ADDITIONAL MATERIAL ON HYPERQUADRICS

This Appendix discusses two issues that were mentioned in Section VII.2. The first relates the notion of tangent space in Section VII.2 to the standard concepts of tangent lines and planes that one encounters in single variable and multivariable calculus. In particular, we show that nonsingularity in the sense of the notes is equivalent to the usual nonsingularity conditions in calculus which are given by the nonvanishing of certain Jacobian determinants, and the tangent hyperplanes defined in these notes coincide with the notions of tangent hyperplanes that one sees in calculus courses, provided the points in question are nonsingular (there is some difference between the notion of tangent space in these notes and standard notions of tangents at singular points, but a discussion of such matters is beyond the scope of these notes). One reference for background material in multivariable calculus is the following standard textbook:

> **J. E. Marsden and A. J. Tromba.** Vector Calculus. (5$^{\text{th}}$ Ed.). *W. H. Freeman & Co., New York NY*, 2003. ISBN: 0–7147–4992–0.

The second topic in this Appendix concerns the determination of which matrices define the same hyperquadric. By Theorem VII.6, in many cases (including all nonsingular hyperquadrics) two symmetric matrices define the same projective hyperquadric if and only if one is a nonzero scalar multiple of the other. In Section E.2 we extend this theorem to some other cases and indicate how it fails in others.

DEFAULT HYPOTHESIS. As in Chapter VII, unless stated otherwise we assume that $\mathbb{F}$ is a (commutative) field such that $1 + 1 \neq 0$ in $\mathbb{F}$.

1. Tangent hyperplanes and differential calculus

We begin by discussing nonsingularity for affine hyperquadrics from the viewpoint of multivariable calculus. More generally, if we are given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ (*i.e.*, continuous partial derivatives), and $V \subset \mathbb{R}^n$ is the set of solutions to the equation $f(\mathbf{x}) = 0$, we want to formulate a mathematical condition which means that $\mathbf{x}$ is not a singular (or exceptional or special) point of V. For example, we expect this to mean that V has no corners or branches at $\mathbf{x}$.

EXAMPLE 1. Let $f(u, v) = v^3 - u^2$, and let $\mathbf{x} = (0, 0)$. Then $\mathbf{x}$ has a 180° corner — or *cusp* — at $\mathbf{x}$
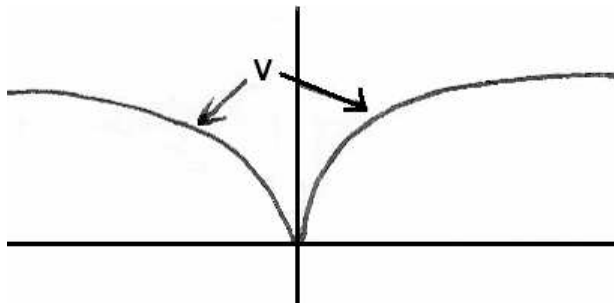

Figure E.1

EXAMPLE 2.   Let $f(u, v) = v^2 - u^2$, and let $\mathbf{x} = (0, 0)$. Then f has two branches (*i.e.*, a **node**) at $\mathbf{x}$
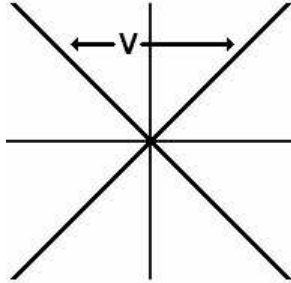

Figure E.2

In both examples, the curve is regular — or *nonsingular* — everywhere else.

