Winter 2012

Review Suggestions for the Midterm Examination

1. Know how to use the lifting criterion and understand what it says and means (e.g., if the domain is simply conn.)

2. Understand the classification ~~res~~ and existence results for connected covering spaces of locally simply connected spaces (as usual, assume all spaces are Hausdorff and locally arcwise connected). Be able to apply it to specific problems; for example, ~~classify an~~ enumerate the equivalence classes of coverings for $\mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2$. * [see below]

3. Know the statement of the Seifert-van Kampen theorem, the characterization of pushouts in group theory, and ways of applying this to specific examples as in the homework exercises.

4. Know the basic definitions of graphs, edge paths, trees, etc. and the proofs of some of their elementary properties like the characterization of connectedness. Also know the proof that a tree is contractible.

5. Know how to work with examples of graphs, using theorems to recognize maximal trees and compute fundamental groups. Likewise for describing the fundamental group of a finite covering of a connected graph.

6. Know how to apply fundamental groups to prove results about maps between spaces like (a) every map from $\mathbb{RP}^2$ to S^1 is homotopic to a constant map [use liftings], (b) if $C \subseteq$ some lens space L of dimension ≥ 3 is homeomorphic to a circle, then C is not a retract of L. [Hint: If $r : L \to C$ is such that $r|C = id$, then what can we say about $\pi_1(C) \to \pi_1(L) \to \pi_1(C)$ and why is this impossible?]. Similarly for showing, say, that S^1 is not a retract of S^n for $n \geq 2$.

7. Rejected problem: Suppose the connected graph X is a union of subgraphs X_1, X_2 s.t. their union is connected. If gen(Y) = # free generators for $\pi_1(Y)$ prove that

$$\text{gen}(X) = \text{gen}(X_1) + \text{gen}(X_2) - \text{gen}(X_1 \cap X_2)$$

[Hint: If $Y \subseteq X$ is a subgroup, find open $U \supseteq Y$ s.t. Y is a strong deformation retract of U; this can be done as in the theorem describing π_1 of a graph.]

Footnote

(*) Count the subgroups of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ as follows: We know they are all $\mathbb{Z}_2$ vector subspaces (why?), so it's only a matter of counting subspaces of dim 0, 1, 2, 3. Note that numbers of 1- and 2-dim subspaces are the same: To see this consider the nondegenerate symmetric bilinear form on $\mathbb{Z}_2^3$ given by

$$F(x_1, x_2, x_3; y_1, y_2, y_3) = \sum x_i y_i$$

If $V \subseteq \mathbb{Z}_2^3$ is a vector subspace, let $V^\perp$ be ~~the~~ set of all x such that $F(x, v) = 0$ for all $v \in V$. Then $V \to V^\perp$ defines a 1-1 onto map from 1- to 2-dim vector subspaces.

Similar problems: Suppose instead we take a 3D lens space s.t. $\pi_1 = \mathbb{Z}_{420}$. How many coverings are there in that case? [Hints: What is the prime factorization of $\mathbb{Z}_{420}$? What can we say about its subgroups?].