Essential Concepts of Projective Geomtry

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(219 pages)

Electronic file index

All files are in the directory `http://math.ucr.edu.~res/progeom`

PREFACE TO THE ORIGINAL 1973 VERSION

The purpose of these lecture notes is to present the basic geometrical properties of projective spaces (**projective geometry**) in a manner reflecting their status in contemporary mathematics. Although projective spaces are no longer studied for their own sake as they were in the early 19th century, they are still a fundamental structure of mathematics. Since a wide range of mathematically related careers chosen by undergraduate mathematics students use geometrical ideas[1], hopefully these notes will be useful to a correspondingly wide range of students.

Of the two basic approaches to projective geometry — the synthetic (in the spirit of classical Euclidean geometry) and the analytic (in the spirit of ordinary analytic geometry) — the latter is generally the more important (and useful) in most modern contexts, and therefore I have tried to emphasize it throughout these notes. Of course, projective geometry (and indeed nearly every subject) should be presented in a relatively efficient manner, and such a treatment of the subject requires the use of both approaches to some extent.[2] Thus the synthetic approach is definitely used at the numerous points where it adds significantly to the exposition, and particularly where it helps motivate the translation of geometrical concepts into algebraic data, but the synthetic approach does have a much less dominant position than in many other texts on the subject.

We have included a large amount of material from affine geometry in these notes. There are several reasons for this. First of all, one of the basic reasons for studying projective geometry is for its applications to the geometry of Euclidean space, and affine geometry is the fundamental link between projective and Euclidean geometry. Furthermore, a discussion of affine geometry allows us to introduce the methods of linear algebra into geometry before projective space is constructed. Each of these is a nontrivial step, and it seems worthwhile to keep them separate. Finally, the linear-algebraic methods of affine geometry have proven to be extremely useful, both in pure mathematics and other subjects. I hope that more emphasis on affine geometry will give students a reasonable understanding of some important ideas that are often difficult to find in a single place.

I have tried to include a reasonably large number of exercises; for the most part, their purpose is not to make the student into a virtuoso at solving projective geometry problems, but rather to develop portions of the subject not treated in the notes and to test

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[1]Including graduate study in mathematics! Compare I. Kaplansky's remark, "Generations of mathematicians are growing up who are on the whole splendidly trained, but suddenly find that, after all, they do need to know what a projective plane is." (*Linear Algebra and Geometry: A Second Course*, p. vii.)

[2]Compare this with comments on pages 104–105 of J. L. Coolidge, *A History of Geometrical Methods*.

the students' understanding of the abstract theory via concrete numerical examples. Numerous books in the bibliography were helpful in the selection of exercises (some of which have been "borrowed").

Much (if not all) of the material in these notes was taught to me many years ago in a similar form by Daniel Moran at the University of Chicago, chiefly using the Second Edition of Birkhoff and MacLane's *Survey of Modern Algebra* and the mimeographed lecture notes, *Fundamental Concepts of Geometry*, by S.-S. Chern (listed in the bibliography). His clear presentation of the subject was very influential (but of course responsibility for these notes is entirely mine). Typists at Purdue University also deserve credit for putting a very patched up manuscript into its original typewritten form.

COMMENTS ON THE 1978 REPRINTING WITH CORRECTIONS

I corrected all the typographical and other mistakes that I discovered (and remembered!). Several sections of supplementary material were also added to treat some questions that had been left unanswered. None of this is really needed to understand the basic material, but the supplementary discussions do lead the way to further topics that are closely related to projective geometry.

Also, I am grateful to James C. Becker for comments on portions of Chapter VII; in particular, these led to a more accurate and coherent treatment of Theorem VII.22.

COMMENTS ON THE 2007-2008 REPRINTING

For several reasons it seemed worthwhile to make these notes available electronically on the World Wide Web, and I have converted the notes to LATEX because this had numerous advantages over posting scans of the original pages. I have corrected all the typographical errors I could find, cleaned up the text in several places, added a substantial number of new exercises, and inserted additional material throughout the notes, but I have not made any major revisions. Some of the changes reflect the evolution of undergraduate courses or new mathematical discoveries during the past 30 years, many others provide additional historical or mathematical background, still others involve references to selected online sites, and finally I have included some comments on more advanced topics that are closely related to the mathematical topics covered in these notes.

Given that 30 years have elapsed since these notes were used to teach projective geometry, it is natural to ask if there are things that could or should be done differently. The limited revisions to these notes reflect my view that few changes were needed. Several new books have appeared during the past 30 years, but the standard mathematical approaches to the subject have not really changed over the past three or four decades; in many cases they remain essentially unchanged since the last part of the 19th century. However, if I had to do everything again, I would probably include material on two important ties between projective geometry and other subjects; namely, the applications of projective geometry to computer graphics and one of the original motivations for the subject — the mathematical

theory of perspective drawing which was developed near the end of the Middle Ages. The Computer Revolution over the past 30 years has had a substantial impact on the applications of projective geometry and its methods to areas like computer graphics; for example, creating an accurate on-screen image often requires extensive calculations using the methods developed in these notes. Since numerous expositions of such material are readily available on the World Wide Web, we shall not try to elaborate on such uses of projective geometry here. Regarding the ties between projective geometry and the mathematical theory of perspective drawing, there is some discussion of the historical setting in the online notes

<div align="center">http://math.ucr.edu/∼res/math153/history08.pdf</div>

with a more mathematical discussion that starts in the online document

<div align="center">http://math.ucr.edu/∼res/math133/geometrynotes4a.pdf</div>

and continues in the document · · · geometrynotes4b.pdf; there is some overlap between the last two documents and the material in these notes.

As indicated by the preceding discussion, I did not try to discuss the historical background for many topics in the course in order to focus on the mathematical content; a comprehensive account of the history could easily take another hundred pages. I have added several bibliographic references that cover some or all of this history, and in some cases I have added comments regarding the viewpoints and accuracy of the individual references. There is also an excellent **MacTutor** online site

<div align="center">http://www-groups.dcs.st-and.ac.uk/∼history/</div>

which contains a great deal of extremely reliable information on the history of mathematics and includes biographies for several hundred contributors to the subject. A clickable list of all World Wide Web links for these notes is available online at the following address:

<div align="center">http://math.ucr.edu/∼res/progeom/pgwww.pdf</div>

PREREQUISITES

We assume that the reader understands the rudiments of set theory, including such things as unions, intersections and Cartesian products. Furthermore, we assume the reader knows the concepts of functions (synonymous with map, mapping, transformation), including one-to-one, onto and inverse functions, and also the algebraic notions of group and subgroup. These may be found in numerous books (for example, Birkhoff and MacLane). Given the number and nature of the mathematical proofs in these notes, clearly we also assume that a reader has developed the ability to follow mathematical arguments at the level of a standard undergraduate level abstract algebra course. Some prior experience with the concepts of isomorphism and automorphism for mathematical systems might be useful, but it is not necessary.

Given a function f from one set X to another set Y and subsets $A \subset X$, $B \subset Y$, we shall denote the image of A (all points $b = f(a)$ for some a) by $f[A]$ and the inverse image of B (all a such that $f(a) \in B$ by $f^{-1}[B]$. As noted in the book by Kelley in the bibliography, this eliminates some potential ambiguities.

Basic material from undergraduate linear algebra courses plays an extremely important role in these notes, and the relevant topics from linear algebra are summarized in the Appendix; detailed treatments also appear in some of the references.

These notes assume some familiarity with high school level deductive geometry as well as an understanding of analytic geoemtry as taught in standard precalculus and calculus courses. A discussion of ordinary Euclidean (and non-Euclidean) geometry at a level compatible with these notes appears in the online files

$$\texttt{http://math.ucr.edu/}{\sim}\texttt{res/math133/geometrynotes}*\texttt{.pdf}$$

where $*$ is one of the following:

$$\texttt{1, 2a, 2b, 3a, 3b, 3c, 4a, 4b, 5a, 5b}$$

There are numerous other files in the directory `http://math.ucr.edu/~res/math133` that may also provide useful background or further information (particular examples are the `geometryintro.pdf`, `mathproofs.pdf` and `metgeom.pdf` files in that directory).

Finally, we assume a few simple facts about counting finite sets; for example, a set with n elements has 2^n subsets,and the number of elements in a disjoint union of two finite sets is given by $\#(A \cup B) = \#(A) + \#(B)$ if $A \cap B = \emptyset$. At some points we shall also use a basic multiplicative principle:

> *Suppose we are given a sequence of r choices $\mathrm{Ch}_1, \cdots, \mathrm{Ch}_r$, and that for each i the number N_i of alternatives at the ith stage doe not depend upon the first $(i-1)$ choices. Then the total number of possible choices is the product $N_1 \cdots N_r$.*

These topics are now covered in most Discrete Mathematics courses for Mathematics or Computer Science students, and virtually any textbook for such courses will cover such material. Two standard texts are listed in the bibliography

SUGGESTIONS FOR USING THESE NOTES

As with most writings, some portions of these notes are more important than others, and this is particularly true since the various sections of the notes serve different functions.

Priorities for coverage of material

The central sections of these notes are I.2–I.3, II.2–II.3, II.5, III.1–III.4, IV.1–IV.3, V.1–V.4 (except Theorem V.26), VI.1–VI.2, and VII.1. These contain definitions of the basic concepts and their fundamental properties. Other sections that should be included in any course on affine and projective geometry, if possible, are II.4, II.6, V.5 and VI.3 (these are closely related), discussion, and this it may be left as reading material for students. Finally, the material in Chapter VII should be understood to the extent that time permits, with the sections taken in order.

The material in the Appendix is mainly for reference purposes and should be consulted whenever the reader is unsure of the linear algebra being used; in keeping with the formulation of many topics in terms of linear algebra with scalars that are not necessarily fields, we have formulated all the basic concepts as generally as possible.

Setting the level of generality

In most sections of these notes, the coordinates in our treatment of analytic projective geometry are assumed to lie in an arbitrary *skew-field* or *division ring* (all the properties of a field aside from the commutative law of multiplication). Some readers may prefer to work with less general coefficients for a variety of reasons, so we shall discuss some of the possibilities.

One option is to restrict attention to analytic projective geometry in which the coordinates lie in a (commutative) field. If this is preferred, then many things simplify immediately, starting with the survey of linear algebra in the Appendix. Furthermore, many of the proofs in Chapter V may be simplified considerably. In particular, one can use Theorem V.5 to simplify the proofs of Pappus' Theorem (the "if" part of Theorem V.19), and the theorems on addition and multiplication in Chapter V, Section 5 (namely, Theorems V.27 and V.28).

Another option would be to restrict further and assume that the coordinates lie in the real or complex numbers, or possibly **only** in the real numbers. If either is chosen, the loss of geometrical insight is relatively small, and various technical qualifications involving commutative multiplication, $1 + 1 \neq 0$ or $1 + 1 + 1 \neq 0$ in $\mathbb{F}$ (the field or skew-field of scalar coordinates), and a few other such issues all become superfluous.

Comments on the notation

Most of this is pretty standard, but some comments on the numbering of results might be in order. The latter are numbered in the form   RESULT Y.n,   where Y denotes the chapter number (in Roman numerals) and the results are numbered sequentially within

each chapter. If the chapter number is missing, then the reference is to a result from the same chapter in which the given statement appears.