In each of the preceding examples, the singular points are precisely the points $\mathbf{x} \in V$ for which the gradient $\nabla f(\mathbf{x})$ is equal to $\mathbf{0}$, and in fact the vanishing of the gradient turns out to be the condition for singularity. The reason of this comes from the Implicit Function Theorem of multivariable calculus (see Section 3.5 of the book by Marsden and Tromba mentioned at the beginning of this Appendix). By this theorem, if $\mathbf{x} \in V$ and $\nabla f(\mathbf{x}) \neq \mathbf{0}$, then for points close to $\mathbf{x}$ one of the coordinates u or v can be expressed as a smooth function of the other (in other words, one can solve for one of the coordinates in terms of the other). Thus if we restrict to points that are sufficiently close to $\mathbf{x}$ the curve V looks locally like the graph of a smooth function and hence is extremely regular. In the figure below we have

$$\frac{\partial F}{\partial v}(\mathbf{x}) \neq 0$$

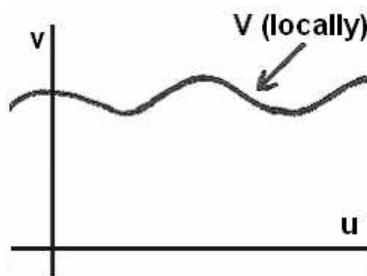and near $\mathbf{x}$ we can solve for v in terms of u.

Figure E.3

It is not always possible to solve *globally* for one coordinate in terms of the other. The simplest example is the circle Γ defined by the equation $f(u, v) = u^2 + v^2 - 1 = 0$. It is nonsingular at every point because $\mathbf{x} \in \Gamma$ imples that $\mathbf{x} \neq \mathbf{0}$ and $\nabla f(u, v) = (2u, 2v)$. However, since many vertical and horizontal lines meet the circle in two points, there is no way that we can view the circle globally as the graph of a reasonable (single valued) function.

Suppose now that $V \subset \mathbb{R}^n$ is the set of all $\mathbf{x}$ such that $f(\mathbf{x}) = 0$, where f is a function with continuous first partial derivatives. We shall say that $\mathbf{x}$ is an *analytically singular* point of V if $\nabla f(\mathbf{x}) = \mathbf{0}$ and $\mathbf{x}$ is an *analytically nonsingular* point otherwise.

Throughout this discussion, if $\mathbf{x}$ is a vector in $\mathbb{R}^n$ then we shall denote its coordinates by $x_1, \cdots, x_n$.

Theorem E.1. *Let Σ be the hyperquadric in $\mathbb{R}^n$ defined by $f(\mathbf{x}) = 0$, where*

$$f(\mathbf{x}) = \sum_{i,j} a_{i,j} x_i x_j + 2 \cdot \sum_k b_k x_k + c .$$

Then Σ is analytically singular at $\mathbf{x}$ if and only if

$$\begin{pmatrix} A & {}^{\mathbf{T}}\mathbf{b} \\ \mathbf{b} & c \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \mathbf{0}$$

where A is the matrix of second degree coefficients of f, $\mathbf{b}$ is the row vector of first degree coefficients, and c is the constant term.

Proof. Direct computation shows that the ith coordinate of $\nabla f(\mathbf{x})$ is equal to

$$\sum_j 2a_{i,j} x_j + 2b_i$$

so that $\mathbf{x}$ is an analytically singular point of Σ if and only if each of these expressions is equal to zero.

If the condition on matrix products

$$\begin{pmatrix} A & {}^{\mathbf{T}}\mathbf{b} \\ \mathbf{b} & c \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \mathbf{0}$$

is true, then by construction and the formula of the previous paragraph we know that $\nabla f(\mathbf{x})$ is twice the vector whose coordinates are the first n entries of the right hand side, and therefore we have $\nabla f(\mathbf{x}) = \mathbf{0}$, proving the "if" implication in the theorem.

