Review — II

Here are some suggestions for review in connection with the final examination to be held on Monday, June 6, 2011. The term "nice space" refers to one which is Hausdorff and locally arcwise connected. One fourth of the exam will consist of problems from material also covered on the midterm examination. In particular, there may be some overlap with questions on the first written assignment and the midterm exmination.

Here are the main topics that may/will be covered:

(0) Questions about classifying covering spaces and subgroups of the fundamental group. Two points about group theory that are worth remembering are Lagrange's result on the order of a subgroup of a finite group and, given a group G with a normal subgroup H, the correspondence between subgroups of G/H and subgroups of G containing H. The concept of a set of generators for a group or subgroup (and normal subgroup generators for a normal subgroup) should also be understood.

(1) Basic definitions and results concerning graphs, trees, and their fundamental groups. In particular, this includes knowing the proofs of (a) the proof that a tree has a "free edge" which intersects the remaining edges in only one vertex, (b) knowing how to derive the formula for the number of free generators for a finite n-sheeted covering of a connected graph in terms of n and the number of free generators for the base graph. Several pieces of algebraic background are needed, most notably the concepts of free group, free product of two groups, and the pushout construction which arises in the Seifert-van Kampen Theorem; there is an extremely good chance that there will be a question involving a relatively simple consequence of the latter result.

(2) Basic definitions of simplices, polyhedra and simplicial complexes, simplicial chain groups and the formula for the boundary mapping, abstract chain complexes and their homology groups, the proof that a morphism of chain maps yields morphisms of homology groups, concepts of chain subcomplexes and quotient complexes, exact sequences, the implications of a zero map in an exact sequence for the adjacent maps on each side (different for the maps to the right and left), proof that a retract of chain complexes yields a direct sum decomposition of homology groups.

(3) Algebraic results on chain groups and homology groups of simplicial complex pairs, including finite generation results, vanishing theorems for homology above a certain point, the structure of homology groups for special examples like simplices and their boundaries. The basic results on exact sequences for simplicial homology groups, most notably the long exact sequence of a pair and the Mayer-Vietoris sequence, should be known, and it is very likely that there will be a question on a relatively simple application of such sequences.

(4) The basic axioms for homology theory, including a general understanding of their naturality properties, compatibility with simplicial homology for simplicial complexes, and the geometric/topological meaning of H_0 and H_1 in terms of arc components and fundamental groups. Also included is knowing basic axioms involving long exact sequences (both for pairs and the Mayer-Vietoris sequence for a space presented as the union of two open subsets), compact supports at least in the case of spaces (but not necessarily pairs), homotopy invariance, and excision at least for maps of the form $H_*(U, U \cap V) \to H_*(U \cup V, V)$ where

U and V are open in some space X. The axioms should be understood well enough to understand their roles in proving fundamental results like the fact that homotopy equivalent spaces have isomorphic homology groups and the fact that local homology groups only depend upon the local structure of a neighborhood of a point (knowing the definition of local homology of a space at a point is implicit in the latter).

(5) Familiarity with some of the basic applications of homology, such as its use to prove nonretraction theorems for certain subspaces of topological spaces (for example, $S^n \subset D^{n+1}$ and $T^2 \subset \mathbf{R}^3 - \{\mathbf{0}\}$ as a surface of revolution), invariance of dimension, invariance of domain, and the Jordan-Brouwer Separation Theorem; the latter is based upon another application — namely, the fact that if $E \subset S^n$ is homeomorphic to a disk then its complement $S^n - E$ has the same homology as a point — but it is not necessary to know the proof of that result. As noted in the lectures, a problem along the lines of one worked out in class (D^n not homeomorphic to D^n if $m \neq n$) is likely to appear on the examination.

