CHAPTER IV

SYNTHETIC PROJECTIVE GEOMETRY

The purpose of this chapter is to begin the study of projective spaces, mainly from the synthetic point of view but with considerable attention to coordinate projective geometry.

1. Axioms for projective geometry

The basic incidence properties of coordinate projective spaces are expressible as follows:

Definition. A geometrical incidence space (S, Π, d) is **projective** if the following hold:

(P-1) : Every line contains at least three points.

(P-2) : If P and Q are geometrical subspaces of S then
$$d(P \star Q) \ = \ d(P) \ + \ d(Q) \ - \ d(P \cap Q) \ .$$

In particular, **(P-2)** is a strong version of the regularity condition **(G-4)** introduced in Section II.5. The above properties were established for $\mathbb{FP}^n$ ($n \geq 2$) in Theorems III.10 and III.9 respectively. It is useful to assume condition **(P-1)** for several reasons; for example, lines in Euclidean geometry have *infinitely many* points, and **(P-1)** implies a high degree of regularity on the incidence structure that is not present in general (compare Exercise 2 below and Theorem IV.11). — In this connection, note that Example 2 in Section II.5 satisfies **(P-2)** and *every* line in this example contains *exactly* two points.

Elementary properties of projective spaces

The following is a simple consequence of the definitions.

THEOREM IV.1. *If S is a geometrical subspace of a geometrical incidence space S', then S is a geometrical incidence space with respect to the subspace incidence structure of Exercise II.5.3.*■

If P is a projective incidence space and $d(P) = n \geq 1$, then P is called a *projective n-space*; if $n = 2$ or 1, then one also says that P is a *projective plane* or *projective line*, respectively.

THEOREM IV.2. *If P is a projective plane and L and M are distinct lines in P, then $L \cap M$ consists of a single point.*■

THEOREM IV.3. *If S is a projective 3-space and P and Q are distinct planes in S, then $P \cap Q$ is a line.*■

These follow from **(P-2)** exactly as Examples 1 and 2 in Section III.4 follow from Theorem III.9.

We conclude this section with another simple but important result:

THEOREM IV.4. *In the definition of a projective space, property* **(P-1)** *is equivalent to the following (provided the space is not a line):*

    (P-1′) **:** *Every plane contains a subset of four points, no three of which are collinear.*

Proof.   Suppose that **(P-1)** holds. Let P be a plane, and let X, Y and Z be noncollinear points in P. Then the lines $L = XY$, $M = XZ$, and $N = YZ$ are distinct and contained in P. Let W be a third point of L, and let V be a third point of M.
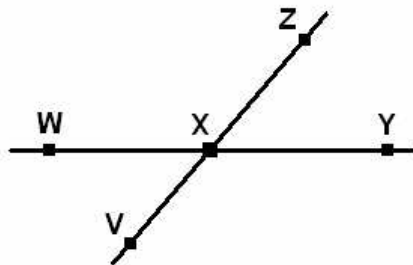

Figure IV.1

Since L and M are distinct and meet at X, it follows that the points V, W, Y, Z must be distinct (if any two are equal then we would have $L = M$; note that there are six cases to check, with one for each pair of letters taken from W, X, Y, Z). Similarly, if any three of these four points were collinear then we would have $L = M$, and therefore no three of the points can be collinear (there are four separate cases that must be checked; these are left to the reader).□

Conversely, suppose that **(P-1′)** holds. Let L be a line, and let P be a plane containing L. By our assumptions, there are four points A, B, C, $D \in P$ such that no three are collinear.
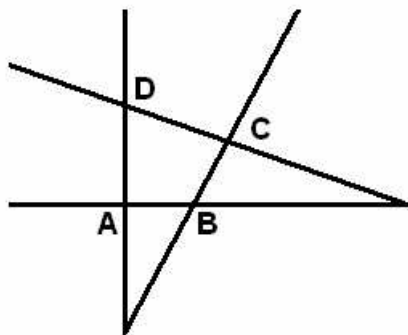

Figure IV.2

Let $M_1 = AB$, $M_2 = BC$, $M_3 = CD$, and $M_4 = AD$. Then the lines M_1 are distinct and coplanar, and no three of them are concurrent (for example, $M_1 \cap M_2 \neq M_3 \cap M_4$, and similarly for the others). It is immediate that M_1 contains at least three distinct points; namely, the

points A and B plus the point where M_1 meets M_3 (these three points are distinct because no three of the lines M_i are concurrent). Similarly, each of the lines M_2, M_3 and M_4 must contain at least three points.

If L is one of the four lines described above, then we are done. Suppose now that L is **not** equal to any of these lines, and let P_i be the point where L meets M_i. If at least three of the points P_1, P_2, P_3, P_4 are distinct, then we have our three distinct points on L. Since no three of the lines M_i are concurrent, it follows that no three of the points P_i can be equal, and therefore if there are not three distinct points among the P_i then there must be two distinct points, with each P_i equal to a unique P_j for $j \neq i$. Renaming the M_i if necessary by a suitable reordering of $\{1, 2, 3, 4\}$, we may assume that the equal pairs are given by $P_1 = P_3$ and $P_2 = P_4$. The drawing below illustrates how such a situation can actually arise.
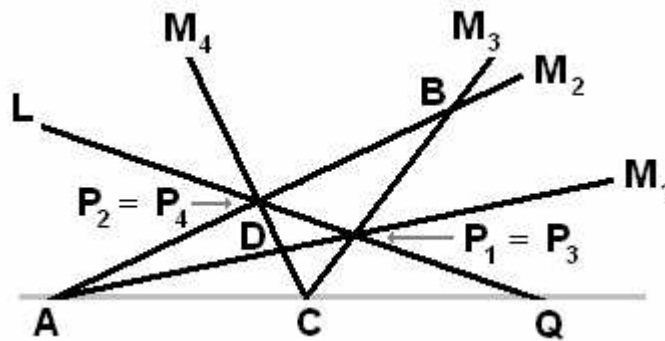

Figure IV.3

We know that $P_1 = P_3$ and $P_2 = P_4$ are two distinct points of L, and Figure IV.3 suggests that the point Q where AC meets L is a third point of L. To **prove** this, we claim it will suffice to verify the following statements motivated by Figure IV.3:

   (i) The point A does not lie on L.

   (ii) The line AC is distinct from M_1 and M_2.

Given these properties, it follows immediately that the three lines AC, M_1 and M_2 — which all pass through the point A which does not lie on L — must meet L in three distinct points (see Exercise 4 below).