Comments on the illustrations

Finally, we include a few words about the illustrations in the notes. Their main purpose is to provide insight into the discussion they accompany; the reader is encouraged to draw his or her own pictures for similar purposes whenever this may seem useful. However, the reader should also recognize that reference to a picture is **not** adequate for purposes of mathematical proof (this is reflected by the quotation at the beginning of Section II.5), and any conclusion suggested by a picture must be established by the usual rules of proof. Further discussion of such issues appears in Section II.2 of the following online notes:

http://math.ucr.edu/~res/math133/geometrynotes2a.pdf

In particular, the discussion at the very end of the Appendix to Section II.2 summarizes the main points, and the Appendix itself gives a standard example from elementary geometry to illustrates how too much reliance on drawings can lead to obviously false conclusions.

CHAPTER I

SYNTHETIC AND ANALYTIC GEOMETRY

The purpose of this chapter is to review some basic facts from classical deductive geometry and coordinate geometry from slightly more advanced viewpoints. The latter reflect the approaches taken in subsequent chapters of these notes.

1. Axioms for Euclidean geometry

During the 19th century, mathematicians recognized that the logical foundations of their subject had to be re-examined and strengthened. In particular, it was very apparent that the axiomatic setting for geometry in Euclid's *Elements* requires nontrivial modifications. Several ways of doing this were worked out by the end of the century, and in 1900 D. Hilbert[1] (1862–1943) gave a definitive account of the modern foundations in his highly influential book, *Foundations of Geometry*.

Mathematical theories begin with primitive concepts, which are not formally defined, and assumptions or axioms which describe the basic relations among the concepts. In Hilbert's setting there are six primitive concepts: *Points, lines,* the notion of *one point lying between two others (betweenness), congruence of segments (same distances between the endpoints)* and *congruence of angles (same angular measurement in degrees or radians).* The axioms on these undefined concepts are divided into five classes: *Incidence, order, congruence, parallelism* and *continuity.* One notable feature of this classification is that only one class (congruence) requires the use of all six primitive concepts. More precisely, the concepts needed for the axiom classes are given as follows:

Axiom class	Concepts required
Incidence	Point, line, plane
Order	Point, line, plane, betweenness
Congruence	All six
Parallelism	Point, line, plane
Continuity	Point, line, plane, betweenness

Strictly speaking, Hilbert's treatment of continuity involves congruence of segments, but the continuity axiom may be formulated without this concept (see Forder, *Foundations of Euclidean Geometry*, p. 297).

As indicated in the table above, congruence of segments and congruence of angles are needed for only one of the axiom classes. Thus it is reasonable to divide the theorems of Euclidean

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[1]DAVID HILBERT made fundamental, important contributions to an extremely groad range of mathematical topics. He is also known for an extremely influential 1900 paper in which he stated 23 problems, and for his formal axiomatic approach to mathematics, which has become the most widely adopted standard for the subject.

geometry into two classes — those which require the use of congruence and those which do not. Of course, the former class is the more important one in classical Euclidean geometry (it is widely noted that "geometry" literally means "earth measurement"). The main concern of these notes is with theorems of the latter class. Although relatively few theorems of this type were known to the classical Greek geometers and their proofs almost always involved congruence in some way, there is an extensive collection of geometrical theorems having little or nothing to do with congruence.[2]

The viewpoint employed to prove such results contrasts sharply with the traditional viewpoint of Euclidean geometry. In the latter subject one generally attempts to prove as much as possible without recourse to the Euclidean Parallel Postulate, and this axiom is introduced only when it is unavoidable. However, in dealing with noncongruence theorems, one assumes the parallel postulate very early in the subject and attempts to prove as much as possible without explicitly discussing congruence. Unfortunately, the statements and proofs of many such theorems are often obscured by the need to treat numerous special cases. PROJECTIVE GEOMETRY provides a mathematical framework for stating and proving many such theorems in a simpler and more unified fashion.

2. Coordinate interpretation of primitive concepts

> As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited. But when these sciences joined company [through analytic geometry], they drew from each other fresh vitality, and thenceforward marched on at a rapid pace towards perfection. — **J.-L. Lagrange (1736–1813)**[3]

Analytic geometry has yielded powerful methods for dealing with geometric problems. One reason for this is that the primitive concepts of Euclidean geometry have precise numerical formulations in Cartesian coordinates. A *point* in 2- or 3-dimensional coordinate space $\mathbb{R}^2$ or $\mathbb{R}^3$ becomes an ordered pair or triple of real numbers. The *line* joining the points $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ becomes the set of all $\mathbf{x}$ expressible in vector form as

$$\mathbf{x} \; = \; \mathbf{a} \; + \; t \cdot (\mathbf{b} - \mathbf{a})$$

for some real number t (in $\mathbb{R}^2$ the third coordinate is suppressed). A plane in $\mathbb{R}^3$ is the set of all $\mathbf{x}$ whose coordinates (x_1, x_2, x_3) satisfy a nontrivial linear equation

$$a_1 x_1 \; + \; a_2 x_2 \; + \; a_3 x_3 \; = \; b$$

for three real numbers a_1, a_2 a_3 that are not all zero. The point $\mathbf{x}$ is *between* $\mathbf{a}$ and $\mathbf{b}$ if

$$\mathbf{x} \; = \; \mathbf{a} \; + \; t \cdot (\mathbf{b} - \mathbf{a})$$

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[2]This discovery did not come from mathematical axiom manipulation for its own sake, but rather from the geometrical theory of *drawing in perspective* begun by Renaissance artists and engineers. This is discussed briefly in Section III.1, and the books by Courant and Robbins, Newman, Kline and Coolidge provide more information on the historical origins; some online references are also given in the comments on the 2007 reprinting of these notes, which appear in the Preface.

[3]JOSEPH-LOUIS LAGRANGE made major contributions in several different areas of mathematics and physics.

where the real number t satisfies $0 < t < 1$. Two *segments* are *congruent* if and only if the distances between their endpoints (given by the usual Pythagorean formula) are equal, and two *angles* $\angle\,\mathbf{abc}$ and $\angle\,\mathbf{xyz}$ are *congruent* if their cosines defined by the usual formula

$$\cos \angle\,\mathbf{uvw} \;=\; \frac{(\mathbf{u}-\mathbf{v})\cdot(\mathbf{w}-\mathbf{v})}{|\mathbf{u}-\mathbf{v}|\,|\mathbf{w}-\mathbf{v}|}$$

are equal. We note that the cosine function and its inverse can be defined mathematically without any explicit appeal to geometry by means of the usual power series expansions (for example, see Appendix F in the book by Ryan or pages 182–184 in the book by Rudin; both books are listed in the bibliography).

In the context described above, the axioms for Euclidean geometry reflect crucial algebraic properties of the real number system and the analytic properties of the cosine function and its inverse.

3. Lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$

We have seen that the vector space structures on $\mathbb{R}^2$ and $\mathbb{R}^3$ yield convenient formulations for some basic concepts of Euclidean geometry, and in this section we shall see that one can use linear algebra to give a unified description of lines and planes.

THEOREM I.1. *Let $P \subset \mathbb{R}^3$ be a plane, and let $\mathbf{x} \in P$. Then*

$$P(\mathbf{x}) \;=\; \{\; \mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} \;=\; \mathbf{z} \;-\; \mathbf{x},\ \text{some } \mathbf{z} \in P \;\}$$

is a 2-dimensional vector subspace of $\mathbb{R}^3$. Furthermore, if $\mathbf{v} \in P$ is arbitrary, then $P(\mathbf{v}) = P(\mathbf{x})$.

Proof. Suppose P is defined by the equation $a_1 x_1 \;+\; a_2 x_2 \;+\; a_3 x_3 \;=\; b$. We claim that

$$P(\mathbf{x}) = \{\; \mathbf{y} \in \mathbb{R}^3 \mid a_1 y_1 \;+\; a_2 y_2 \;+\; a_3 y_3 \;=\; 0 \;\}\;.$$

Since the coefficients a_i are not all zero, the set $P(\mathbf{x})$ is a 2-dimensional vector subspace of $\mathbb{R}^3$ by Theorem A.10. To prove that $P(\mathbf{x})$ equals the latter set, note that $\mathbf{y} \in P(\mathbf{x})$ implies

$$\sum_{i=1}^{3} a_i y_i \;=\; \sum_{i=1}^{3} a_i(z_i - x_i) \;=\; \sum_{i=1}^{3} a_i z_i \;-\; \sum_{i=1}^{3} a_i x_i \;=\; b - b \;=\; 0$$

and conversely $\sum_{i=1}^{3} a_i y_i \;=\; 0$ implies

$$0 \;=\; \sum_{i=1}^{3} a_i z_i \;-\; \sum_{i=1}^{3} a_i x_i \;=\; \sum_{i=1}^{3} a_i z_i \;-\; b\;.$$

This shows that $P(\mathbf{x})$ is the specified vector subspace of $\mathbb{R}^3$.

To see that $P(\mathbf{v}) = P(\mathbf{x})$, notice that both are equal to $\{\; \mathbf{y} \in \mathbb{R}^3 \mid \sum_{i=1}^{3} a_i y_i \;=\; 0\}$ by the reasoning of the previous paragraph.■

Here is the corresponding result for lines.

THEOREM I.2. *Let $n = 2$ or 3, let $L \subset \mathbb{R}^n$ be a line, and let $\mathbf{x} \in L$. Then*

$$L(\mathbf{x}) = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{y} = \mathbf{z} - \mathbf{x}, \text{ some } \mathbf{z} \in L \}$$

is a 1-dimensional vector subspace of $\mathbb{R}^n$. Furthermore, if $\mathbf{v} \in L$ is arbitrary, then $L(\mathbf{v}) = L(\mathbf{x})$.

Proof.   Suppose P is definable as

$$\{ \mathbf{z} \in \mathbb{R}^n \mid \mathbf{z} = \mathbf{a} - t(\mathbf{b} - \mathbf{a}), \text{ some } t \in \mathbb{R} \}$$

where $\mathbf{a} \neq \mathbf{b}$. We claim that

$$L(\mathbf{x}) = \{ \mathbf{z} \in \mathbb{R}^{\mathbf{n}} \mid \mathbf{z} = \mathbf{a} - \mathbf{t}(\mathbf{b} - \mathbf{a}), \text{ some } \mathbf{t} \in \mathbb{R} \} .$$

Since the latter is a 1-dimensional subspace of $\mathbb{R}^n$, this claim implies the first part of the theorem. The second part also follows because both $L(\mathbf{v})$ and $L(\mathbf{x})$ are then equal to this subspace.