For the sake of completeness we shall reproduce the solution to the preceding question: Suppose that D^m and D^n are homeomorphic and $h : D^m \to D^n$ is a homeomorphism. Let $U = D^n - S^{n-1}$ and $V = h\left[D^m - S^{m-1}\right] = D^n - h\left[S^{m-1}\right]$. Then U and V are dense subsets of D^n, and they are both open in D^n because they are complements of compact subsets. It follows that $U \cap V = U - h\left[S^{m-1}\right]$ is open in both U and V, and since U and V are dense it follows that $U \cap V$ is nonempty. The inclusions $U \cap V \subset U, V$ imply that $U \cap V$ is homeomorphic to an open subset of $\mathbf{R}^n$ (by the first inclusion) and $\mathbf{R}^m$ (by the second). By Invariance of Dimension we must have $m = n$.

Homework exercises which may be especially useful to review. We shall keep this list short.

1.1, 1.3, 1.4, 2.9

3.1, 3.9, 3.11, 4.5, 4.6, 4.11, 5.1*ab*[only], 5.3, 5.7, 5.11

Relevant files in the course directory:

We shall only list file on material covered since the cutoff for the midterm examination.

`brouwer.pdf` (Geometric details for the proof of the Brouwer Fixed Point Theorem and its applications; of secondary importance.)

`cell-euler.pdf` (Not covered in the course or examination.)

`chainboundary.pdf` (Supplement to main notes with details; it is important to know the basic formulas and $d\,d = 0$.)

`computingHq.pdf` (Supplement to main notes on computability of homology; not covered in the examination.)

`convexbodies.pdf`
`convexbodies2.pdf` (Know the basic and important homeomorphism theorem; in the examination it will not be necessary to know the details of the proof.)

`exercises03.pdf`
`exercises03a.pdf`
`exercises04.pdf`
`exercises04a.pdf`
`exercises05.pdf`
`exercises99.pdf` (Self-explanatory; see above for suggestions on priorities.)

`fishmaze.pdf`
`fishmaze1.pdf`
`fishmaze2.pdf` (Illustrations for the Jordan Curve Theorem.)

`graphs.pdf`
`graphs2.pdf`
`graphs3.pdf`
`graphs4.pdf` (Some of these files have been added since the cutoff for the midterm examination; the first three are useful or important but the last may be viewed as reference material which will not appear on the examination).

`homology-axioms.pdf` (Important formalization of this topic; see above for priorities.)

`hwassignment1.pdf`
`hwassignment1solns.pdf` (Worth reviewing.)

`hwassignment2.pdf` (This needs to be turned in before the examination. Note that the hypothesis on the sequence of compact subsets in Problem 1 has been strengthened.)

`intmatrices.pdf` (Background material for computing homology groups; aside from the results on subgroups of finitely generated free abelian groups, this will not be covered in the examination.)

`longexact.pdf` (Detailed proofs; the statements are important, but it is not necessary to know the details of the arguments.)

`math205Ctopic07.pdf`
`math205Ctopic08.pdf`
`math205Ctopic09.pdf`
`math205Ctopic09a.pdf`
`math205Ctopic10.pdf`
`math205Ctopic11.pdf`
`math205Ctopic12.pdf`
`math205Ctopic13.pdf`
`math205Ctopic14.pdf`
`math205Ctopic15.pdf`
`math205Ctopic15a.pdf`
`math205Ctopic16.pdf`
`math205Ctopic16a.pdf` (Basic course material; the last file was not covered in class, and nothing from the last two files will appear on the examination.)

`math246Aalgtopcontents.pdf`
`math246Aalgtopnotes2010.pdf` (Reference material; a few pages are cited in one file from the previous list, but otherwise it is not necessary to read through these files.)

`prism-dissection.pdf` (Illustration to show how one can decompose a solid triangular prism into 3 solid tetrahedra.)

`reducedMVseq.pdf`
`reducedVrelative.pdf` (Important additional material needed for the discussion of the Jordan-Brouwer Theorem.)

`review1.pdf` (Worth reviewing since some material will be on the examination.)

`solutions03.pdf`
`solutions04.pdf`
`solutions04a.pdf`
`solutions05.pdf`
`solutions05a.pdf` (Solutions to the exercises; see above for priorities involving exercises.)

`triangulation-pictures.pdf` (Illustrations to indicate how one can triangulate some familiar geometrical figures.)