Assertion (i) follows because $A \in L$ implies

$$A \in M_2 \cap L = M_4 \cap M_2 \cap L$$

and since $A \in M_1 \cap M_2$ this means that M_1, M_2, and M_4 are concurrent at A. However, we know this is false, so we must have $A \in L$. To prove assertion (ii), note that if $AC = M_1 = AB$, then A, B, C are collinear, and the same conclusion will hold if $AC = M_2 = BC$. Since the points A, B, C are noncollinear by construction, it follows that (ii) must also hold, and as noted above this completes the proof that L has at least three points.∎

EXERCISES

1. Let (S, Π, d) be an n-dimensional projective incidence space $(n \geq 2)$, let P be a plane in S, and let $X \in P$. Prove that there are at least three distinct lines in P which contain X.

2. Let $n \geq 3$ be an integer, let P be the set $\{0, \ 1, \ \cdots \ , n\}$, and take the family of subsets $\mathcal{L}$ whose elements are $\{1, \ \cdots \ , n\}$ and all subsets of the form $\{0, \ k\}$, where $k > 0$. Show that $(P, \mathcal{L})$ is a regular incidence plane which satisfies **(P-2)** but not **(P-1)**. [*Hint:* In this case **(P-2)** is equivalent to the conclusion of Theorem IV.2.]

3. This is a generalization of the previous exercise. Let S be a geometrical incidence space of dimension $n \geq 2$, and let ∞_S be an object not belonging to S (the axioms for set theory give us explicit choices, but the method of construction is unimportant). Define the **cone** on S to be $S^\bullet = S \cup \{\infty_S\}$, and define a subset Q of $S^\bullet$ to be a k-planes of $S^\bullet$ if and only if **either** Q is a k-plane of S **or** $Q = Q_0 \cup \{\infty_S\}$, where Q_0 is a $(k-1)$-plane in S (as usual, a 0-plane is a set consisting of exactly one element). Prove that $S^\bullet$ with these definitions of k-planes is a geometrical incidence $(n+1)$-space, and that $S^\bullet$ satisfies **(P-2)** if and only if S does. Explain why $S^\bullet$ does not satisfy **(P-1)** and hence is not projective.

4. Let $(P, \mathcal{L})$ be an incidence plane, let L be a line in P, let X be a point in P which does not lie on L, and assume that $M_1, \ \cdots \ M_k$ are lines which pass through X and meet L in points $Y_1, \ \cdots \ Y_k$ respectively. Prove that the points $Y_1, \ \cdots \ Y_k$ are distinct if and only if the lines $M_1, \ \cdots \ M_k$ are distinct.

5. Let (S, Π, d) be a regular incidence space of dimension ≥ 3, and assume that every plane in S is projective (so it follows that **(P-1)** holds). Prove that S is projective. [*Hint:* Since S is regular, condition **(P-2)** can only fail to be true for geometrical subspaces Q and R such that $Q \cap R = \varnothing$. If $d(R) = 0$, so that R consists of a single point, then condition **(P-2)** holds by Theorem II.30. Assume by induction that **(P-2)** holds whenever $d(R_0) \leq k - 1$, and suppose that $d(R) = k$. Let $R_0 \subset R$ be $(k-1)$-dimensional, and choose $\mathbf{y} \in R$ such that $\mathbf{y} \notin R_0$. Show that $Q \star R_0 \subset Q \star R$, and using this prove that $d(Q \star R)$ is equal to $d(Q) + k$ or $d(Q) + k + 1$. The latter is the conclusion we want, so assume it is false. Given $\mathbf{x} \in Q$, let $\mathbf{x} R$ denote the join of $\{\mathbf{x}\}$ and R, and define $\mathbf{y} Q$ similarly. Show that $\mathbf{x} R \cap Q$ is a line that we shall call L, and also show that $R \cap \mathbf{y} Q$ is a line that we shall call M. Since $L \subset Q$ and $M \subset R$, it follows that $L \cap M = \varnothing$. Finally, show that $\mathbf{x} R \ \cap \ \mathbf{y} Q$ is a plane, and this plane contains both L and M. Since we are assuming all planes in S are projective, it follows that $L \cap M$ is nonempty, contradicting our previous conclusion about this intersection. Why does this imply that $d(Q \star R) = d(Q) + k + 1$?]

2. Desargues' Theorem

In this section we shall prove a synthetic version of a fundamental result of plane geometry due to G. Desargues (1591–1661). The formulation and proof of Desargues' Theorem show that projective geometry provides an effective framework for proving nontrivial geometrical theorems.

THEOREM IV.5. (Desargues' Theorem) *Let P be a projective incidence space of dimension at least three, and let $\{A, B, C\}$ and $\{A', B', C'\}$ be triples of noncollinear points such that the lines AA', BB' and CC' are concurrent at some point X which does not belong to either of $\{A, B, C\}$ and $\{A', B', C'\}$. Then the points*

$$
\begin{aligned}
D &\in BC \cap B'C' \\
E &\in AC \cap A'C' \\
F &\in AB \cap A'B'
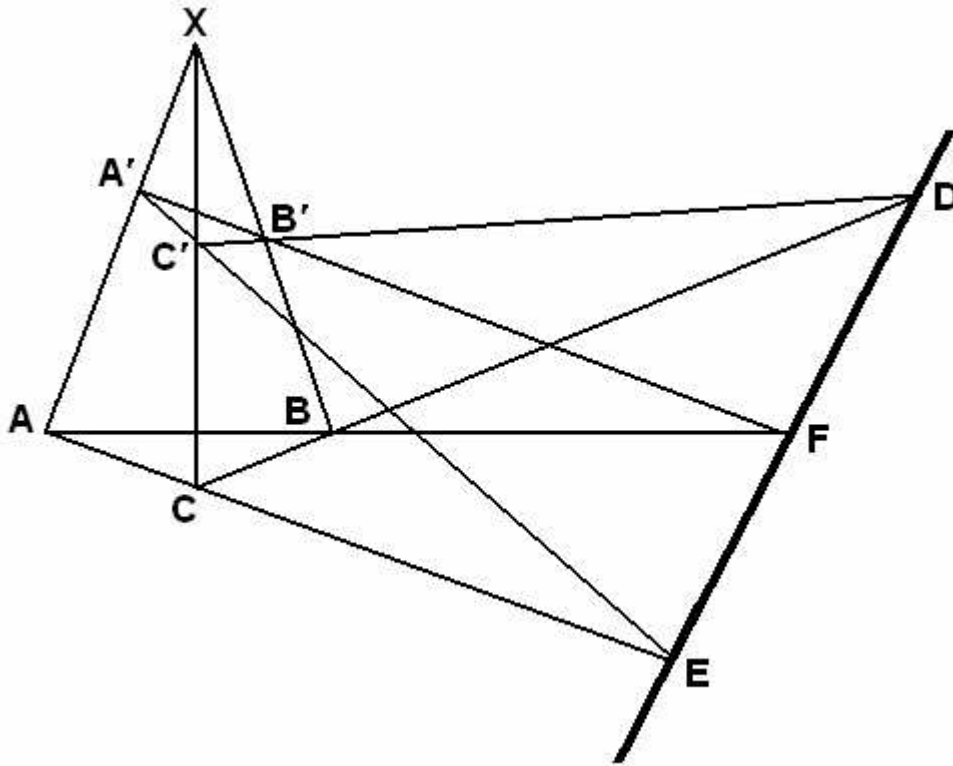\end{aligned}
$$

are collinear.