Conversely, suppose that $\mathbf{x}$ is an analytically singular point. Then the same reasoning as before shows that the first n entries of

$$\begin{pmatrix} A & {}^{\mathbf{T}}\mathbf{b} \\ \mathbf{b} & c \end{pmatrix}$$

are zero, so the proof of this implication reduces to showing that the last entry equals zero, which translates to $\sum_i b_i x_i + c = 0$. We already know that

$$\sum_j a_{i,j} x_j + b_i = 0$$

for each i, and if we multiply each such equation by the associated coordinate x_i and sum over i we obtain

$$\sum_{i,j} a_{i,j} x_i x_j + b_i x_i = 0$$

and if we subtact this from the given equation $f(\mathbf{x}) = 0$ we obtain the desired equation $\sum_i b_i x_i + c = 0.\blacksquare$

Theorem E.2. *The affine hyperquadric in the previous theorem is totally analytically nonsingular if and only if the last column of the matrix*

$$\begin{pmatrix} A & {}^{\mathrm{T}}\mathbf{b} \\ \mathbf{b} & c \end{pmatrix}$$

is not expressible as a linear combination of the preceding ones.

Proof. Suppose first that the hyperquadric has a singular point. Let $\mathbf{x}$ be the singular point, and let $\mathbf{k}_i$ be the ith column of the matrix in the theorem. Then we have

$$\mathbf{k}_{n+1} \;=\; -\sum_i x_i \mathbf{k}_i \;.$$

Conversely, suppose that the last column is a linear combination of the others, so that (say) we have $\mathbf{k}_{n+1} \;=\; \sum_i y_i \mathbf{k}_i$. Then

$$\begin{pmatrix} A & {}^{\mathrm{T}}\mathbf{b} \\ \mathbf{b} & c \end{pmatrix} \cdot \begin{pmatrix} -\mathbf{y} \\ 1 \end{pmatrix} \;=\; \mathbf{0}$$

Explicitly, these imply that

$$
\begin{array}{ll}
(\mathbf{A}_i) & \sum a_{i,j}(-y_j) \;+\; b_i \;=\; 0, \\
(\mathbf{B}) & \sum b_j(-y_j) \;+\; c \;=\; 0 \;.
\end{array}
$$

In particular we have $0 \;=\; \sum (\mathbf{A}_i)(-y_i) \;+\; (\mathbf{B})$, which is just the equation

$$0 \;=\; \sum_{i,j} a_{i,j}(-y_i)(-y_j) \;+\; 2 \cdot \sum_k b_k(-y_k) \;+\; c \;.$$

Therefore $-\mathbf{y} \in \Sigma$. Since the ith coordinate of $\nabla f(-\mathbf{y})$ is the left hand side of $2(\mathbf{A}_i)$, it follows also that $\nabla f(-\mathbf{y}) \;=\; \mathbf{0}$, and therefore $-\mathbf{y}$ is a singular point of Σ.■

Using the concepts of Section VII.2, we may reinterpret the preceding results as follows:

> Let Σ be an affine hyperquadric in $\mathbb{R}^n$, and let $\mathbb{P}(\Sigma)$ denote its projective extension. Then the set of singular points of $\Sigma^* \cap \mathrm{J}(\mathbb{R}^n)$ in the sense of Section VII.2 is equal to the set of analytically singular points of Σ as defined above.

EXAMPLE. There are nonsingular affine quadrics Σ such that Σ has no singular points but the projective extension Σ^* has singular points. In particular, the cylinder $\Sigma \subset \mathbb{R}^3$ defined by the equation $x^2 + y^2 \;=\; 1$ has no singular points, but its projective extension has a unique singular point; namely, the ideal point on the z-axis. For this example, the singular point is the intersection of Σ^* with the ideal plane.

Tangent spaces and tangents to curves

We begin by recalling the analytic definition of tangent line to a hyperquadric Σ in $\mathbb{R}^n$. Namely, it was given by $\mathbf{x} + \mathbb{R} \cdot \gamma'(t_0)$, where γ is a smooth curve lying totally in Σ such that $\gamma(t_0) \;=\; \mathbf{x}$.

Theorem E.3. *Let $\Sigma \subset \mathbb{R}^n$ be a nonsingular hyperquadric, let $\mathbf{x} \in \Sigma$, and let Σ be a line through $\mathbf{x}$. Then the following are equivalent:*

(i) The line L is tangent to Σ in the analytic sense.