Since $\mathbf{x} \in L$, there is a real number s such that $\mathbf{x} = \mathbf{a} + s(\mathbf{b} - \mathbf{a})$. If $\mathbf{y} \in L(\mathbf{x})$, write $\mathbf{y} = \mathbf{z} - \mathbf{x}$, where $\mathbf{z} \in L$; since $\mathbf{z} \in L$, there is a real number r such that $\mathbf{z} = \mathbf{a} + r(\mathbf{b} - \mathbf{a})$. If we subtract $\mathbf{x}$ from $\mathbf{z}$ we obtain

$$\mathbf{y} = \mathbf{z} - \mathbf{x} = (r - s)(\mathbf{b} - \mathbf{a}) .$$

Thus $L(\mathbf{x})$ is contained in the given subspace. Conversely, if $\mathbf{y} = t(\mathbf{b} - \mathbf{a})$, set $\mathbf{z} = \mathbf{x} + \mathbf{y}$. Then

$$\mathbf{z} = \mathbf{x} + \mathbf{y} = \mathbf{a} + s(\mathbf{b} - \mathbf{a}) + t(\mathbf{b} - \mathbf{a}) = \mathbf{a} + (s + t)(\mathbf{b} - \mathbf{a}) .$$

Thus $\mathbf{y} \in L(\mathbf{x})$, showing that the given subspace is equal to $L(\mathbf{x})$.∎

The following definition will yield a unified reformulation of the theorems above:

Definition.   Let V be a vector space over a field $\mathbb{F}$, let $S \subset V$ be a nonempty subset, and let $\mathbf{x} \in V$. The *translate of S by $\mathbf{x}$*, written $\mathbf{x} + S$, is the set

$$\{ \mathbf{x} \in S \mid \mathbf{y} = \mathbf{x} + \mathbf{s}, \text{ some } \mathbf{s} \in S \} .$$

The fundamental properties of translates are given in the following theorems; the proof of the first is left as an exercise.

THEOREM I.3. *If $\mathbf{z}$, $\mathbf{x} \in V$ and $S \subset V$ is nonempty, then $\mathbf{z} + (\mathbf{x} + S) = (\mathbf{z} + \mathbf{x}) + S$.*∎

THEOREM I.4. *Let V be a vector space, let W be a vector subspace of V, let $\mathbf{x} \in V$, and suppose $\mathbf{y} \in \mathbf{x} + W$. Then $\mathbf{x} + W = \mathbf{y} + W$.*

Proof.   If $\mathbf{z} \in \mathbf{y} + W$, then $\mathbf{z} = \mathbf{y} + \mathbf{u}$, where $\mathbf{u} \in W$. But $\mathbf{y} = \mathbf{x} + \mathbf{v}$, where $\mathbf{v} \in W$, and hence $\mathbf{z} = \mathbf{x} + \mathbf{u} + \mathbf{v}$, where $\mathbf{u} + \mathbf{v} \in W$. Hence $\mathbf{y} + W \subset \mathbf{x} + W$.

On the other hand, if $\mathbf{z} \in \mathbf{x} + W$, then $\mathbf{z} \in \mathbf{x} + \mathbf{w}$, where $\mathbf{w} \in W$. Since $\mathbf{y} = \mathbf{x} + \mathbf{v}$ (as above), it follows that

$$\mathbf{x} + \mathbf{w} = (\mathbf{x} + \mathbf{v}) + (\mathbf{w} - \mathbf{v}) = \mathbf{y} + (\mathbf{w} - \mathbf{v}) \in \mathbf{y} + W .$$

Consequently, we also have $\mathbf{x} + W \subset \mathbf{y} + W$.∎

We shall now reformulate Theorem 1 and Theorem 2.

THEOREM I.5. *Every plane in $\mathbb{R}^3$ is a translate of a 2-dimensional vector subspace, and every line in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a translate of a 1-dimensional vector subspace.*

Proof. If A is a line or plane with $\mathbf{x} \in A$ and $A(\mathbf{x})$ is defined as above, then it is easy to verify that $A = \mathbf{x} + A(\mathbf{x})$. ∎

The converse to Theorem 5 is also true.

THEOREM I.6. *Every translate of a 2-dimensional subspace of $\mathbb{R}^3$ is a plane, and every translate of a 1-dimensional vector subspace of $\mathbb{R}^2$ or $\mathbb{R}^3$ is a line.*

Proof. CASE 1. Two-dimensional subspaces. Let $\mathbf{b}$ and $\mathbf{c}$ form a basis for W, and let $\mathbf{a} = \mathbf{c} \times \mathbf{b}$ (the usual cross product; see Section A.5 in Appendix A). Then $\mathbf{y} \in W$ if and only if $\mathbf{a} \cdot \mathbf{y} = 0$ by Theorem A.10 and the cross product identities at the beginning of Section A.5 in Appendix A. We claim that $\mathbf{z} \in \mathbf{x} \in W$ if and only if $\mathbf{a} \cdot \mathbf{z} = \mathbf{a} \cdot \mathbf{x}$.

If $\mathbf{z} \in \mathbf{x} + W$, write $\mathbf{z} = \mathbf{x} + \mathbf{w}$, where $\mathbf{w} \in W$. By distributivity of the dot product we have

$$\mathbf{a} \cdot \mathbf{z} = \mathbf{a} \cdot (\mathbf{x} + \mathbf{w}) = (\mathbf{a} \cdot \mathbf{x}) + (\mathbf{a} \cdot \mathbf{w}) = (\mathbf{a} \cdot \mathbf{x})$$

the latter following because $\mathbf{a} \cdot \mathbf{w} = 0$. Conversely, if $\mathbf{a} \cdot \mathbf{z} = (\mathbf{a} \cdot \mathbf{x})$, then

$$\mathbf{a} \cdot (\mathbf{w} - \mathbf{x}) = (\mathbf{a} \cdot \mathbf{z}) - (\mathbf{a} \cdot \mathbf{x}) = 0$$

and hence $\mathbf{z} - \mathbf{x} \in W$. Since $\mathbf{z} = \mathbf{x} + (\mathbf{z} \cdot \mathbf{x})$, clearly $\mathbf{x} \in \mathbf{z} + W$.

CASE 2. One-dimensional subspaces. Let $\mathbf{w}$ be a nonzero (hence spanning) vector in W, and let $\mathbf{y} \in \mathbf{x} + \mathbf{w}$. Then the line $\mathbf{xy}$ is equal to $\mathbf{x} + W$. ∎

The theorems above readily yield an alternate characterization of lines in $\mathbb{R}^2$ which is similar to the characterization of planes in $\mathbb{R}^3$.

THEOREM I.7. *A subset of $\mathbb{R}^2$ is a line if ane only if there exist a_1, a_2, $b \in \mathbb{R}$ such that not both a_1 and a_2 are zero and the point $\mathbf{x} = (x_1, x_2)$ lies in the subset if and only if $a_1 x_1 + a_2 x_2 = b$.*

Proof. Suppose that the set S is defined by the equation above. Let W be the set of all $\mathbf{y} = (y_1, y_2)$ such that $a_1 y_1 + a_2 y_2 = 0$. By Theorem A.10 we know that W is a 1-dimensional subspace of $\mathbb{R}^2$. Thus if $\mathbf{y} \in W$, the argument proving Theorem 1 shows that $S = \mathbf{y} + W$.

On the other hand, suppose that $\mathbf{y} + W$ is a line in $\mathbb{R}^2$, where W is a 1-dimensional vector subspace of $\mathbb{R}^2$. Let $\mathbf{w} = (w_1, w_2)$ be a nonzero vector in W; then $J(\mathbf{w}) = (w_2, -w_1)$ is also nonzero, and $\mathbf{z} \in W$ if and only if it is perpendicular to $J(\mathbf{w})$ by Theorem A.10. A modified versionof the proof of Theorem 6, Case 1, shows that $\mathbf{x} \in \mathbf{y} + W$ if and only if

$$J(\mathbf{w}) \cdot \mathbf{x} = J(\mathbf{w}) \cdot \mathbf{y} .$$

Thus it suffices to take $(a_1, a_2) = J(\mathbf{w})$ and $b = \mathbf{w} \cdot \mathbf{y}$. ∎

EXERCISES

1. Prove Theorem 3.

2. Verify the assertion $S = \mathbf{x} + S(\mathbf{x})$ made in Theorem 5.

3. Let V be a vector space, let $W \subset V$ be a vector subspace, and suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in V. Prove that the sets $\mathbf{u} + W$ and $\mathbf{v} + W$ are either disjoint or equal.

4.   Fill in the details of the proof of Theorem 7.

5.   Let P be the unique plane through the given triples of points in each of the following cases. Find an equation defining P, and determine the 2-dimensional vector subspace of which P is a translate.

(i) $(1, 3, 2)$, $(4, 1, -1)$, $(2, 0, 0)$.

(ii) $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$.

(iii) $(2, -1, 3)$, $(1, 1, 1)$, $(3, 0, 4)$.

6.   Suppose that we are given two distinct lines L, M in $\mathbb{R}^3$ which meet at the point $\mathbf{x}_0$, and write these lies as $L = \mathbf{x}_0 + V$ and $M = \mathbf{x}_0 + W$, where V and W have bases given by $\{\,\mathbf{a}\,\}$ and $\{\,\mathbf{b}\,\}$ respectively. Explain why there is a plane containing L and M [*Hint:*   Why do $\mathbf{a}$ and $\mathbf{b}$ span a 2-dimensional vector subspace?]

7.   Let $\{\,\mathbf{a},\,\mathbf{b},\,\mathbf{c}\,\}$ be a basis for $\mathbb{R}^3$, let V be the 1-dimensional vector subspace spanned by $\mathbf{a}$, and let W be the 1-dimensional vector space spanned by $\mathbf{c} - \mathbf{b}$. Prove that the lines V (the unique line containing $\mathbf{0}$ and $\mathbf{a}$) and $\mathbf{b} + W$ (the unique line containing $\mathbf{b}$ and $\mathbf{c}$) have no points in common. [*Hint:*   If such a point exists, then by the preceding exercise the two lines are coplanar and lie in some plane $\mathbf{z} + X$, where X is a 2-dimensional vector subspace. Why do the vectors $\mathbf{0}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ all lie in X, and why does this imply that $\mathbf{z} + X = X$? Derive a contradiction from this and the preceding two sentences.]

CHAPTER II

AFFINE GEOMETRY

In the previous chapter we indicated how several basic ideas from geometry have natural interpretations in terms of vector spaces and linear algebra. This chapter continues the process of formulating basic geometric concepts in such terms. It begins with standard material, moves on to consider topics not covered in most courses on classical deductive geometry or analytic geometry, and it concludes by giving an abstract formulation of the concept of geometrical incidence and closely related issues.

1. Synthetic affine geometry

In this section we shall consider some properties of Euclidean spaces which only depend upon the axioms of incidence and parallelism

Definition. A *three-dimensional incidence space* is a triple $(S, \mathcal{L}, \mathcal{P})$ consisting of a nonempty set S (whose elements are called *points*) and two nonempty disjoint families of proper subsets of S denoted by $\mathcal{L}$ (*lines*) and $\mathcal{P}$ (*planes*) respectively, which satisfy the following conditions:

(I-1) Every line (element of $\mathcal{L}$) contains at least two points, and every plane (element of $\mathcal{P}$) contains at least three points.

(I-2) If $\mathbf{x}$ and $\mathbf{y}$ are distinct points of S, then there is a unique line L such that $\mathbf{x}$, $\mathbf{y} \in L$.