Figure IV.4

Proof. The proof splits into two cases, depending upon whether or not the sets $\{A, B, C\}$ and $\{A', B', C'\}$ are coplanar. One feature of the proof that may seem counter-intuitive is that the noncoplanar case is the easier one. In fact, we shall derive the coplanar case using the validity of the result in the noncoplanar case.

CASE 1. *Suppose that the planes determined by $\{A, B, C\}$ and $\{A', B', C'\}$ are distinct.* Then the point X which lies on AA', BB' and CC' cannot lie in either plane. On the other hand, it follows that the points $\{A', B', C'\}$ lie in the 3-space S determined by $\{X, A, B, C\}$, and therefore we also know that the planes ABC and $A'B'C'$ are contained in S. These two planes are distinct (otherwise the two triples of noncollinear points would be the same), and hence their intersection is a line. By definition, all three of the points D, E, F all lie in the intersection of the two planes, and therefore they all lie on the two planes' line of intersection.

CASE 2. *Suppose that the planes determined by $\{A, B, C\}$ and $\{A', B', C'\}$ are identical.* The idea is to realize the given configuration as the photographic projection of a similar noncoplanar configuration on the common plane. Since photographic projections onto planes preserve collinearity, this such a realization will imply that the original three points D, E, F are all collinear.

Under the hypothesis of Case 2, all the points under consideration lie on a single plane we shall call P. Let Y be a point not on P, and let $Z \in AY$ be another. Consider the line $A'Z$; since A' and Z both lie on the plane AXY, the whole line $A'Z$ lies in AXY. Thus $A'Z$ and XY meet in a point we shall call Q.
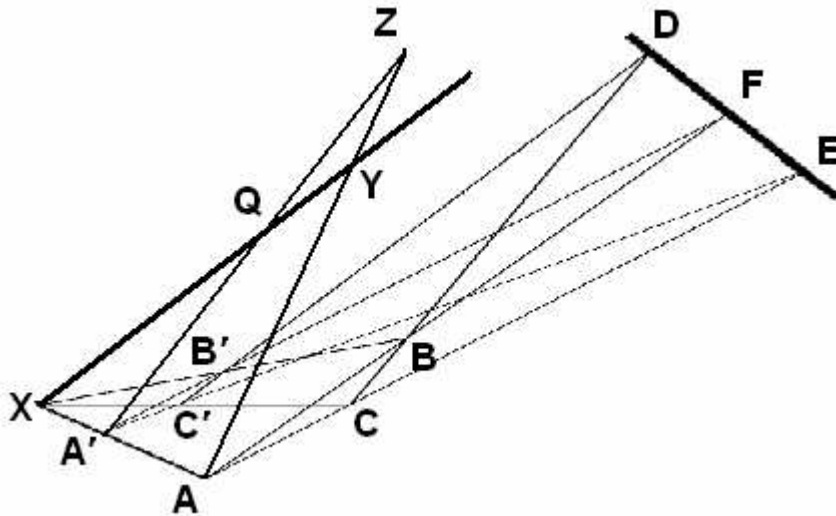

Figure IV.5

Consider the following three noncoplanar triangle pairs:

   (*i*) $C'QB'$ and CYB.

   (*ii*) $C'QA'$ and CYA.

   (*iii*) $B'QA'$ and BYA.

Since AA', BB' and CC' all meet at X, the nonplanar case of the theorem applies in all thre cases. Let $G \in BY \cap B'Q$ and $H \in CY \cap C'Q$, and note that $Z \in AR \cap A'Q$. Then the truth of the theorem in the noncoplanar case implies that each of the triples

$$\{D, H, G\}, \qquad \{F, Y, G\}, \qquad \{E, H, Z\}$$

is collinear.

Let P' be the plane DFZ. Then $G \in P'$ by the collinearity of the second triple, and hence $H \in P'$ by the collinearity of the first triple. Since $Z, H \in P'$, we have $E \in P'$ by the collinearity of the third triple. All of this implies that

$$E \ \in \ P' \ \cap \ P \ = \ DF$$

which shows that the set $\{D, E, F\}$ is collinear. ∎

Definition. A projective plane P is said to be *Desarguian* if Theorem 5 is always valid in P.

By Theorem 5, every projective plane that is isomorphic to a plane in a projective space of higher dimension is Desarguian. In particular, if $\mathbb{F}$ is a skew-field, then $\mathbb{F}\mathbb{P}^2$ is Desarguian because $\mathbb{F}\mathbb{P}^2$ is isomorphic to the plane in $\mathbb{F}\mathbb{P}^3$ consisting of all points having homogeneous coordinates in $\mathbb{F}^4$ of the form $(x_1, x_2, x_3, 0)$. In Section 4 we shall note that, conversely, every Desarguian plane is isomorphic to a plane in a projective 3-space.

An example of a non-Desarguian projective plane (the *Moulton plane*)[1] can be given by taking the real projective plane $\mathbb{R}\mathbb{P}^2$ as the underlying set of points, and modifying the definition of lines as follows: The new lines will include the line at infinity, all lines which have slope ≤ 0 or are parallel to the y-axis, and the broken lines defined by the equations

$$y \ = \ m(x - a), \qquad x \ \leq \ a \ (i.e., \ y \ \leq 0)$$
$$y \ = \ \tfrac{1}{2}m(x - a), \qquad x \ \geq \ a \ (i.e., \ y \ \geq 0)$$

where $m > 0$. As the points at infinite of the latter lines we take those belonging to $y = m(x-a)$. A straightforward argument shows that the axioms for a projective plane are satisfied (see Exercise 4 below). However, as Figure IV.6 suggests, Desargues' Theorem is false in this plane.
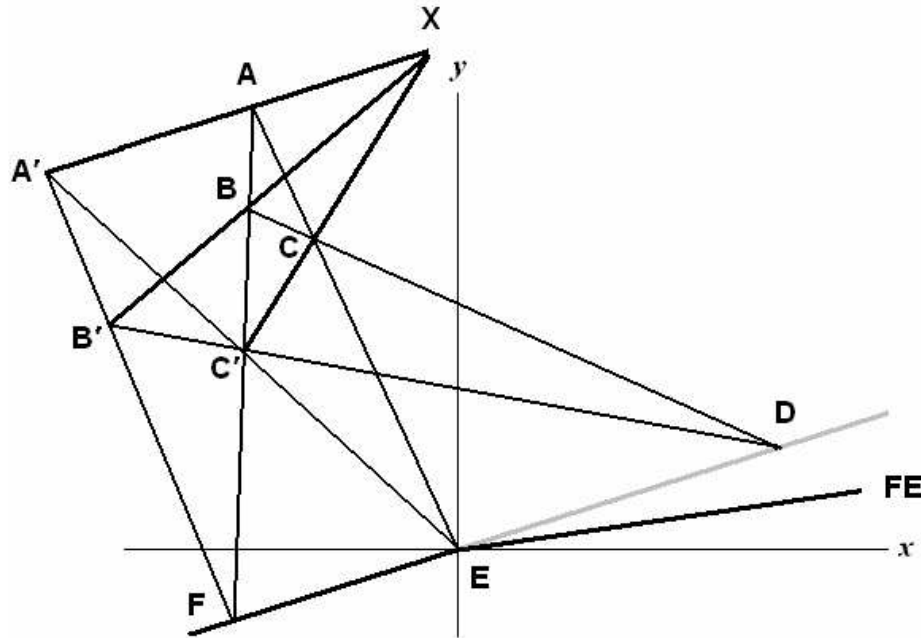