(ii) The line L lines on the affine hyperplane of $\mathbb{R}^n$ defined by the equation

$$\sum_{i,j} a_{i,j} x_i u_j \; + \; \sum_k b_k(v_k + x_k) \; + \; c \; = \; 0 \; .$$

The second property tells us that the set of points on tangent lines in the analytic sense is a hyperplane whose projective extension has homogeneous coordinates $\sum a_{i,j} x_j \; + \; b_j$ and $\sum b_k x_k \; + \; c$; i.e., its homogeneous coordinates are given by the following vector:

$$\begin{pmatrix} ^{\mathbf{T}}\mathbf{x} & 1 \end{pmatrix} \cdot \begin{pmatrix} A & ^{\mathbf{T}}\mathbf{b} \\ \mathbf{b} & c \end{pmatrix}$$

Note that these are just the homogeneous coordinates for the tangent hyperplane to Σ^* at $J(\mathbf{x})$ as defined in Section VII.2.

Proof of the theorem. The fist condition implies the second. For each i let $x_i(t)$ denote the ith coordinate of $\gamma(t)$, where γ satisfies the conditions in (i). Then by our assumptions we have

$$\sum_{i,j} a_{i,j} x_i(t) x_j(t)) \; + \; 2 \sum_k b_k x_k(t) \; + \; c \; = \; 0 \; .$$

Differentiation with respect to t implies that

$$2 \sum_{i,j} x_i(t) x_j'(t) \; + \; 2 \sum_k x_k'(t) \; = \; 0 \; .$$

Let $\gamma'(t_0) \; = \; (u_1, \; \cdots \;, u_n)$ and evaluate the expression above at $t = t_0$. This implies the equation

$$2 \sum a_{i,j} x_i u_j \; + \; \sum_k b_k u_k \; = \; 0 \; .$$

To show that $L \; = \; \mathbf{x} + \mathbb{R} \cdot \mathbf{u}$ lies in the set defined by the equation in (ii), it suffices to show that $\mathbf{v} = \mathbf{x} + \mathbf{u}$ lies in this subset. If we divide the equation above by 2 and add

$$f(\mathbf{x}) \; = \; \sum_{i,j} a_{i,j} x_i x_j \; + \; 2 \cdot \sum_k b_k x_k \; + \; c$$

to it, the resulting equation is the one displayed in (ii). To see that this equation is a hyperplane, observe that $\nabla f(\mathbf{x}) \; = \; \mathbf{0}$ implies that $\sum a_{i,j} x_j \; + \; b_i \neq 0$ for some i.$\square$

Proof that the second condition implies the first. This is a consequence of the following corollary to the Implicit Function Theorem:

> *If f is a smooth function of n variables with $f(\mathbf{x}) = 0$ but $\nabla f(\mathbf{x}) \neq \mathbf{0}$, then there is a smooth curve γ such that $\gamma(t)$ lies in the zero seto of f for all t, and we also have $\gamma(t_0) = \mathbf{x}$, $\gamma'(t_0) = \mathbf{v}$.*

(See Marsden and Tromba, pp. 248–250, for more about this.)

Since $\nabla f(\mathbf{x}) \neq \mathbf{0}$ if Σ is nonsingular, the statement above is applicable.∎

2. Matrices defining the same singular hyperquadric

By Theorem VII.6 we know that if Σ is a nonempty nonsingular hyperquadric in $\mathbb{FP}^n$ which is defined by the nonzero symmetric matrices A and B, then A and B are nonzero scalar multiples of each other. In fact, we know this is also the case if Σ has at least one nonsingular point. It is natural to ask whether the existence of such a point is needed to prove such a result on symmetric matrices defining the same quadric. We shall show that the answer depends upon the algebraic properties of the field $\mathbb{F}$. In particular, one can drop the assumption about a nonsingular point if $\mathbb{F}$ is the complex numbers, but one cannot drop the assumption if $\mathbb{F}$ is the real numbers or the finite field $\mathbb{Z}_p$, where p is an odd prime.