Notation. The line given by **(I-2)** is called the *line joining* $\mathbf{x}$ *and* $\mathbf{y}$ and denoted by $\mathbf{xy}$.

(I-3) If $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are distinct points of S and $\mathbf{z} \notin \mathbf{xy}$, then there is a unique plane P such that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in P$.

(I-4) If a plane P contains the distinct points $\mathbf{x}$ and $\mathbf{y}$, then it also contains the line $\mathbf{xy}$.

(I-5) If P and Q are planes with a nonempty intersection, then $P \cap Q$ contains at least two points.

Of course, the standard example in $\mathbb{R}^3$ with lines and planes defined by the formulas in Chapter I (we shall verify a more general statement later in this chapter). A list of other simple examples appears in Prenowitz and Jordan, *Basic Concepts of Geometry*, pp. 141–146.

A few theorems in Euclidean geometry are true for every three-dimensional incidence space. The proofs of these results provide an easy introduction to the synthetic techniques of these notes. In the first six results, the triple $(S, \mathcal{L}, \mathcal{P})$ denotes a fixed three-dimensional incidence space.

Definition. A set B of points in S is *collinear* if there is some line L in S such that $B \subset L$, and it is *noncollinear* otherwise. A set A of points in S is *coplanar* if there is some plane P in S such that $A \subset P$, and it is *noncoplanar* otherwise. — Frequently we say that the points $\mathbf{x}$, $\mathbf{y}$, $\cdots$ (*etc.*) are collinear or coplanar if the set with these elements is collinear or coplanar respectively.

THEOREM II.1. *Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be distinct points of S such that $\mathbf{z} \notin \mathbf{xy}$. Then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a noncollinear set.*

Proof.   Suppose that L is a line containing the given three points. Since $\mathbf{x}$ and $\mathbf{y}$ are distinct, by **(I-2)** we know that $L = \mathbf{xy}$. By our assumption on L it follows that $\mathbf{z} \in L$; however, this contradicts the hypothesis $\mathbf{z} \notin \mathbf{xy}$. Therefore there is no line containing $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.∎

THEOREM II.2. *There is a subset of four noncoplanar points in S.*

Proof.   Let P be a plane in S. We claim that P contains three noncollinear points. By **(I-1)** we know that P contains three distinct points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}_0$. If these three points are noncollinear, let $\mathbf{c} = \mathbf{c}_0$. If they are collinear, then the line L containing them is a subset of P by **(I-4)**, and since $\mathcal{L}$ and $\mathcal{P}$ are disjoint it follows that L must be a proper subset of P; therefore there is some point $\mathbf{c} \in P$ such that $\mathbf{c} \notin L$, and by the preceding result the set $\{\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}\}$ is noncollinear. Thus in any case we know that P contains three noncollinear points.

Since P is a proper subset of S, there is a point $\mathbf{d} \in S$ such that $\mathbf{d} \notin P$. We claim that $\{\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}\ \mathbf{d}\}$ is noncoplanar. For if Q were a plane containing all four points, then $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in P$ would imply $P = Q$, which contradicts our basic stipulation that $\mathbf{d} \notin P$.∎

THEOREM II.3. *The intersection of two distinct lines in S is either a point or the empty set.*

Proof.   Suppose that $\mathbf{x} \neq \mathbf{y}$ but both belong to $L \cap M$ for some lines L and M. By property **(I-2)** we must have $L = M$. Thus the intersection of distinct lines must consist of at most one point.∎

THEOREM II.4. *The intersection of two distinct planes in S is either a line or the empty set.*

Proof.   Suppose that P and Q are distinct planes in S with a nonempty intersection, and let $\mathbf{x} \in P \cap Q$. By **(I-5)** there is a second point $\mathbf{y} \in P \cap Q$. If L is the line $\mathbf{xy}$, then $L \subset P$ and $L \subset Q$ by two applications of **(I-4)**; hence we have $L \subset P \cap Q$. If there is a point $\mathbf{z} \in P \cap Q$ with $\mathbf{z} \notin L$, then the points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are noncollinear but contained in both of the planes P and Q. By **(I-3)** we must have $P = Q$. On the other hand, by assumption we know $P \neq Q$, so we have reached a contradiction. The source of this contradiction is our hypothesis that $P \cap Q$ strictly contains L, and therefore it follows that $P \cap Q = L$.∎

THEOREM II.5. *Let L and M be distinct lines, and assume that $L \cap M \neq \varnothing$. Then there is a unique plane P such that $L \subset P$ and $M \subset P$.*

In less formal terms, given two intersecting lines there is a unique plane containing them.

Proof.   Let $\mathbf{x} \in L \cap M$ be the unique common point (it is unique by Theorem 3). By **(I-2)** there exist points $\mathbf{y} \in L$ and $\mathbf{z} \in M$, each of which is distinct from $\mathbf{x}$. The points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are noncollinear because $L = \mathbf{xy}$ and $\mathbf{z} \in M - \{\mathbf{x}\} = M - L$. By **(I-3)** there is a unique plane P such that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in P$, and by **(I-4)** we know that $L \subset P$ and $M \subset P$. This proves the existence of a plane containing both L and M. To see this plane is unique, observe that every plane Q containing both lines must contain $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$. By **(I-3)** there is a unique such plane, and therefore we must have $Q = P$.∎

THEOREM II.6. *Given a line L and a point $\mathbf{z}$ not on L, there is a unique plane P such that $L \subset P$ and $\mathbf{z} \in P$.*

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be distinct points of L, so that $L = \mathbf{xy}$. We then know that the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is noncollinear, and hence there is a unique plane P containing them. By **(I-4)** we know that $L \subset P$ and $\mathbf{z} \in P$. Conversely, if Q is an arbitrary plane containing L and $\mathbf{z}$, then Q contains the three noncollinear points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, and hence by **(I-3)** we know that $Q = P$.∎

Notation. We shall denote the unique plane in the preceding result by $L\mathbf{z}$.

Of course, all the theorems above are quite simple; their conclusions are probably very clear intuitively, and their proofs are fairly straightforward arguments. One must add Hilbert's Axioms of Order or the Euclidean Parallelism Axiom to obtain something more substantial. Since our aim is to introduce the parallel postulate at an early point, we might as well do so now (a thorough treatment of geometric theorems derivable from the Axioms of Incidence and Order appears in Chapter 12 of Coxeter, *Introduction to Geometry*; we shall discuss the Axioms of Order in Section VI.6 of these notes).

Definition. Two lines in a three-dimensional incidence space S are *parallel* if they are disjoint and coplanar (note particularly the second condition). If L and L' are parallel, we shall write $L||L'$ and denote their common plane by LL'. — Note that if $L||M$ then $M||L$ because the conditions in the definition of parallelism are symmetric in the two lines.

Affine three-dimensional incidence spaces

Definition. A three-dimensional incidence space $(S, \mathcal{L}, \mathcal{P})$ is an *affine three-space* if the following holds:

(EPP) For each line L in S and each point $\mathbf{x} \notin L$ there is a unique line $L' \subset L\mathbf{x}$ such that $\mathbf{x} \in L$ and $L \cap L' = \varnothing$ (in other words, there is a unique line L' which contains $\mathbf{x}$ and is parallel to L).

This property is often called *the Euclidean Parallelism Property*, the *Euclidean Parallel Postulate* or *Playfair's Postulate*.[1] Actually, Euclid's *Elements* employed a logically equivalent statement which also requires the concepts of betweenness and congruence, and the advantages of using the purely incidence-theoretical statement **(EPP)** were noted explicitly by Proclus Diadochus (412–485).[2]

A discussion of the origin of the term "affine" appears in Section II.5 of the following online site:

$$\texttt{http://math.ucr.edu/}\sim\texttt{res/math133/geometrynotes2b.pdf}$$

Many nontrivial results in Euclidean geometry can be proved for arbitrary affine three-spaces. We shall limit ourselves to two examples here and leave others as exercises. In Theorems 7 and 8 below, the triple $(S, \mathcal{L}, \mathcal{P})$ will denote an arbitrary affine three-dimensional incidence space.

THEOREM II.7. *Two lines which are parallel to a third line are parallel.*

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[1] JOHN PLAYFAIR (1748–1819) was a Scottish scientist who is also known for an influential book on the philosophy of science; in his geometrical writings, he acknowledged that others had previously considered **EPP** as an axiom for geometry.

[2] The writings of Proclus provide valuable information on important, but apparently lost, works of earlier Greek mathematicians.

Proof.   There are two cases, depending on whether or not all three lines lie in a single plane; to see that the three lines need not be coplanar in ordinary 3-dimensional coordinate geometry, consider the three lines in $\mathbb{R}^3$ given by the z-axis and the lines joining $(1, 0, 0)$ and $(0, 1, 0)$ to $(1, 0, 1)$ and $(0, 1, 1)$ respectively.

THE COPLANAR CASE. Suppose that we have three distinct lines L, M, N in a plane P such that $L||N$ and $M||N$; we want to show that $L||N$.

If L is not parallel to N, then there is some point $\mathbf{x} \in L \cap N$, and it follows that L and N are distinct lines through $\mathbf{x}$, each of which is parallel to M. However, this contradicts the Euclidean Parallel Postulate. Therefore the lines L and N cannot have any points in common.

THE NONCOPLANAR CASE. Let α be the plane containing L and M, and let β be the plane containing M and N. By the basic assumption in this case we have $\alpha \neq \beta$. We need to show that $L \cap N = \varnothing$ but L and N are coplanar.

The lines L and N are disjoint.   Assume that the L and N have a common point that we shall call $\mathbf{x}$. Let γ be the plane determined by $\mathbf{x}$ and N (since $L||M$ and $\mathbf{x} \in L$, clearly $\mathbf{x} \notin M$). Since $\mathbf{x} \in L \subset \alpha$ and $M \subset \alpha$, Theorem 6 implies that $\alpha = \gamma$. A similar argument shows that $\beta = \gamma$ and hence $\alpha = \beta$; the latter contradicts our basic stipulation that $\alpha \neq \beta$, and therefore it follows that L and N cannot have any points in common.
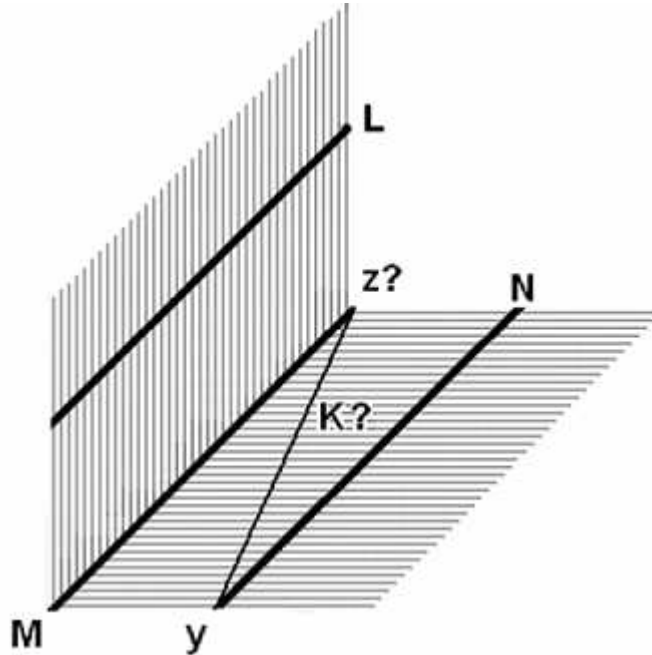
Figure II.1

The lines L and N are coplanar.   Let $\mathbf{y} \in N$, and consider the plane $L\mathbf{y}$. Now L cannot be contained in β because $\beta \neq \alpha = LM$ and $M \subset \beta$. By construction the planes $L\mathbf{y}$ and β have the point $\mathbf{y}$ in common, and therefore we know that $L\mathbf{y}$ meets β in some line K. Since L and K are coplanar, it will suffice to show that $N = K$. Since N and K both contain $\mathbf{y}$ and all three lines M, N and K are contained in β, it will suffice to show that $K||M$.