Figure IV.6

[1]FOREST RAY MOULTON (1872–1952) was an American scientist who worked mostly in astronomy but is also recognized for his contributions to mathematics.

Informally speaking, Desargues' Theorem fails to hold for some projective planes because there is not enough room in two-dimensional spaces to apply standard techniques.[2] Perhaps surprisingly, there are several other significant examples of geometrical problems in which higher dimensional cases (say $n \geq N$, where N depends upon the problem) are simpler to handle than lower dimensional ones (see the paper of Gorn for another example).

Finally, we note that Theorems II.27–29 are basically special cases of Desargues' Theorem.

EXERCISES

1. Explain why the results mentioned above are essentially special cases of Desargues' Theorem.

2. Is it possible to "plant ten trees in ten rows of three?" Explain your answer using Desargues' Theorem.[3]

3. Explain why each of the following pairs of triangles in Euclidean 3-space satisfies the hypotheses of Desargues' Theorem.

(i) Two coplanar triangles such that the lines joining the corresponding vertices are parallel.

(ii) A triangle and the triangle formed by joining the midpoints of its sides.

(iii) Two congruent triangles in distinct planes whose corresponding sides are parallel.

4. Prove that the Moulton plane (defined in the notes) is a projective plane.

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[2]For an example of a *finite* Non-Desarguian plane. see pages 158–159 of Hartshorne's book.

[3]For more information on the relevance of projective geometry to counting and arrangement problems, see Section IV.3.

3. Duality

> By the principle of duality ... geometry is at one stroke [nearly] doubled in extent with no
> expenditure of extra labor, — **Eric Temple Bell (1883–1960)**, *Men of Mathematics*[4]

Consider the following fundamental properties of projective planes:

(1) Given two distinct points, there is a unique line containing both of them.

(1*) Given two distinct lines, there is a unique point contained in both of them.

(2) Every line contains at least three distinct points.

(2*) Every point is contained in at least three distinct lines.

The important point to notice is that Statement (n^*) is obtained from Statement (n) by inter-
changing the following words and phrases:

(i)   *point* $\longleftrightarrow$ *line*

(ii)   *is contained in* $\longleftrightarrow$ *contains*

Furthermore, Statement (n) is obtained from Statement (n^*) by exactly the same process. Since
the four properties $(1) - (1^*)$ and $(2) - (2^*)$ completely characterize projective planes (see
Exercise 1 below), one would expect that points and lines in projective planes behave somewhat
symmetrically with respect to each other.

This can be made mathematically precise in the following manner: Given a projective plane
$(P, \mathcal{L})$, we define a **dual plane** $(P^*, \mathcal{L}^*)$ such that P^* is the set $\mathcal{L}$ of lines in P and $\mathcal{L}^*$ is in 1–1
correspondence with P. Specifically, for each $x \in P$ we define the *pencil of lines with vertex x*
to be the set

$$\mathfrak{p}(x) \ = \ \{ \, L \ \in \ \mathcal{L} \ = \ P^* \mid x \ \in \ L \, \} \, .$$

In other words, $L \in \mathfrak{p}(x)$ if and only if $X \in L$. Let P^{**} denote the set of all pencils associated
to the projective plane whose points are given by P and whose lines are given by P^*.

THEOREM IV.6. *If (P, P^*) satisfies properties $(1) - (1^*)$ and $(2) - (2^*)$, then (P^*, P^{**}) also does.*

Proof.   There are four things to check:

(a) Given two lines, there is a unique pencil containing both of them.

(a*) Given two pencils, there is a unique line contained in both of them.

(b) Every pencil contains at least three lines.

(b*) Every line is contained in at least three pencils.

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[4]See the bibliography for more information and comments on this book.

However, it is clear that (a) and (a^*) are rephrasings of (1^*) and (1) respectively, and likewise (b) and (b^*) are rephrasings of (2^*) and (2) respectively. Thus $(1) - (1^*)$ and $(2) - (2^*)$ for (P^*, P^{**}) are logically equivalent to $(1) - (1^*)$ and $(2) - (2^*)$ for (P, P^*).■

As indicated above, we call (P^*, P^{**}) the *dual projective plane* to (P, P^*).

By Theorem 6 we can similarly define P^{***} to be the set of all pencils in P^{**}, and it follows that (P^{**}, P^{***}) is also a projective plane. However, repetition of the pencil construction does not give us anything new because of the following result:

THEOREM IV.7. *Let (P, P^*) be a projective plane, and let $\mathfrak{p}_0 : P \to P^{**}$ be the map sending a point x to the pencil $\mathfrak{p}(x)$ of lines through x. Then $\mathfrak{p}_0$ defines an isomorphism of incidence planes from (P, P^*) to the double dual projective plane (P^{**}, P^{***}).*

Proof.   By construction the map $\mathfrak{p}_0$ is onto. It is also 1–1 because $\mathfrak{p}(x) = \mathfrak{p}(y)$ implies every line passing through x also passes through y. This is impossible unless $x = y$.

This it remains to show that L is a line in P if and only if $\mathfrak{p}(L)$ is a line in P^{**}. But lines in P^{**} have the form
$$\mathfrak{p}(L) = \{ w \in P^{**} \mid L \in \mathfrak{p}(x) \} \, .$$
where L is a line in P. Since $L \in \mathfrak{p}(x)$ if and only if $x \in L$, it follows that $\mathfrak{p}(x) \in \mathfrak{p}(L)$ if and only if $x \in L$. Hence $\mathfrak{p}(L)$ is the image of L under the map $\mathfrak{p}_0$, and thus the latter is an isomorphism of (projective) geometrical incidence planes.■

The preceding theorems yield the following important phenomenon[5] which was described in the first paragraph of this section; it was discovered independently by J.-V. Poncelet (1788–1867) and J. Gergonne (1771–1859).