At certain points in this section we shall use results from Section VII.4, and Theorem E.7 also uses some basic facts about fields.

The first result of this section shows that Theorem VII.6 generalizes if $\mathbb{F}$ is the field $\mathbb{C}$ of complex numbers.

Theorem E.4. *Let $\mathbb{F}$ be a field which satisfies the Default Hypothesis, and assume that $\mathbb{F}$ is closed under taking square roots, so that for each $a \in \mathbb{F}$ there is some $b \in \mathbb{F}$ such that $b^2 = a$. Let Σ be a nonempty quadric in $\mathbb{FP}^n$, and let A and B be nonzero symmetric matrices such that $\Sigma_A = \Sigma = \Sigma_B$. Then B is a nonzero scalar multiple of A.*

By Theorem VII.6, the proof of this theorem reduces to showing the following result.

Theorem E.5. *Let $\mathbb{F}$ be a field which satisfies the Default Hypothesis, assume that $\mathbb{F}$ is closed under taking square roots, and let Σ be a nonempty quadric in $\mathbb{FP}^n$. Then Σ has a nonsingular point.*

Proof. By the same argument employed to classify quadrics over the complex numbers in Section VII.4, there is a projective collineation T of $\mathbb{FP}^n$ such that $T[\Sigma]$ is defined by a homogeneous quadratic equation of the form

$$x_1^2 + \cdots + x_r^2 = 0$$

where $2 \leq r \leq n + 1$ (the proof of Theorem VII.15 only used the existence of square roots and $1 + 1 \neq 0$). If we can prove the theorem for these special examples of hyperquadrics, then the general case will follow because T defines a 1–1 correspondence between the nonsingular points of Σ and the nonsingular points of $T[\Sigma]$.

Finding a nonsingular point for one of the special hyperquadrics is elementary; specifically, take the point with $x_1 = 1$, $x_2 = \sqrt{-1}$, and all other coordinates equal to zero.■

Although the field of complex numbers is closed under taking square roots, many fields — including the real numbers — do not have this property, and for such fields the conclusion of the preceding theorem is almost never valid. More precisely, we have the following result:

Theorem E.6. *Let $\mathbb{F}$ be a field satisfying the Default Condition at the beginning of this section, and assume further that at least one element in $\mathbb{F}$ is not a perfect square but $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_3$. Then for each $n \geq 2$ there is a nonempty hyperquadric $\Sigma \subset \mathbb{FP}^n$ and symmetric nonzero $(n+1) \times (n+1)$ matrices A and B such that $\Sigma_A = \Sigma = \Sigma_B$ (i.e., Σ is defined by both A and B), but A and B are not multiples of each other.*

The proof of this theorem depends upon the following purely algebraic result:

Theorem E.7. *Let $\mathbb{F}$ be a field such that $\mathbb{F}$ contains exactly one element which is not a perfect square. Then $\mathbb{F}$ is isomorphic to $\mathbb{Z}_3$.*

Two comments about this theorem and its proof are worth noting: (i) The proof of the theorem also goes through if $1 + 1 = 0$ in $\mathbb{F}$. (ii) If $\mathbb{F}$ is isomorphic to $\mathbb{Z}_3$, then -1 is not a perfect square, and it is the only element with this property.

Proof of Theorem E.7. Let $\mathbb{F}^\times$ be the multiplicative abelian group of nonzero elements of $\mathbb{F}$, let Γ be the quotient of $\mathbb{F}^\times$ by the subgroup of nonzero elements that are perfect squares in $\mathbb{F}$, and let $\theta : \mathbb{F}^\times \to \Gamma$ be the canonical projection homomorphism. By construction, if $\alpha \in \Gamma$ then $\alpha^2 = 1$. Let K be the kernel of θ. By our assumption on $\mathbb{F}$, we know that K is a proper subgroup of $\mathbb{F}^\times$. Let $a \in F^\times$ be the unique element which does not lie in K.