Suppose the lines are not parallel, and let $\mathbf{z} \in K \cap M$. Since $L||M$ it follows that $\mathbf{z} \notin L$. Furthermore, $L \cup K \subset L\mathbf{y}$ implies that $\mathbf{z} \in L\mathbf{y}$, and hence $\mathbf{y} = L\mathbf{z}$. Since $\mathbf{z} \in M$ and L and M are coplanar, it follows that $M \subset L\mathbf{z}$. Thus M is contained in $L\mathbf{y} \cap \beta$, and since the latter is the

line K, this shows that $M = K$. On the other hand, by construction we know that $M \cap N = \varnothing$ and $K \cap N \neq \varnothing$, so that M and K are obviously distinct. This contradiction implies that $K \| M$ must hold.∎

The next result is an analog of the Parallel Postulate for parallel planes.

THEOREM II.8. *If P is a plane and $\mathbf{x} \notin P$, then there is a unique plane Q such that $\mathbf{x} \in Q$ and $P \cap Q = \varnothing$.*

Proof.   Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in P$ be the noncollinear points, and consider the lines A', B' through $\mathbf{x}$ which are parallel to $A = \mathbf{bc}$ and $B = \mathbf{ac}$. Let Q be the plane determined by A' and B', so that $\mathbf{x} \in Q$ by hypothesis. We claim that $P \cap Q = \varnothing$.

Assume the contrary; since $\mathbf{x} \in QA$ and $\mathbf{x} \notin P$, the intersection $P \cap Q$ is a line we shall call L.
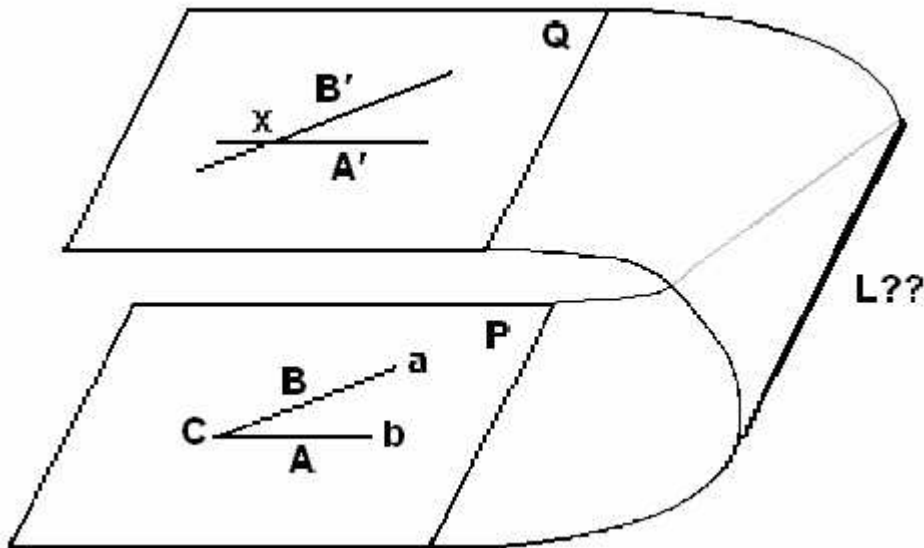
Figure II.2

STEP 1.   We shall show that $L \neq A$, B. The proof that $L \neq A$ and $L \neq B$ are similar, so we shall only show $L \neq A$ and leave the other part as an exercise. — If $L = A$, then $L \subset Q$. Since A' is the unique line in Q which is parallel to A, there must be a point $\mathbf{u} \in B' \cap A$. Consider the plane $B'\mathbf{c}$. Since $\mathbf{c} \in A$, it follows that $A \subset B'\mathbf{c}$. Hence $B'\mathbf{c}$ is a plane containing A and B. The only plane which satisfies these conditions is P, and hence $B' \subset P$. But $\mathbf{x} \in B'$ and $\mathbf{x} \notin P$, so we have a contradiction. Therefore we must have $L \neq A$.

STEP 2.   We claim that either $A' \cap L$ and $A \cap L$ are both nonempty or else $B' \cap L$ and $B \cap L$ are both nonempty. — We shall only show that if either $A' \cap L$ is $A \cap L$ is empty then both $B' \cap L$ and $B \cap L$ are both nonempty, since the other case follows by reversing the roles of A and B. Since L and A both lie in the plane P, the condition $A \cap L = \varnothing$ implies $A \| L$. Since $A \| A'$, by Theorem 7 and Theorem 1 we know that $A' \| L$. Since $B \neq A$ is a line through the point $\mathbf{c} \in A$, either $B = L$ or $B \cap L \neq \varnothing$ holds by the Parallel Postulate (in fact, $B \neq L$ by Step

1). Likewise, B and B' are lines through $\mathbf{x}$ in the plane Q and $L \subset Q$, so that the $A'||L$ and the Parallel Postulate imply $B' \cap L \neq \varnothing$.

STEP 3.   There are two cases, depending upon whether $A' \cap L$ and $A \cap L$ are both nonempty or $B' \cap L$ and $B \cap L$ are both nonempty. Only the latter will be considered, since the former follows by a similar argument. Let $\mathbf{y} \in B \cap L$ and $\mathbf{z} \in B' \cap L$; since $B \cap B' = \varnothing$, if follows that $\mathbf{y} \neq \mathbf{z}$ and hence $L = \mathbf{yz}$. Let β be the plane BB'. Then $L \subset \beta$ since $\mathbf{z}$, $\mathbf{y} \in \beta$. Since $L \neq B$, the plane β is the one determined by L and B. But L, $B \subset P$ by assumption, and hence $\beta = P$. In other words, B' is contained in P. But $\mathbf{x} \in B'$ and $\mathbf{x} \notin P$, a contradiction which shows that the line L cannot exist.∎

Following standard terminology, we shall say that the plane Q is *parallel* to P or that it is the plane parallel to P which passes through $\mathbf{x}$.

Corresponding definitions for incidence planes and affine planes exist, and analogs of Theorems 1, 2, 3 and 7 hold for these objects. However, incidence planes have far fewer interesting properties than their three-dimensional counterparts, and affine planes are best studied using the methods of projective geometry that are developed in later sections of these notes.

EXERCISES

Definition.   A line and a plane in a three-dimensional incidence space are *parallel* if they are disjoint.

> *Exercises **1**–**4** are to be proved for arbitrary 3-dimensional incidence spaces.*

1.   Suppose that each of two intersecting lines is parallel to a third line. Prove that the three lines are coplanar.

2.   Suppose that the lines L and L' are coplanar, and there is a line M not in this plane such that $L||M$ and $L'||M$. Prove that $L||L'$.

3.   Let P and Q be planes, and assume that each line in P is parallel to a line in Q. Prove that P is parallel to Q.

4.   Suppose that the line L is contained in the plane P, and suppose that $L||L'$. Prove that either L' is parallel to P or else $L \subset P$.

> *In exercises **5**–**6**, assume the incidence space is affine.*

5.   Let P and Q be parallel planes, and let L be any line which contains a point of Q and is parallel to a line in P. Prove that L is contained in Q. [*Hint:* Let M be the line in P, and let $\mathbf{x} \in L \cap Q$. Prove that $L = M\mathbf{x} \cap Q$.]

6.   Two lines are said to be **skew lines** if they are not coplanar. Suppose that L and M are skew lines. Prove that there is a unique plane P such that $L \subset P$ and P is parallel to M. [*Hint:* Let $\mathbf{x} \in L$, let M' be a line parallel to M which contains $\mathbf{x}$, and consider the plane LM'.]

2. Affine subspaces of vector spaces

Let $\mathbb{F}$ be a field, and let V be a vector space over $\mathbb{F}$ (in fact, everything in this section goes through if we take $\mathbb{F}$ to be a **skew-field** as described in Section A.1 in Appendix A). Motivated by Section I.3, we define *lines* and *planes* in V to be translates of 1- and 2-dimensional vector subspaces of V. Denote these families of lines and planes by $\mathcal{L}_V$ and $\mathcal{P}_V$ respectively. If $\dim V \geq 3$ we shall prove that $(V, \mathcal{L}_V, \mathcal{P}_V)$ satisfies all the conditions in the definition of an affine incidence 3-space except perhaps for the incidence axiom $\mathbf{I - 5}$, and we shall show that the latter also holds if $\dim V = 3$.

THEOREM II.9. *If V, etc. are as above and $\dim V \geq 3$, then $\mathcal{L}_V$ and $\mathcal{P}_V$ are nonempty disjoint families of proper subsets of V.*

Proof.   Since $\dim V \geq 3$ there are 1- and 2-dimensional vector subspaces of V, and therefore the families $\mathcal{L}_V$ and $\mathcal{P}_V$ are both nonempty. If we have $Y \in \mathcal{L}_V \cap \mathcal{P}_V$, then we may write
$$Y \;=\; \mathbf{x} \;+\; W_1 \;=\; \mathbf{y} \;+\; W_2$$
where $\dim W_i = i$. By Theorem I.4 we know that $\mathbf{y} \in \mathbf{x}+W_1$ implies the identity $\mathbf{x}+W_1 = \mathbf{y}+W_1$, and therefore Theorem I.3 implies
$$W_2 \;=\; -\mathbf{y} \;+\; (\mathbf{y} \;+\; W_1) \;=\; -\mathbf{y} \;+\; (\mathbf{y} \;+\; W_2) \;=\; W_2 \;.$$
Since $\dim W_1 \neq \dim W_2$ this is impossible. Therefore the families $\mathcal{L}_V$ and $\mathcal{P}_V$ must be disjoint. To see that an element of either family is a proper subset of V, suppose to the contrary that $\mathbf{x} + W = V$, where $\dim W = 1$ or 2. Since $\dim W < 3 \leq \dim V$, it follows that W is a proper subset of V; let $\mathbf{v} \in V$ be such that $\mathbf{v} \notin W$. By our hypothesis, we must have $\mathbf{x} + \mathbf{v} \in \mathbf{x} + W$, and thus we also have
$$\mathbf{v} \;=\; -\mathbf{x} \;+\; (\mathbf{x} \;+\; \mathbf{v}) \;\in\; -\mathbf{x} \;+\; (\mathbf{x} \;+\; W) \;=\; W$$
which contradicts our fundamental condition on $\mathbf{x}$. The contradiction arises from our assumption that $\mathbf{x}+W = V$, and therefore this must be false; therefore the sets in $\mathcal{L}_V$ and $\mathcal{P}_V$ are all proper subsets of V.∎