METATHEOREM IV.8. (Principle of Duality) *A theorem about projective planes remains true if one interchanges the words* point *and* line *and also the phrases* contains *and* is contained in.■

The justification for the Duality Principle is simple. The statement obtained by making the indicated changes is equivalent to *a statement about duals of projective planes* which corresponds to the *original statement for projective planes*. Since duals of projective planes are also projective planes, the modified statement must also hold.

Definition.   Let **A** be a statement about projective planes. The *dual statement* is the one obtained by making the changes indicated in Metatheorem 8, and it is denoted by $\mathfrak{D}(\mathbf{A})$ or $\mathbf{A}^*$.

EXAMPLE 1.   The phrase *three points are collinear* (contained in a common single line) dualizes to *three lines are concurrent* (containing a common single point).

EXAMPLE 2.   The property $(\mathbf{P\text{-}1'})$, which assumes the existence of four points, not three of which are collinear, dualizes to the statement, *There exist four lines, no three of which are concurrent.* This statement was shown to follow from $(\mathbf{P\text{-}1'})$ in the course of proving Theorem 4.

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[5]This is called a **Metatheorem** because it is really a statement *about* mathematics rather than a theorem *within* mathematics itself. In other words it is a *theorem about theorems*.

By the metatheorem, a statement $\mathbf{A}^*$ is true for all projective planes if $\mathbf{A}^*$ is. On the other hand, Theorem 7 implies that $\mathbf{A}^{**}$ is logically equivalent to $\mathbf{A}$. This is helpful in restating the Principle of Duality in a somewhat more useful form:

Modified Principle of Duality. *Suppose that* $\mathbf{A}_1, \cdots, \mathbf{A}_n$ *are statements about projective planes. Then* $\mathbf{A}_n$ *is true in all projective planes satisfying* $\mathbf{A}_1, \cdots, \mathbf{A}_{n-1}$ *if and only if* $\mathbf{A}_n^*$ *is true in all projective planes satisfying* $\mathbf{A}_1^*, \cdots, \mathbf{A}_{n-1}^*$.∎

In practice, one often knows that the statements $\mathbf{A}_1, \cdots, \mathbf{A}_{n-1}$ imply their own duals. Under these circumstances, the Modified Principle of Duality shows that $\mathbf{A}_n^*$ is true if $\mathbf{A}_n$ is true, and conversely. The next two theorems give significant examples of such statements; in both cases we have $(n-1) = 1$.

THEOREM IV.9. *If a projective plane contains only finitely many points, then it also contains only finitely many lines.*

Proof. Suppose the plane contains n elements. Then there are 2^n subsets of the plane. But lines are subsets of the plane, and hence there are at most 2^n lines.∎

In Theorems 11-15 we shall prove very strong duality results for projective planes with only finitely many points. Recall that examples of such systems are given by the coordinate projective planes $\mathbb{Z}_p\mathbb{P}^2$, where p is a prime.

We shall now give a considerably less trivial example involving duality.

THEOREM IV.10. *If a projective plane (P, P^*) is Desarguian, then the dual of Desargues' Theorem is also true in (P, P^*).*

Proof. The first step in the proof is to describe the dual result in terms of (P, P^*).

The dualization of two abstract triples of noncollinear points is two distinct triples of non-concurrent lines, which we denote by $\{\alpha, \beta, \gamma\}$ and $\{\alpha', \beta', \gamma'\}$. Next, the concurrency hypothesis for AA', BB' and CC' dualizes to a hypothesis that $E \in \alpha \cap \alpha'$, $F \in \beta \cap \beta'$, and $D \in \gamma \cap \gamma'$ are collinear. Finally, the collinearity conclusion in Desargues' Theorem dualizes to a statement that three lines are concurrent. Specifically, if we set

$$
\begin{array}{ll}
A \in \beta \cap \gamma & A' \in \beta' \cap \gamma' \\
B \in \alpha \cap \gamma & B' \in \alpha' \cap \gamma' \\
C \in \alpha \cap \beta & C' \in \alpha' \cap \beta'
\end{array}
$$

then we wish to show that AA', BB' and CC' are concurrent. As the drawing below illustrates, the data for the dual theorem are similar to the data for the original theorem, the key difference being that Desargues' Theorem assumes concurrency of the lines AA', BB' and CC', using this to prove the collinearity of D, E and F, while the dual theorem assumes collinearity of the three points and aims to prove concurrency of the three lines.
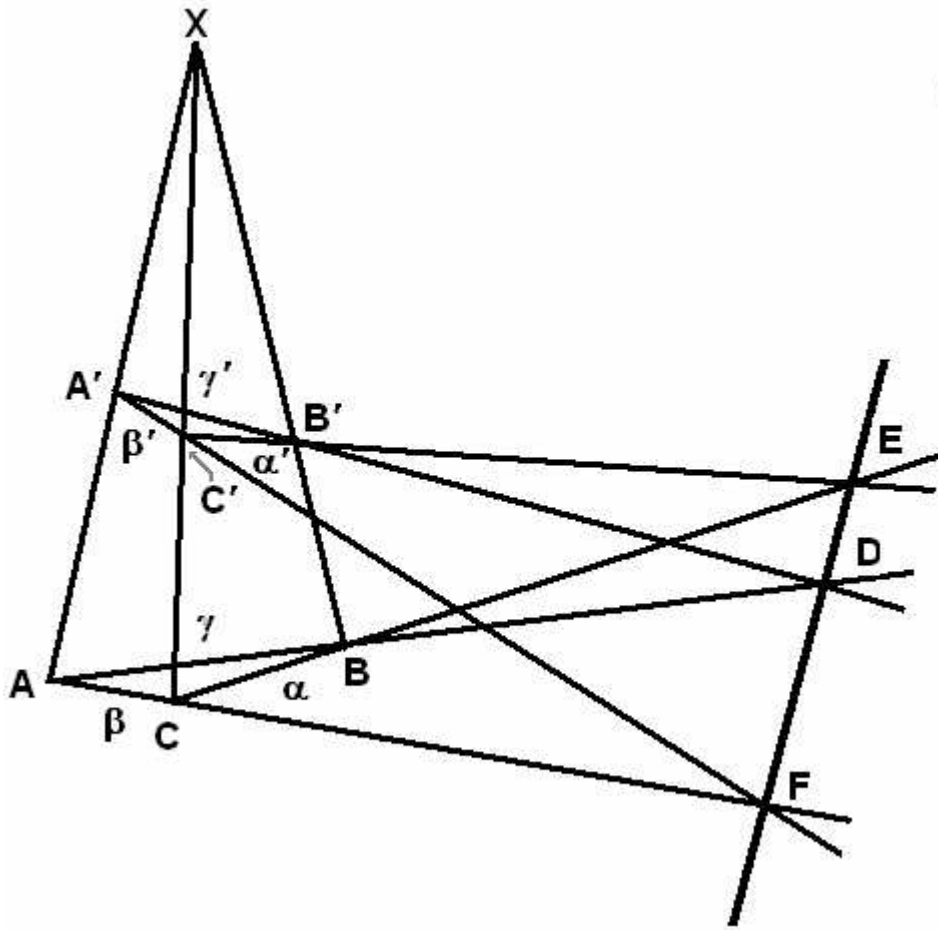
Figure IV.7

By construction, the lines $B'C'$, BC and DF all meet at E. Therefore Desargues' Theorem applies to the triples $\{C', C, F\}$ and $\{B', B, D\}$, and hence we may conclude that the three points

$$X \in BB' \cap CC'$$
$$G \in BD \cap CF$$
$$H \in B'D \cap C'F$$

are collinear. Using this, we obtain the following additional conclusions:

(1) $B, D \in \gamma$, and therefore $\gamma = BD$.