We claim that the subgroup K is trivial; if this were not the case and b^2 is a nontrivial element of K, then a and ab^2 would be distinct elements of $\mathbb{F}^\times$ that do not lie in K. Combining this with the discussion in the previous paragraph, we see that every element $c \in \mathbb{F}^\times$ satisfies $c^2 = 1$.

Standard results on roots of polynomials over fields imply that the nontrivial quadratic polynomial $x^2 - 1$ over a field $\mathbb{F}$ has at most two roots in $\mathbb{F}$. Since the preceding paragraph shows that the square of every element of $\mathbb{F}^\times$ is equal to 1, it follows that $\mathbb{F}^\times$ contains at most two elements and hence $\mathbb{F}$ contains either two or three elements. It is an elementary exercise to show that every field of this type is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$ depending upon whether $\mathbb{F}$ has two or three elements. If the field has two elements, then every element is a perfect square, so the only remaining possibility is that $\mathbb{F}$ is isomorphic to $\mathbb{Z}_3$.∎

Proof of Theorem E.6. By Theorem E.7, one can find distinct elements c, $d \in \mathbb{F}$ such that neither is a perfect square in $\mathbb{F}$. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & -c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & -d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-1} \end{pmatrix}$$

where $\mathbf{0}_{n-1}$ is an $(n-1) \times (n-1)$ zero matrix. It is straightforward to show that both Σ_A and Σ_B are equal to the $(n-2)$-plane in $\mathbb{FP}^n$ whose homogeneous coordinates satisfy $x_1 = x_2 = 0$. However, the matrices A and B are not multiples of each other.∎

EXAMPLES. **1.** If $\mathbb{F}$ is the real or rational numbers and a is a negative integer, then a is not a perfect square in $\mathbb{F}$.

2. More generally, suppose that $r < n$, and consider sequences α of positive (real or rational) numbers $a_2, \cdots, a_{r+1}$ in $\mathbb{F}$. For each such sequence α let $D(\alpha)$ be the $(n+1) \times (n+1)$ diagonal matrix with diagonal entries

$$1, \ a_2, \ \cdots, \ a_{r+1}, \ 0, \ \cdots, \ 0 \ .$$

For all possible choices of α the hyperquadric $\Sigma_{D(\alpha)}$ is the same subset of $\mathbb{FP}^n$ — namely the $(n-r)$-plane defined by $x_{r+1} = \cdots = x_{n+1} = 0$ — but no two of the matrices $D(\alpha)$ can be scalar multiples of each other (since the entries in the first row and first column are always 1, it follows that the only possible choice of scalar is 1, and this is impossible if the two sequences of numbers are different).

3. Also, if $\mathbb{F} = \mathbb{Z}_p$ where p is a prime not equal to 2 or 3, then there are $\frac{1}{2}(p-1)$ elements of $\mathbb{F}$ which are not perfect squares in $\mathbb{Z}_p$. For example, if $p = 5$ then 2 and 3 are not perfect squares, while if $p = 7$ then 3, 5 and 6 are not perfect squares. Further discussion of the general case can be found in the books by Davenport (pp. 62–68) and LeVeque (pp. 45–46) cited at the end of this Appendix.

Final remarks

We shall conclude this section with three very loosely related observations. The first provides still further examples of real symmetric matrices which define the same hyperquadric but are not multiples of each other. The second shows that the conclusion to Theorem E.6 is also valid if $\mathbb{F}$ is isomorphic to $\mathbb{Z}_3$; this yields a converse to Theorem E.4: *If a field $\mathbb{F}$ is* not *closed under taking square roots, then some nonempty hyperquadric in $\mathbb{FP}^n$ $(n \geq 2)$ is defined by two symmetric matrices that are not scalar multiples of each other.* In contrast, the third observation shows that if a hyperquadric is definable by two such matrices, then it is a "degenerate hyperquadric" which is equal to a k-plane for some positive integer k.

