SOLUTIONS TO FIRST

TAKE–HOME ASSIGNMENT

Winter 2018

1. Let q be defined as in the statement of the problem.

Suppose first that f has a lifting as described, so that $f = q\circ g$ for some $g : (X, x) \to (S^1, 1)$. If $u \in \pi_1(X, x)$, this implies that $f_*(u) = (q\circ g)_*(u) = q_*(g_*(u)) = 2g_*(u)$, so that the image of f_* is contained in $2 \cdot \pi_1(S^1, 1) \approx 2 \cdot \mathbb{Z}$.

Conversely, suppose that the image of f_* is contained in $2 \cdot \pi_1(S^1, 1)$. The latter is equal to the image of q_*, so by the Lifting Theorem there is a unique lifting $g : (X, x) \to (S^1, 1)$ such that $f = q\circ g$.■

2. (a) There are two parts to the proof. One is to show that $p|E_0$ maps onto B. The easiest way to do this is by the path lifting property. If $b \in B$, let γ be a continuous curve joining b_0 to b, and let Γ be the unique lifting to E such that $\Gamma(0) = e_0$. Then the image of Γ is contained in the arc component E_0, and in particular $\Gamma(1) \in E_0$. By construction $p\circ\Gamma(1) = \gamma(1) = b$, and hence p is onto.

The second part is to show that the restriction $p|E_0 : E_0 \to B$ has the property of a covering space projection. Once again let $b \in B$. Since $p : E \to B$ is a covering space projection and both spaces are locally arcwise connected, there is an arcwise connected open neighborhood U of b in B such that $p^{-1}[U]$ is a union of pairwise disjoint subsets W_β such that each restriction $p|W_\beta$ is a homeomorphism onto U. Let $\{V_\gamma\}$ be the subfamily of $\{W_\beta\}$ consisting of all sets such that $W_\beta \cap E_\alpha \neq \emptyset$; then consideration of arc components shows that each V_γ is contained in E_0 and the remaining sets W_β are all disjoint from E_0. The first paragraph implies that the subfamily $\{V_\gamma\}$ is nonempty, and therefore we know that $p|E_0$ is a covering space projection onto B.■

(b) Since E is locally arcwise connected, it is the union of its pairwise disjoint arc components E_α, and each of these is an open, closed, arcwise connected subspace. Furthermore, by (a) we know that each restriction $p|E_\alpha$ is a covering space projection.■

(c) Write $p : (E, e_0) \to (B, b_0)$ be as in (a), and use (b) to express E as a union of pairwise disjoint open closed subspaces E_α such that each E_α is arcwise connected and each restriction $p|E_\alpha$ is a covering space projection. Pick points $e_\alpha \in E_\alpha$ which map to b_0 (these exist by (a)). Then for each α the induced map of fundamental groups

$$(p_\alpha)_* : \pi_1(E_\alpha, e_\alpha) \quad \longrightarrow \quad \pi_1(B, b_0) \quad = \quad \{1\}$$

is injective, and therefore each covering p_α is 1-sheeted. This means that each p_α is a homeomorphism.

It will be convenient to reformuate this as follows: Let $F = p^{-1}[\{b_0\}]$. Then the sheets of $p : E \to B$ can be written as a union of pairwise disjoint open and closed subsets E_x such that E_x is the unique arc component containing x and x ranges over the elements of $F = p^{-1}[\{b_0\}]$. Furthermore, the covering space projection determines homeomorphisms $p|E_x : E_x \to B$ for all x. It follows that the assembled mapping $h : E \to B \times F$ such that $h|E_x = p|E_x$ (for each $x \in F$) is a 1–1, onto, continuous and open mapping, and therefore h is a homeomorphism.■