(2) $B', D \in \gamma'$, and therefore $\gamma' = B'D$.

(3) $C, F \in \beta$, and therefore $\beta = CF$.

(4) $C', F \in \beta'$, and therefore $\beta' = C'F$.

Therefore $G \in \gamma \cap \beta$ and $H \in \gamma' \cap \beta'$. Since the common points of these lines are A and A' by definition, we see that X, A and A' are collinear. In other words, we have $X \in AA' \cap BB' \cap CC'$, and hence the three lines are concurrent, which is what we wanted to prove.∎

Duality and finite projective planes

We shall illustrate the usefulness of duality by proving a few simple but far-reaching results on projective planes which contain only finitely many points. We have already noted that for each prime p there is a corresponding projective plane $\mathbb{Z}_p\mathbb{P}^2$. As noted below, these are more than abstract curiosities, and they play an important role in combinatorial theory (the study of counting principles) and its applications to experimental design and error-correcting codes.

By Theorem 9, it follows that a projective plane is finite if and only if its dual plane is finite. In fact, one can draw much stronger conclusions.

THEOREM IV.11. *Let P be a finite projective plane. Then all lines in P contain exactly the same number of points.*

Proof.   Let L and M be the lines. Since there exist four point, no three of which are collinear, there must exist a point $\mathbf{p}$ which belongs to neither L nor M.
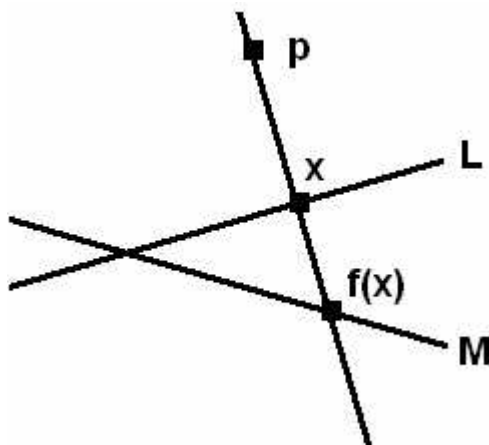

Figure IV.8

Define a map $f : L \to M$ by sending $\mathbf{x} \in L$ to the point $f(\mathbf{x})$ where $\mathbf{px}$ meets M. It is a straightforward exercise to verify that f is 1–1 and onto (see Exercise 6 below).∎

Dualizing the preceding, we obtain the following conclusion.

THEOREM IV.12. *Let P be a finite projective plane. Then all points in P are contained in exactly the same number of lines.*∎

Observe that the next result is self-dual, with the dual statement logically equivalent to the original one.

THEOREM IV.13. *Let P be a finite projective plane. Then the number of points on each line is equal to the number of lines containing each point.*

Proof. Let L be a line in P and let $\mathbf{x} \notin L$. Define a map from lines through $\mathbf{x}$ to points on L by sending a line M with $\mathbf{x} \in M$ to its unique intersection point with L. It is again a routine exercise to show this map is 1–1 and onto (again see Exercise 6 below).∎

Definition. The *order* of a finite projective plane is the positive integer $n \geq 2$ such that every line contains $n + 1$ points and every point lies on $n + 1$ lines. The reason for the subtracting one from the common number is as follows: If $\mathbb{F}$ is a finite field with q elements, then the order of $\mathbb{FP}^2$ will be equal to q.

The results above yield the following interesting and significant restriction on the number of points in a finite projective plane:

THEOREM IV.14. *Let P be a finite projective plane of order n. Then P contains exactly $n^2 + n + 1$ points.*

In particular, for most positive integers m it is not possible to construct a projective plane with exactly m points. More will be said about the possibilities for m below.

Proof. We know that $n + 1$ is the number or points on every line and the number of lines through every point. Let $\mathbf{x} \in P$. If we count all pairs $(\mathbf{y}, L)$ such that L is a line through $\mathbf{x}$ and $\mathbf{y} \in L$, then we see that there are exactly $(n + 1)^2$ of them. In counting the pairs, some points such as $\mathbf{x}$ may appear more than once. However, $\mathbf{x}$ is the only point which does so, for $\mathbf{y} \neq \mathbf{x}$ implies there is only one line containing both points. Furthermore, by the preceding result we know that $\mathbf{x}$ appears exactly $n + 1$ times. Therefore the correct number of points in P is given by subtracting n (not $n + 1$) from the number of ordered pairs, and it follows that P contains exactly

$$(n+1)^2 \ - \ n \ = \ n^2 \ + \ n \ + \ 1$$

distinct points.∎

The next theorem follows immediately by duality.

THEOREM IV.15. *Let P be a finite projective plane of order n. Then P contains exactly $n^2 + n + 1$ lines.*∎

Further remarks on finite projective planes

We shall now consider two issues raised in the preceding discussion:

(1) The possible orders of finite projective planes.

(2) The mathematical and nonmathematical uses of finite projective planes.

ORDERS OF FINITE PROJECTIVE PLANES. The theory of finite fields is completely understood and is presented in nearly every graduate level algebra textbook (for example, see Section V.5 of the book by Hungerford in the bibliography). For our purposes it will suffice to note that for each prime number p and each positive integer n, there is a field with exactly $q = p^n$ elements. It follows that *every prime power is the order of some projective plane*.

The possible existence of projective planes with other orders is an open question. However, many possible orders are excluded by the following result, which is known as the **Bruck–Ryser–Chowla Theorem:** *Suppose that the positive integer n has the form $4k + 1$ or $4k + 2$ for some positive integer k. Then n is the sum of two (integral) squares.*

References for this result include the books by Albert and Sandler, Hall (the book on group theory), and Ryser in the bibliography as well as the original paper by Bruck-Ryser (also in the bibliography) and the following online reference[6]:

$$\text{http://www.math.unh.edu/}\sim\text{dvf/532/7proj-plane.pdf}$$

Here is a list of the integers between 2 and 100 which cannot be orders of finite projective planes by the Bruck-Ryser-Chowla Theorem:

6	14	21	22	30	33	38	42
45	46	54	57	62	66	69	70
74	77	78	82	86	93	94	97

The smallest positive integer ≥ 2 which is not a prime power and not excluded by the Bruck-Ryser-Chowla Theorem is 10. In the late nineteen eighties a substantial argument — which used sophisticated methods together with involved massive amounts of computer calculations — showed that no projective planes of order 10 exist. Two papers on this work are listed in the bibliography; the article by one author (C. W. H. Lam) in the *American Mathematical Monthly* was written to explain the research on this problem to a reasonably broad general audience of mathematicians and students.