1. If $\mathbb{F} = \mathbb{R}$, then additional examples as in Theorem 6 are given by the matrix pairs

$$A \;=\; \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-1} \end{pmatrix} \qquad B \;=\; \begin{pmatrix} 1 & 1 & \mathbf{0} \\ 1 & d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-1} \end{pmatrix}$$

where $\mathbf{0}_{n-1}$ is as above and $d > 1$. This is true because $x_1^2 + 2x_1 x_2 + dx_2^2 = 0$ if and only if $x_1 = x_2 = 0$.

2. For similar reasons, if $\mathbb{F} = \mathbb{Z}_3$ the matrices given by

$$A \;=\; \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-1} \end{pmatrix} \qquad B_{\pm} \;=\; \begin{pmatrix} 1 & \pm 1 & \mathbf{0} \\ \pm 1 & -1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-1} \end{pmatrix}$$

define the same hyperquadric — namely, the $(n-2)$-plane in $\mathbb{Z}_3\mathbb{P}^n$ given by the linear equations $x_1 = x_2 = 0$ — even though none of these three matrices is a scalar multiple of another. In particular, this means that *Theorem 6 extends to the case where $\mathbb{F}$ is isomorphic to $\mathbb{Z}_3$*, and thus it is true for all fields $\mathbb{F}$ such that $1 + 1 \neq 0$ in $\mathbb{F}$ but $\mathbb{F}$ is not closed under taking square roots.■

3. By Theorem VII.6, if Σ is a quadric defined by two matrices A and B which are not multiples of each other, then then every point of Σ is singular. The next to last result of this section implies that such quadrics are in fact (projective) geometrical subspaces of $\mathbb{FP}^n$.

Theorem E.8. *Suppose that the symmetric matrix A defines a hyperquadric $\Sigma \subset \mathbb{FP}^n$ such that every point of Σ is singular. Let $W \subset \mathbb{F}^{n+1,1}$ be the kernel of (left multiplication by) A. Then Σ is equal to $\mathcal{S}_1(W)$, and hence is a $(\dim W - 1)$-plane in $\mathbb{FP}^n$.*

Proof. Given $X \in \mathbb{FP}^n$, let ξ be a set of homogeneous coordinates for X. If $X \in \mathcal{S}_1(W)$, then $A\xi = \mathbf{0}$ and hence $^{\mathbf{T}}\xi A \xi = {}^{\mathbf{T}}\xi \cdot \mathbf{0} = 0$, so that $X \in \Sigma$. Conversely, if $X \in \Sigma$, then since every point of Σ is singular we must have $^{\mathbf{T}}\xi A = {}^{\mathbf{T}}\mathbf{0}$, and since A is symmetric this is equivalent to $A\xi = \mathbf{0}$, so that $\xi \in W$ and $X \in \mathcal{S}_1(W)$.■

If we combine his result with Theorem VII.6, we obtain the following conclusion, which yields a fairly definitive statement on the relationship between hyperquadrics and their defining matrices:

Theorem E.9. *Let $\mathbb{F}$ be a field in which $1+1 \neq 0$, let $n \geq 2$, and let $\Sigma \subset \mathbb{FP}^n$ be a hyperquadric which is* nondegenerate *in the sense that it is* **NOT** *a (projective) geometrical subspace of $\mathbb{FP}^n$. Suppose that A and B are symmetric $(n{+}1){\times}(n{+}1)$ matrices over $\mathbb{F}$ such that $\Sigma_A = \Sigma = \Sigma_B$. Then B and A are (nonzero) scalar multiples of each other.*∎

In other words, *every nondegenerate hyperquadric is defined by a matrix which is unique up to multiplication by a nonzero scalar.*

References

H. Davenport. *The Higher Arithmetic.* Harper, New York, 1960.

W. LeVeque. *Topics in Number Theory, Volume* I. Addison-Wesley, Reading, MA, 1956.