THEOREM II.10. *Every line in V contains at least two points, and every plane contains at least three points.*

Proof.   Let $\mathbf{x}+W$ be a line or plane in V, and let $\{\, \mathbf{w}_1 \,\}$ or $\{\, \mathbf{w}_1, \mathbf{w}_2 \,\}$ be a basis for W depending upon whether $\dim W$ equals 1 or 2. Take the subsets $\{\, \mathbf{v}, \mathbf{v} + \mathbf{w}_1, \,\}$ or $\{\, \mathbf{v}, \mathbf{v} + \mathbf{w}_1, \mathbf{v} + \mathbf{w}_2 \,\}$ in these respective cases.∎

THEOREM II.11. *Given two distinct points in V, there is a unique line containing them.*

Proof.   Let $\mathbf{x} \neq \mathbf{y}$ be distinct points in V, and let L_0 be the 1-dimensional vector subspace spanned by the nonzero vector $\mathbf{y} - \mathbf{x}$. Then $L = \mathbf{x} + L_0$ is a line containing $\mathbf{x}$ and $\mathbf{y}$. Suppose now that M is an arbitrary line containing $\mathbf{x}$ and $\mathbf{y}$. Write $M = \mathbf{z}+W$ where $\dim W = 1$. Then Theorem I.4 and $\mathbf{x} \in M = \mathbf{z} + W$ imply that $M = \mathbf{x} + W$. Furthermore, $\mathbf{y} \in M = \mathbf{x} + W$ then implies that $\mathbf{y} - \mathbf{x} \in W$, and since the latter vector spans L_0 it follows that $L_0 \subset W$. However, $\dim L_0 = \dim W$, and therefore $L_0 = W$ (see Theorem A.8). Thus the line $M = \mathbf{z} + W$ must be equal to $\mathbf{x} + L_0 = L$.∎

THEOREM II.12. *Given three points in V that are not collinear, there is a unique plane containing them.*

Proof. Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be the noncollinear points. If $\mathbf{y} - \mathbf{x}$ and $\mathbf{z} - \mathbf{x}$ were linearly dependent, then there would be a 1-dimensional vector subspace W containing them and hence the original three points would all lie on the line $\mathbf{x} + W$. Therefore we know that $\mathbf{y} - \mathbf{x}$ and $\mathbf{z} - \mathbf{x}$ are **linearly independent**, and thus the vector subspace W they span is 2-dimensional. If $P = \mathbf{x} + W$, then it follows immediately that P is a plane containing $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$. To prove uniqueness, suppose that $\mathbf{v} + U$ is an arbitrary plane containing all three points. As before, we must have $\mathbf{v} + U = \mathbf{x} + U$ since $bfx \in \mathbf{v} + U$, and since we also have $\mathbf{y}$, $\mathbf{z} \in \mathbf{v} + U = \mathbf{x} + U$ it also follows as in earlier arguments that $\mathbf{y} - \mathbf{x}$ and $\mathbf{z} - \mathbf{x}$ lie in U. Once again, since the two vectors in question span the subspace W, it follows that $W \subset U$, and since the dimensions are equal it follows that $W = U$. Thus we have $\mathbf{v} + U = \mathbf{x} + W$, and hence there is only one plane containing the original three points.∎

THEOREM II.13. *If P is a plane in V and $\mathbf{x}$, $\mathbf{y} \in P$, then the unique line containing $\mathbf{x}$ and $\mathbf{y}$ is a subset of P.*

Proof. As before we may write $P = \mathbf{x} + W$, where W is a 2-dimensional subspace; we also know that the unique line joining $\mathbf{x}$ and $\mathbf{y}$ has the form $L = \mathbf{x} + L_0$, where L_0 is spanned by $\mathbf{y} - \mathbf{x}$. The condition $\mathbf{y} \in P$ implies that $\mathbf{y} - \mathbf{x} \in W$, and since W is a vector subspace it follows that $L_0 \subset W$. But this immediately implies that $L = \mathbf{x} + L_0 \subset \mathbf{x} + W = P$, which is what we wanted to prove.∎

THEOREM II.14. (Euclidean Parallelism Property) *Let L be a line in V, and let $\mathbf{y} \notin L$. Then there is a unique line M such that (i) $\mathbf{y} \in M$, (ii) $L \cap M = \varnothing$, (iii) L and M are coplanar.*

Proof. Write $L = \mathbf{x} + L_0$ where $\dim L_0 = 1$, and consider the line $M = \mathbf{y} + L_0$. Then M clearly satisfies the first condition. To see it satisfies the second condition, suppose to the contrary that there is some common point $\mathbf{z} \in L \cap M$. Then the identities

$$\mathbf{z} \in L = \mathbf{x} + L_0 \qquad \mathbf{z} \in M = \mathbf{y} + L_0$$

imply that $L = \mathbf{x} + L_0 = \mathbf{z} + L_0 = \mathbf{y} + L_0 = M$, which contradicts the basic conditions that $\mathbf{y} \in M$ but $\mathbf{y} \notin L$. Therefore $L \cap M = \varnothing$. Finally, to see that M also satisfies the third condition, let W be the subspace spanned by L_0 and $\mathbf{y} - \mathbf{x}$; the latter vector does not belong to L_0 because $\mathbf{y} \notin L$, and therefore W must be a 2-dimensional vector subspace of V. If we now take $P = \mathbf{x} + W$, it follows immediately that $L \subset P$ and also

$$M = \mathbf{y} + L_0 \, \mathbf{x} + (\mathbf{y} - \mathbf{x}) + L_0 \subset \mathbf{x} + W = P$$

so that L and M are coplanar. Therefore M satisfies all of the three conditions in the theorem.

To complete the proof, we need to show that *there is only one line which satisfies all three conditions in the theorem.* — In any case, we know there is only one plane which contains L and $\mathbf{y}$, and hence it must be the plane $P = \mathbf{x} + W$ from the preceding paragraph. Thus if N is a line satisfying all three conditions in the theorem, it must be contained in P. Suppose then that $N = \mathbf{y} + L_1$ is an arbitrary line in P with the required properties. Since $\mathbf{y} + L_1 \in \mathbf{x} + W = \mathbf{y} + W$, it follows that $L_1 \subset W$. Therefore, if $\mathbf{0} \neq \mathbf{u} \in L_1$ we can write $\mathbf{u} = s(\mathbf{y} - \mathbf{x}) + t\mathbf{v}$, where $\mathbf{v}$ is a

nonzero vector in L_0 and s, $t \in \mathbb{F}$. The assumption that $L \cap N = \varnothing$ implies there are no scalars p and q such that

$$\mathbf{x} + p\mathbf{v} = \mathbf{y} + q\mathbf{u}$$

holds. Substituting for $\mathbf{u}$ in this equation, we may rewrite it in the form

$$p\mathbf{v} = (\mathbf{y} - \mathbf{x}) + q\mathbf{u} = (1 + sq)(\mathbf{y} - \mathbf{x}) + qt\mathbf{v}$$

and hence by equating coefficients we cannot find p and q such that $p = qt$ and $1 + sq = 0$. Now if $s \neq 0$, then these two equations have the solution $q = -s^{-1}$ and $p = -s^{-1}t$. Therefore, if there is no solution then we must have $s = 0$. The latter in turn implies that $\mathbf{u} = t\mathbf{v}$ and hence $L_1 = L_0$, so that $N = \mathbf{y} + L_1 = \mathbf{y} + L_0 = M$.∎

THEOREM II.15. *If* $\dim V = 3$ *and two planes in* V *have a nonempty intersection, then their intersection contains a line.*

Proof. Let P and Q be planes, and let $\mathbf{x} \in P \cap Q$. Write $P = \mathbf{x} + W$ and $Q = \mathbf{x} + U$, where $\dim W = \dim U = 2$. Then

$$P \cap Q = (\mathbf{x} + W) \cap (\mathbf{x} + U)$$

clearly contains $\mathbf{x} + (W \cap U)$, so it suffices to show that the vector subspace $W \cap U$ contains a 1-dimensional vector subspace. However, we have

$$\dim(W \cap U) = \dim W - \dim U - \dim(W + U) = 4 - \dim(W + U)$$

and since $\dim(W + U) \leq \dim V = 3$, the displayed equation immediately implies $\dim(W \cap U) \geq 4 - 3 = 1$. Hence the intersection of the vector subspaces is nonzero, and as such it contains a nonzero vector $\mathbf{z}$ as well as the 1-dimensional subspace spanned by $\mathbf{z}$.∎

The preceding results imply that Theorems 7 and 8 from Section 1, and the conclusions of Exercises 5 and 6 from that section, are all true for the models $(V, \mathcal{L}_V, \mathcal{P}_V)$ described above provided $\dim V = 3$. In some contexts it is useful to interpret the conclusions of the theorems or exercises in temrs of the vector space structure on V. For example, in Theorem 8 if P is the plane $\mathbf{x} + W$, then the parallel plane Q will be $\mathbf{y} + W$. Another example is discussed in Exercise 5 below.

Generalizing incidence to higher dimensions

The characterization of lines and planes as translates of 1- and 2-dimensional subspaces suggests a simple method for generalizing incidence structures to dimensions greater than three.[3] Namely, define a *k-plane* in a vector space V to be a translate of a k-dimensional vector subspace.

The following quotation from Winger, *Introduction to Projective Geometry*,[4] may help the reader understand the reasons for formulating the concepts of affine geometry in arbitrary dimensions.[5]

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[3]The explicit mathematical study of higher dimensional geometry began around the middle of the 19th century, particularly in the work of LUDWIG SCHLÄFLI (1814–1895), who also made contributions to other areas of mathematics. Many ideas in his work on higher dimensional geometry were independently discovered by others with the development of linear algebra during the second half of that century.

[4]See page 15 of that book.

[5]Actually, spaces of higher dimensions play an important role in theoretical physics. Einstein's use of a four-dimensional space-time is of course well-known, but the use of spaces with dimensions ≥ 4 in physics was at least implicit during much of the 19th century. In particular, 6-dimensional configuration spaces were implicit in work on celestial mechanics, and spaces of various dimensions were widely used in classical dynamics.

> The timid may console themselves with the reflection that the geometry of four and higher dimensions is, if not a necessity, certainly a convenience of language — a translation of the algebra — and let the philosophers ponder the metaphysical questions involved in the idea of a point set of higher dimensions.

We shall conclude this section with a characterization of k-planes in V, where V is finite-dimensional and $1 + 1 \neq 0$ in $\mathbb{F}$; in particular, the result below applies to the $V = \mathbb{R}^n$. An extension of this characterization to all fields except the field $\mathbb{Z}_2$ with two elements is given in Exercise 1 below.

Definition. Let V be a vector space over the field $\mathbb{F}$, and let $P \subset V$. We shall say P is a *flat subset* of V if for each pair of distinct points $\mathbf{x}$, $\mathbf{y} \in \mathbb{F}$ the line $\mathbf{xy}$ is contained in $\mathbb{F}$.