By the preceding discussion, the existence of projective planes of order n is understood for $n \leq 11$, and the first open case is $n = 12$. Very little is known about this case.

PROJECTIVE GEOMETRY AND FINITE CONFIGURATIONS. Of course, one can view existence problems about finite projective planes as extremely challenging puzzles similar to Magic Squares (including the *Sudoku* puzzles that have recently become extremely popular), but one important reason for studying them is their relevance to questions of independent interest.

One somewhat whimsical "application" (*plant ten trees in ten rows of three*) was mentioned in Exercise IV.2.2, where the point was that such configurations exist by Desargues' Theorem. In fact, the study of projective spaces — especially finite ones – turns out to have many useful consequences in the study of finite *tactical configurations* or *block designs*, which is part of *combinatorial theory* or *combinatorics*. Specifically, finite projective planes are of interest as examples of *Latin squares*, or square matrices whose entries are in a finite set such that each element appears in every row and every column exactly once. As noted earlier, such objects play a significant role in the areas of statistics involving the design of experiments and in the theory of error-correcting codes. Although further comments are well outside the scope of these notes, the references in the bibliography by Buekenhout, Crapo and Rota, Hall (the book on combinatorial theory), Kárteszi, Lindner and Rodgers, and (the previously cited book by) Ryser are all sources for further information. There are also several online web sites dedicated to questions about finite geometry.

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[6]To update this document, the conjecture of E. Catalan (1814–1894) has recently been shown to be true bu P. Mihăilescu. The proof is at a very advanced level, but for the sake of completeness here is a reference: P. Mihăilescu, *Primary cyclotomic units and a proof of Catalan's Conjecture*, [Crelle] Journal für die reine und angewandte Mathematik **572** (2004), 167–195.

Duality in higher dimensions

The concept of duality extends to all projective n-spaces (where $n \geq 2$), but the duality is more complicated for $n \geq 3$ than it is in the 2-dimensional case. Specifically, if S is a projective n-space, then points of the dual space S^* are given by the *hyperplanes* of S. for each k-plane P in S, there is an associated *(linear) bundle of hyperplanes with center* P, which we shall denote by $\mathfrak{b}(P)$, and the dimension of $\mathfrak{b}(P)$ is set equal to $n - k - 1$. We shall also denote the set of all such bundles by Π^* the associated dimension function by d^*. The following result is the appropriate generalization of Theorems 6 and 7 which allows extension of the principle of duality to higher dimensions.

THEOREM IV.16. *If (S, Π, d) is a projective n-space for some $n \geq 2$, then so is the dual object (S^*, Π^*, d^*). Furthermore, the map $\mathcal{E}$ sending $\mathbf{x} \in S$ to the bundle of hyperplanes $\mathfrak{b}(\mathbf{x})$ with center x defines an isomorphism of geometrical incidence spaces.*

We shall not give a direct proof of this result for two reasons.

  1. Although the proof is totally elementary, it is a rather long and boring sequence of routine verifications.

  2. The result follows from a coordinatization theorem in the next section (Theorem 18) and the results of Section VI.1, at least if $n \geq 3$ (and we have already done the case $n = 2$).

An excellent direct proof of Theorem 16 is given in Sections 4.3 and 4.4 of Murtha and Willard, *Linear Algebra and Geometry* (see the bibliography for further information).

EXERCISES

1. Prove that properties $(1) - (1^*)$ and $(2) - (2^*)$ completely characterize projective planes. In other words, if $(P, \mathcal{L})$ is a pair consisting of a set P and a nonempty collection of proper subsets $\mathcal{L}$ satisfying these, then there is a geometrical incidence space structure $(P, \mathcal{L}, d)$ such that the incidence space is a projective plane and $\mathcal{L}$ is the family of lines in Π.

2. Write out the plane duals of the following phrases, and sketch both the given data and their plane duals.

  (i) Two lines, and a point on neither line.

  (ii) Three collinear points and a fourth point not on the line of the other three.

  (iii) Two triples of collinear points not on the same line.

  (iv) Three nonconcurrent lines, and three points such that each point lies on exactly one of the lines.

3. Draw the plane duals of the illustrated finite sets of points which are marked heavily in the two drawings below.
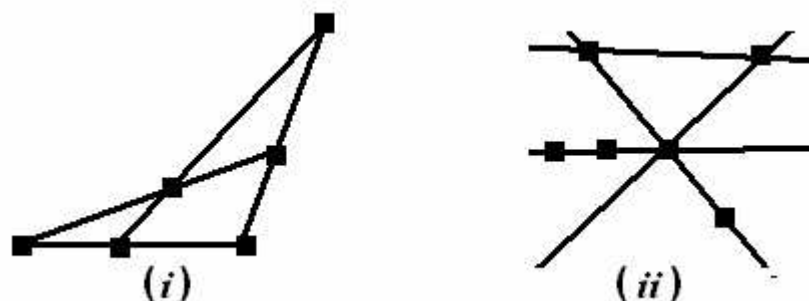

Figure IV.9

4. Suppose that $f : S \to T$ is an incidence space isomorphism from one projective n-space ($n \geq 2$) to another. Prove that f induces a 1–1 correspondence $f^* : S^* \to T^*$ taking a hyperplane $H \subset S$ to the image hyperplane $f^*(H) = f[H] \subset T$. Prove that this correspondence has the property that B is an r-dimensional bundle of hyperplanes (in S^*) if and only if $f^*[B]$ is such a subset of T^* (hence it is an isomorphism of geometrical incidence spaces, assuming Theorem 16).

5. Prove that the construction in the preceding exercise sending f to f^* has the following properties:

(i) If $g : T \to U$ is another isomorphism of projective n-spaces, then $(g \circ f)^* = g^* \circ f^*$.

(ii) For all choices of S the map $(\mathrm{id}_S)^*$ is equal to the identity on S^*.

(iii) For all f we have $\left(f^{-1} \right)^* = \left(f^* \right)^{-1}$.

6. Complete the proof of Theorem 11.

7. Let $(P, \mathcal{L})$ be a finite affine plane. Prove that there is a positive integer n such that

(i) every line in P contains exactly n points,

(ii) every point in P lies on exactly n lines,

(iii) the plane P contains exactly n^2 points.

