Mathematics 205B, Winter 2019, Assignment 2

This will be due on **Friday, March 19, 2019, at 9:10 A.M,** which is the beginning of the second examination. If you wish to use some version of $\TeX$ in writing up your answers, please feel free to do so. The plain $\TeX$ file for this document is in the course directory.

You must show the work behind or reasons for your answers.

1. (a) Consider a solid pyramid in 3-space whose base is a square with vertices A, B, C, D and whose apex is the vertex E. Describe a simplicial decomposition of the pyramid whose vertices are these five points (and no others).

(b) Write down an explicit 3-dimensional chain for the simplicial complex in (a) such that the coefficients of all the 3-simplices are nonzero and the boundary is a linear combination of the lateral and base 2-simplices such that all the coefficients are nonzero.

2. (a) Suppose that we have pairs of spaces $(A, B) \subset (U, V)$ so that inclusion is a continuous map of pairs, and assume that U and V are open subsets of some larger space X. Prove that if A is a strong deformation retract of U and B is a strong deformation retract of V, then the pair inclusion induces singular homology isomorphisms $H_q(A, B) \to H_q(U, V)$ for all integers q. [*Warning:* It does not follow that (A, B) is a deformation retract of (U, V), so an algebraic argument is needed.]

(b) The following result leads to a connection between the excision properties (and Mayer-Vietoris sequences) in simplicial and singular homology: *Let $(A, B) \subset (U, V)$ as above where A and B are closed in X, U and V are open in X, and the spaces A, B, $A \cap B$ and $A \cup B$ are strong deformation retracts of U, V, $U \cap V$ and $U \cup V$ respectively, then the pair inclusion $(A, A \cap B) \to (A \cup B, B)$ induces singular homology isomorphisms in all dimensions.* — Prove this result using the axioms for singular homology.

3. (a) Let $\mathbb{R}^n_+ \subset \mathbb{R}^n$ be the set of points whose first coordinates are nonnegative. Prove that if $A = A_0 \times \{0\}$ where $A_0 \subset \mathbb{R}^{n-1}$, then $\mathbb{R}^n_+ - A$ is starshaped with respect to the standard unit vector $\mathbf{e}_n$ whose last coordinate is 1 and whose other coordinates are 0 (more precisely, if x lies in the set then so does the closed line segment joining x to $\mathbf{e}_n$). What does this imply about the homology of $\mathbb{R}^n_+ - A$, and why does this imply that $\mathbb{R}^n_+$ is not homeomorphic to $\mathbb{R}^n$? [*Hint:* Consider the local homology at a point in $\mathbb{R}^n$ and the local homology at $0 \in \mathbb{R}^n_+$.]

(b) Suppose that $A \subset S^n$ is homeomorphic to the figure Y tree. Prove that $H_q(S^n - A) = 0$ for all $q > 0$.