THEOREM II.16. *Let V be a finite-dimensional vector space over a field $\mathbb{F}$ in which $1 + 1 \neq 0$. Then a nonempty set $P \subset V$ is a flat subset if and only if it is a k-plane for some integer k satisfying $0 \leq k \leq \dim V$.*

Definition. A subset $S \subset V$ is said to be an *affine subspace* if can be written as $\mathbf{x} + W$, where $\mathbf{x} \in V$ and W is a vector subspace of V. With this terminology, we can restate the theorem to say that if $1 + 1 \neq 0$ in $\mathbb{F}$, then a nonempty subset of V is an affine subspace if and only if it is a flat subset.

Proof. We split the proof into the "if" and "only if" parts.

Every k-plane is a flat subset. Suppose that W is a k-dimensional vector subspace and $\mathbf{x} \in V$. Let $\mathbf{y}$, $\mathbf{z} \in \mathbf{x} + V$. Then we may write $\mathbf{y} = \mathbf{x} + \mathbf{u}$ and $\mathbf{z} = \mathbf{x} + \mathbf{v}$ for some distinct vectors $\mathbf{u}$, $\mathbf{v} \in W$. A typical point on the line $\mathbf{yz}$ has the form $\mathbf{y} + t(\mathbf{z} - \mathbf{y})$ for some scalar t, and we have

$$\mathbf{y} \; + \; t(\mathbf{z} - \mathbf{y}) \; = \; \mathbf{x} \; + \; \mathbf{u} \; + \; t(\mathbf{v} - \mathbf{u}) \; \in \; \mathbf{x} \; + \; W$$

which shows that $\mathbf{x} + W$ is a flat subset. Note that this implication does not require any assumption about the nontriviality of $1 + 1$.

Every flat subset has the form $\mathbf{x} + W$ for some vector subspace W.   If we know this, we also know that $k = \dim W$ is less than or equal to $\dim V$. — Suppose that $\mathbf{x} \in P$, and let $W = (-\mathbf{x}) + P$; we need to show that W is a vector subspace. To see that W is closed under scalar multiplication, note first that $\mathbf{w} \in W$ implies $\mathbf{x} + \mathbf{w} \in P$, so that flatness implies every point on the line $\mathbf{x}(\mathbf{x} + \mathbf{w})$ is contained in P. For each scalar t we know that $\mathbf{x} + t\mathbf{w}$ lies on this line, and thus each point of this type lies in $P = \mathbf{x} + W$. If we subtract $\mathbf{x}$ we see that $t\mathbf{w} \in W$ and hence W is closed under scalar multiplication.

To see that W is closed under vector addition, suppose that $\mathbf{w}_1$ and $\mathbf{w}_2$ are in W. By the previous paragraph we know that $2\mathbf{w}_1$ and $2\mathbf{w}_2$ also belong to W, so that $\mathbf{u}_1 = \mathbf{x} + 2\mathbf{w}_1$ and $\mathbf{u}_2 = \mathbf{x} + 2\mathbf{w}_2$ are in P. Our hypothesis on $\mathbb{F}$ implies the latter contains an element $\frac{1}{2} = (1+1)^{-1}$, so by flatness we also know that

$$\tfrac{1}{2}\left(\, \mathbf{u}_1 + \mathbf{u}_2 \,\right) \; + \; \tfrac{1}{2}\left(\, \mathbf{u}_2 - \mathbf{u}_1 \,\right) \; \in \; P \; .$$

If we now expand and simplify the displayed vector, we see that it is equal to $\mathbf{v} + \mathbf{w}_1 + \mathbf{w}_2$. Therefore it follows that $\mathbf{w}_1 + \mathbf{w}_2 \in W$, and hence W is a vector subspace of V.■

Not surprisingly, if S is an affine subspace of the finite-dimensional vector space V, then we define its *dimension* by $\dim S = \dim W$, where W is a vector subspace of V such that $S = \mathbf{x} + W$.

— *This number is well defined* because $S = \mathbf{x} + W = \mathbf{y} + U$ implies $\mathbf{y} \in \mathbf{x} + W$, so that $\mathbf{y} + W = \mathbf{x} + W = \mathbf{y} + U$ and hence

$$W = -\mathbf{y} + (\mathbf{y} + W) = -\mathbf{y} + (\mathbf{y} + U) = U .\blacksquare$$

Hyperplanes

One particularly important family of affine subspaces in a finite-dimensional vector space V is the set of all **hyperplanes** in V. We shall conclude this section by defining such objects and proving a crucial fact about them.

Definition. Let n be a positive integer, and let V be an n-dimensional vector space over the field $\mathbb{F}$. A subset $H \subset V$ is called a *hyperplane* in V if H is the translate of an $(n-1)$-dimensional subspace. — In particular, if $\dim V = 3$ then a hyperplane is just a plane and if $\dim V = 2$ then a hyperplane is just a line.

One reason for the importance of hyperplanes is that if $k < \dim V$ then every k-plane is an intersection of finitely many hyperplanes (see Exercise II.5.4).

We have seen that planes in $\mathbb{R}^3$ and lines in $\mathbb{R}^2$ are describable as the sets of all points $\mathbf{x}$ which satisfy a nontrivial first degree equation in the coordinates (x_1, x_2, x_3) or (x_1, x_2) respectively (see Theorems I.1, I.5 and I.7). The final result of this section is a generalization of these facts to arbitrary hyperplanes in $\mathbb{F}^n$, where $\mathbb{F}$ is an arbitrary field.

THEOREM II.17. *In the notation of the previous paragraph, let H be a nonempty subset of $\mathbb{F}^n$. Then H is a hyperplane if and only if there exist scalars $c_1, \cdots, c_n$ which are not all zero such that H is the set of all $\mathbf{x} \in \mathbb{F}^n$ whose coordinates $x_1, \cdots, x_n$ satisfy the equation*

$$\sum_{i=0}^{n} c_i x_i = b$$

for some $b \in \mathbb{F}$.

Proof. Suppose that H is defined by the equation above. If we choose j such that $a_j \neq 0$ and let $\mathbf{e}_j$ be the unit vector whose j^{th} coordinate is 1 and whose other coordinates are zero, then we have $-a_j^{-1} b \, \mathbf{e}_j \in H$, and hence the latter is nonempty. Set W equal to $(-\mathbf{z}) + H$, where $\mathbf{z} \in H$ is fixed. As in Chapter I, it follows that $\mathbf{y} \in W$ if and only if its coordinates $y_1, \cdots, y_n$ satisfy the equation $\sum_i a_i y_i = 0$. Since the coefficients a_i are not all zero, it follows from Theorem A.10 that W is an $(n - 1)$-dimensional vector subspace of $\mathbb{F}^n$, and therefore $H = \mathbf{x} + W$ is a hyperplane.$\square$

Conversely, suppose H is a hyperplane and write $\mathbf{x} + W$ for a suitable vector $\mathbf{x}$ and $(n - 1)$-dimensional subspace W. Let $\{\mathbf{w}_1, \cdots, \mathbf{w}_{n-1}\}$ be a basis for W, and write these vectors out in coordinate form:

$$\mathbf{w}_i = (w_{i,1}, \cdots, w_{i,n})$$

If B is the matrix whose rows are the vectors $\mathbf{w}_i$, then the rank of B is equal to $(n - 1)$ by construction. Therefore, by Theorem A.10 the set Y of all $\mathbf{y} = (y_1, \cdots, y_n)$ which solve the system

$$\sum_j y_j w_{i,j} = 0 \qquad \left(1 \leq i \leq (n-1)\right)$$

is a 1-dimensional vector subspace of $\mathbb{F}^n$. Let $\mathbf{a}$ be a nonzero (hence spanning) vector in Y.

We claim that $\mathbf{z} \in W$ if and only if $\sum_i a_i z_i = 0$. By construction, W is contained in the subspace S of vectors whose coordinates satisfy this equation. By Theorem A.10 we know that $\dim S = (n-1)$, which is equal to $\dim W$ by our choice of the latter; therefore Theorem A.9 implies that $W = S$, and it follows immediately that H is the set of all $\mathbf{z} \in \mathbb{F}^n$ whose coordinates $z_1, \cdots, z_n$ satisfy the nontrivial linear equation $\sum_i a_i z_i = \sum_i a_i x_i$ (where $\mathbf{x} = (x_1, \cdots, x_n) \in W$ is the fixed vector chosen as in the preceding paragraph).■

EXERCISES

1. Prove that Theorem 16 remains true for every field **except** $\mathbb{Z}_2$. Give an example of a flat subspace of $(\mathbb{Z}_2)^3$ which is not a k-plane for some k.

2. Let $\mathbb{F}$ be a field. Prove that the lines in $\mathbb{F}^2$ defined by the equations $ax + by + cz = 0$ and $a'x + by + c'z = 0$ (compare Theorem I.7 and Theorem 17 above) are parallel or identical if and only if $ab' - ba' = 0$.

3. Find the equation of the hyperplane in $\mathbb{R}^4$ passing through the (noncoplanar) points $(1,0,1,0)$, $(0,1,0,1)$, $(0,1,1,0)$, and $(1,0,0,1)$.

4. Suppose that $\mathbf{x}_1 + W_1$ and $\mathbf{x}_2 + W_2$ are k_1- and k_2-planes in a vector space V such that $\mathbf{x}_1 + W_1 \cap \mathbf{x}_2 + W_2 \neq \varnothing$. Let $\mathbf{z}$ be a common point of these subsets. Prove that their intersection is equal to $\mathbf{z} + W_1 \cap \mathbf{z} + W_2$, and generalize this result to arbitrary finite intersections.

5. Let V be a 3-dimensional vector space over $\mathbb{F}$, and suppose we are given the configuration of Exercise II.1.6 (L and M are skew lines, and P is a plane containing L but parallel to M). Suppose that the skew lines are given by $\mathbf{x} + W$ and $\mathbf{y} + U$. Prove that the plane P is equal to $\mathbf{x} + (U + W)$ [*Hint:* Show that the latter contains L and is disjoint from M.].

6. Suppose that $\dim V = n$ and $H = \mathbf{x} + W$ is a hyperplane in V. Suppose that $\mathbf{y} \in V$ but $\mathbf{y} \notin H$. Prove that $H' = \mathbf{y} + W$ is the unique hyperplane K such that $\mathbf{y} \in K$ and $H \cap K = \varnothing$. [*Hints:* If $\mathbf{z} \in H \cap H'$ then $N = \mathbf{x} + W = \mathbf{z} + W = \mathbf{y} + W = H'$. If $K = \mathbf{y} + U$ where U is some $(n-1)$-dimensional vector subspace different from W, explain why $\dim W \cap U = n - 2$. Choose a basis A of $(n-2)$ vectors for this intersection, and let $\mathbf{u}_0 \in U$, $\mathbf{w}_0 \in W$ such that $\mathbf{u}_0, \mathbf{w}_0 \notin U \cap W$. Show that $A \cup \{\mathbf{u}_0, \mathbf{w}_0\}$ is a basis for V, write $\mathbf{y} - \mathbf{x}$ in terms of this basis, and use this equation to find a vector which lies in $H \cap K = (\mathbf{x} + W) \cap (\mathbf{y} + U)$.]