4. Conditions for coordinatization

Since the results and techniques of linear algebra are applicable to coordinate projective n-spaces $\mathcal{S}_1(V)$ (where $\dim V = n + 1$), these are the most conveniently studied of all projective spaces. Thus it is desirable to know when a projective n-space is isomorphic to one having the form $\mathcal{S}_1(V)$, where $\dim V = n + 1$. The following remarkable theorem shows that relatively weak hypotheses suffice for the existence of such an isomorphism.

THEOREM IV.17. *Let P be a projective n-space in which Desargues' Theorem is valid (for example, $n \geq 3$ or P is a Desarguian plane). Then there is a skew-field $\mathbb{F}$ such that P is isomorphic to $\mathbb{F}\mathbb{P}^n$ (where we view $\mathbb{F}^{n+1}$ as a right vector space over $\mathbb{F}$. If $\mathbb{E}$ is another skew-field such that P is isomorphic to $\mathbb{E}\mathbb{P}^n$, then $\mathbb{E}$ and $\mathbb{F}$ are isomorphic as skwe-fields.*

A well-illustrated proof of Theorem 17 from first principles in the case $n = 2$ appears on pages 175–193 of the book by Fishback listed in the bibliography. Other versions of the proof appear in several other references from the bibliography. Very abstract approaches to the theorem when $n = 2$ appear in Chapter III of the book by Bumcrot and also in the book by Artzy. The proof in Chapter 6 of Hartshorne's book combines some of the best features of the other proofs. There is also a proof of Theorem 17 for arbitrary values of $n \geq 2$ in Chapter VI from Volume I of Hodge and Pedoe. Yet another reference is Sections 4.6 and 4.7 of Murtha and Willard. For more information on the uniqueness statement, see Theorem V.10.

Theorem 17 yields a classification for projective n-spaces ($n \geq 3$) that is parallel to Theorem II.38 (see Remark 3 below for further discussion). Because of its importance, we state this classification separately.

THEOREM IV.18. *Let P be a projective n-space, where $n \geq 3$. Then there is a skew-field $\mathbb{F}$, unique up to algebraic isomorphism, such that P and $\mathbb{F}\mathbb{P}^n$ are isomorphic as geometrical incidence spaces.* ∎

Theorem 17 also implies the converse to a remark following the definition of a Desarguian projective plane in Section IV.2.

THEOREM IV.19. *If a projective plane is Desarguian, then it is isomorphic to a plane in a projective 3-space.*

Proof.   By Theorem 17, the plane is isomorphic to $\mathbb{F}\mathbb{P}^2$ for some skew-field $\mathbb{F}$. But $\mathbb{F}\mathbb{P}^2$ is isomorphic to the plane in $\mathbb{F}\mathbb{P}^3$ defined by $x_4 = 0$. ∎

REMARK 1.   Suppose that P is the Desarguian plane $\mathbb{F}\mathbb{P}^2$. By Theorem 10, P^* is also Desarguian and hence is isomorphic to some plane $\mathbb{E}\mathbb{P}^2$. As one might expect, the skew-fields $\mathbb{F}$ and $\mathbb{E}$ are closely related. In fact, by Theorem V.1 the dual plane P^* is isomorphic to $\mathcal{S}_1(\mathbb{F}_3)$, where $\mathbb{F}_3$ is a *left* vector space over $\mathbb{F}$, and left vector spaces over $\mathbb{F}$ correespond to right vector spaces over the **opposite skew-field** $\mathbb{F}^{\mathsf{OP}}$, whose elements are the same as $\mathbb{F}$ and whose multiplication is given by reversing the multiplication in $\mathbb{F}$. More precisely, one defines a new product $\otimes$ in $\mathbb{F}$ via $a \otimes b = b \cdot a$, and if V is a left vector space over $\mathbb{F}$ define vector space operations via the vector addition on $\mathbb{F}$ and the right scalar product $\mathbf{x} \otimes a = a \cdot \mathbf{x}$ (here the dot represents the

original multiplication). It is a routine exercise to check that $\mathbb{F}^{\mathsf{OP}}$ is a skew-field and $\otimes$ makes left $\mathbb{F}$-vector spaces into right vector spaces over $\mathbb{F}^{\mathsf{OP}}$. Thus, since $\mathbb{F}_3$ is a 3-dimensional right $\mathbb{F}^{\mathsf{OP}}$-vector space, it follows that $\mathbb{E}$ must be isomorphic to $\mathbb{F}^{\mathsf{OP}}$.

REMARK 2. If P is the coordinate projective plane $\mathbb{F}\mathbb{P}^2$ and multiplication in $\mathbb{F}$ is commutative, then the preceding remark implies that P and P^* are isomorphic because $\mathbb{F}$ and $\mathbb{F}^{\mathsf{OP}}$ are identical in such cases. However, the reader should be warned that the isomorphisms from P to P^* are much less "natural" than the standard isomorphisms from P to P^{**} (this is illustrated by Exercise V.1.5). More information on the noncommutative case appears in Appendix C.

EXAMPLE 3. Theorem II.18 follows from Theorem 18. For if S is an affine n-space, let $\overline{S}$ be its synthetic projective extension as defined in the Addendum to Section III.4. By Theorem III.16, we know that $\overline{S}$ is an n-dimensional projective incidence space, so that $\overline{S}$ is isomorphic to $\mathbb{F}\mathbb{P}^n$ for some skew-field $\mathbb{F}$ and $S \subset \overline{S}$ is the complement of some hyperplane. Since there is an element of the geometric symmetry group of $\mathbb{F}\mathbb{P}^n$ taking this hyperplane to the one defined by $x_{n+1} = 0$ (see Exercise III.4.14), we can assume that S corresponds to the image of $\mathbb{F}^n$ in $\mathbb{F}\mathbb{P}^n$. Since k-planes in S are given by intersections of k-planes in $\overline{S}$ with S and similarly the k-planes in $\mathbb{F}^n$ are given by intersections of k-planes in $\mathbb{F}\mathbb{P}^n$ with the image of $\mathbb{F}$, it follows that the induced 1–1 correspondence between S and $\mathbb{F}^n$ is a geometrical incidence space isomorphism.

A general coordinatization theory for projective planes that are not necessarily Desarguian exists; the corresponding algebraic systems are generalizations of skew-fields known as *planar ternary rings*. Among the more accessible references for this material are the previously cited book by Albert and Sandler, Chapter 4 of the book by Artzy, Chapters III and VI of the book by Bumcrot, and Chapter 17 of the book by Hall on group theory. One of the most important examples of a planar ternary ring (the **Cayley numbers**) is described on pages 195–196 of Artzy[7] As one might expect, the Cayley numbers (also called *octonions*) are associated to a projective plane called the *Cayley projective plane*.

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[7]One brief and accurate online summary is given in `http://en.wikipedia.org/wiki/Octonion`. The reliability issue for Wikipedia articles is discussed in a footnote for Appendix C, and it also applies here.