space and $f : X \longrightarrow S^1$ is a continuous map such that $f(x_0) = 1$ for some $x_0 \in X$. Prove that f is homotopic to a constant map if and only if the induced map of fundamental groups from $\pi_1(X, x_0)$ to $\pi_1(S^1, 1)$ is trivial.

5. Let A be a simplex in $\mathbb{R}^n$ for some n, and let $B \subset A$ be a union of faces in A. Prove that the homology groups $H_q(B)$ are isomorphic to the homology groups $H_{q+1}(A, B)$ for all $q > 0$. Give a counterexample to the corresponding statement if $q = 0$ and A is a 1-simplex.

6. Prove that if $f : S^n \longrightarrow S^n$ is continuous and $1-1$, then f is a homeomorphism using Invariance of Domain. Also prove that if $m > n$, then there is no continuous $1-1$ mapping from S^m to S^n.

Part III

7. Consider the map

$$\phi_t : \mathbb{R}^3 \to \mathbb{R}^3 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} (\cos t)x - (\sin t)y \\ (\sin t)x + (\cos t)y \\ z \end{pmatrix}$$

a. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere: show that ϕ_t, when restricted to S^2, defines a one-parameter family of diffeomorphisms. (Note: this entails showing that for a *fixed* t each ϕ_t is 1. bijective *in* S^2; 2. smooth with smooth inverse; 3. $\phi_s \circ \phi_t = \phi_{s+t}$.)

b. Determine the vector field generating this one-parameter family of diffeomorphisms.

8. Show that the subset of $\mathbb{R}^3$ defined by the equation

$$(1 - z^2)(x^2 + y^2) = 1$$

is a smooth manifold.

9. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$F(x, y) = (x + y^2, x^2).$$

Denoting by u, v the Cartesian coordinates of the target, determine $F^*(vdu + dv)$.