7. Given an example of two planes P_1, P_2 in $\mathbb{R}^4$ whose intersection consists of a single point. [*Hint:* There are many examples for which the P_i are vector subspaces.]

3. Affine bases

We shall need analogs of linear independence and spanning that apply to arbitrary k-planes and not just k-dimensional vector subspaces. The starting points are two basic observations.

THEOREM II.18. *Suppose that S is a k-plane in a vector space V over the field $\mathbb{F}$. Given $\mathbf{a}_1, \cdots, \mathbf{a}_r \in S$, let $t_1, \cdots, t_r \in S$ be such that $\sum_j t_j = 1$. Then $\sum_j t_j \mathbf{a}_j \in S$.*

Proof. Write $S = \mathbf{x} + W$, where $\mathbf{x} \in S$ and W is a k-dimensional vector subspace, and for each i write $\mathbf{a}_i = \mathbf{x} + \mathbf{w}_i$ where $\mathbf{w}_i \in \mathbb{F}$. Then

$$\sum_j t_j \mathbf{a}_j = \sum_j t_j(\mathbf{x} + \mathbf{w}_j) = \sum_j t_j \mathbf{x} + \sum_j t_j \mathbf{w}_j = \mathbf{x} + \sum_j t_j \mathbf{w}_j$$

where the latter holds because $\sum_j t_j = 1$. Since W is a vector subspace, we know that $\sum_j t_j \mathbf{w}_j \in W$, and therefore it follows that $\sum_j t_j \mathbf{a}_j \in W$.∎

THEOREM II.19. *If V is as above and $T \subset V$ is an arbitrary subset, define the **affine hull** of T by*

$$\mathcal{H}(T) = \{\, \mathbf{x} \in V \mid \mathbf{x} = \sum_j t_j \mathbf{w}_j, \text{ where } \mathbf{v}_i \in T \text{ for all } t \text{ and } \sum_j t_j = 1\}$$

(note that the sum is finite, but T need not be finite). Then $\mathcal{H}(T)$ is an affine subspace of V.

Sometimes we shall also say that $\mathcal{H}(T)$ is the *affine span* of T, and we shall say that T *affinely spans* an affine subspace S if $S = \mathcal{H}(T)$.

Proof. Suppose that $\mathbf{x}, \mathbf{y} \in \mathcal{H}(T)$, and write

$$\mathbf{x} = \sum_i s_i \mathbf{u}_i = \sum_j t_j \mathbf{v}_j$$

where $\mathbf{u}_i \in T$ and $\mathbf{v}_j \in T$ for all i and j, and the coefficients satisfy $\sum_i s_i = \sum_j t_j = 1$. We need to show that $\mathbf{x} + c(\mathbf{y} - \mathbf{x}) \in \mathcal{H}(T)$. But

$$\mathbf{x} + c(\mathbf{y} - \mathbf{x}) = \sum_i s_i \mathbf{u}_i + c \cdot \left(\sum_j t_j \mathbf{v}_j - \sum_i s_i \mathbf{u}_i \right) =$$

$$(1 - c) \cdot \sum_i s_i \mathbf{u}_i + c \cdot \sum_j t_j \mathbf{v}_j = \sum_i s_i (1 - c) \mathbf{u}_i + \sum_j t_j c \mathbf{v}_j \,.$$

We have no a *priori* way of knowing whether any of the vectors $\mathbf{u}_i$ are equal to any of the vectors $\mathbf{v}_j$, but in any case we can combine like terms to rewrite the last expression as $\sum_q r_q \mathbf{w}_q$, where the vectors $\mathbf{w}_q$ run through all the vectors $\mathbf{u}_i$ and $\mathbf{v}_j$, and the coefficients r_q are given accordingly; by construction, we then have

$$\sum_q r_q = \sum_i s_i (1 - c) + \sum_j t_j c = (1 - c) \cdot \sum_i s_i + c \cdot \sum_j t_j = (1 - c) \cdot 1 + c \cdot 1 = 1$$

and therefore it follows that the point $\mathbf{x} + c(\mathbf{y} - \mathbf{x})$ belongs to $\mathcal{H}(T)$.∎

Thus *a linear combination of points in an affine subspace will also lie in the subspace* **provided** *the coefficients add up to* 1, and by Theorem 19 this is the most general type of linear combination one can expect to lie in S.

Definition.   A vector $\mathbf{v}$ is an *affine combination* of the vectors $\mathbf{x}_0, \cdots, \mathbf{x}_n$ if we have $\mathbf{x} = \sum_j t_j \mathbf{x}_j$, where $\sum_j t_j = 1$. Thus the affine hull $\mathcal{H}(T)$ of a set T is the set of all (finite) affine combinations of vectors in T.

AFFINE VERSUS LINEAR COMBINATIONS. If a vector $\mathbf{y}$ is a linear combination of the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$, then it is automatically an affine combination of $\mathbf{0}, \mathbf{v}_1, \cdots, \mathbf{v}_n$, for if $\mathbf{y} = \sum_i t_i \mathbf{x}_i$ then

$$\mathbf{y} \;=\; \left(1 \;-\; \sum_j t_j\right) \cdot \mathbf{0} \;+\; \sum_i t_i \mathbf{x}_i \;.$$

Definition.   Let V be a vector space over a field $\mathbb{F}$. and let $X \subset V$ be the set $\{\mathbf{x}_0, \cdots, \mathbf{x}_n\}$. We shall say that X is *affinely dependent* if one element of X is expressible as an affine combination of the others and *affinely independent* otherwise. By convention, one point subsets are affinely independent.

The next result gives the fundamental relationship between affine dependence and linear dependence (and, by taking negations, it also gives the fundamental relationship between affine independence and linear independence).

THEOREM II.20. *In the setting above, the finite set $X = \{\mathbf{x}_0, \cdots, \mathbf{x}_n\} \subset V$ is affinely dependent if and only if the set $X' = \{\mathbf{x}_1 - \mathbf{x}_0, \cdots, \mathbf{x}_n - \mathbf{x}_0\}$ is linearly dependent. Likewise, the finite set $X = \{\mathbf{x}_0, \cdots, \mathbf{x}_n\} \subset V$ is affinely independent if and only if the set $X' = \{\mathbf{x}_1 - \mathbf{x}_0, \cdots, \mathbf{x}_n - \mathbf{x}_0\}$ is linearly independent.*

Proof.   Since affine dependence and affine independence are the negations of each other and similarly for linear dependence and linear independence, it will suffice to prove the first conclusion in the theorem.

Proof that X is affinely dependent if X' is linearly dependent.   By linear dependence there is some $k > 0$ such that

$$\mathbf{x}_k - \mathbf{x}_0 \;=\; \sum_{i \neq 0, k} c_i \left(\mathbf{x}_i - \mathbf{x}_0\right)$$

and therefore we also have

$$\mathbf{x}_k \;=\; \mathbf{x}_0 \;+\; \sum_{i \neq 0, k} c_i \, \mathbf{x}_i \;-\; \sum_{i \neq 0, k} c_i \, \mathbf{x}_0 \;=\; \left(1 \;-\; \sum_{i \neq 0, k} c_i\right) \mathbf{x}_0 \;+\; \sum_{i \neq 0, k} c_i \, \mathbf{x}_i \;.$$

Therefore $\mathbf{x}_k$ is also an affine combination of all the $\mathbf{x}_j$ such that $j \neq k$.

Proof that X' is linearly dependent if X is affinely dependent.   By assumption there is some k such that

$$\mathbf{x}_k \;=\; \sum_{j \neq k} c_j \, \mathbf{x}_j$$

where $\sum_{j \neq k} c_j = 1$. Therefore we also have

$$\mathbf{x}_k - \mathbf{x}_0 = \left(\sum_{j \neq k} c_j \, \mathbf{x}_j \right) - \mathbf{x}_0 = \sum_{j \neq k} c_j (\mathbf{x}_j - \mathbf{x}_0) \ .$$

Note that we can take the summation on the right hand side to run over all j such that $j \neq k$, 0 because $\mathbf{x}_0 - \mathbf{x}_0 = \mathbf{0}$.

There are now two cases depending on whether $k > 0$ or $k = 0$. In the first case, we have obtained an expression for $\mathbf{x}_k - \mathbf{x}_0$ in terms of the other vectors in X', and therefore X' is linearly dependent. Suppose now that $k = 0$, so that the preceding equation reduces to

$$\mathbf{0} = \sum_{j > 0} c_j (\mathbf{x}_j - \mathbf{x}_0) \ .$$

Since $\sum_{j > 0} c_j = 1$ it follows that $c_m \neq 0$ for some $m > 0$, and this now implies

$$\mathbf{x}_m - \mathbf{x}_0 = \sum_{j \neq m, 0} \frac{-c_j}{c_m} (\mathbf{x}_j - \mathbf{x}_0)$$

which shows that X' is linearly dependent.$\blacksquare$

One important characterization of linear independence for a set Y is that an arbitrary vector has at most one expression as a linear combination of the vectors in Y. There is a similar characterization of affine independence.

THEOREM II.21. *A (finite) set X of vectors in a given vector space V is affinely independent if and only if every vector in V has* **at most one** *expression as an affine combination of vectors in X.*

Proof. Suppose X is affinely independent and that

$$\mathbf{y} = \sum_j t_j \mathbf{v}_j = \sum_j s_j \mathbf{v}_j$$

where $\mathbf{v}_j$ runs through the vectors in V and

$$\sum_j t_j = \sum_j s_j = 1 \ .$$

Then we have

$$\left(\sum_j t_j \mathbf{v}_j \right) - \mathbf{v}_0 = \left(\sum_j s_j \mathbf{v}_j \right) - \mathbf{v}_0$$

which in turn implies

$$\left(\sum_j t_j \mathbf{v}_j \right) - \left(\sum_j t_j \right) \mathbf{v}_0 = \left(\sum_j s_j \mathbf{v}_j \right) - \left(\sum_j s_j \right) \mathbf{v}_0$$

so that we have

$$\sum_{j > 0} t_j (\mathbf{v}_j - \mathbf{v}_0) = \sum_j s_j (\mathbf{v}_j - \mathbf{v}_0) \ .$$

Since the vectors $\mathbf{v}_j - \mathbf{v}_0$ (where $j > 0$) is linearly independent, it follows that $t_j = s_j$ for all $j > 0$. Once we know this, we can also conclude that $t_0 = 1 - \sum_{j>0} t_j$ is equal to $1 - \sum_{j>0} s_j = s_0$, and therefore all the corresponding coefficients in the two expressions are equal. $\square$

Conversely, suppose X satisfies the uniqueness condition for affine combinations, and suppose that we have

$$\sum_{j>0} c_j(\mathbf{v}_j - \mathbf{v}_0) = \mathbf{0}.$$

We then need to show that $c_j = 0$ for all j. But the equation above implies that

$$\mathbf{v}_0 = \left(\sum_{j>0} c_j(\mathbf{v}_j - \mathbf{v}_0) \right) + \mathbf{v}_0$$

and if we simplify the right hand side we obtain the equation

$$\mathbf{v}_0 = \left(1 - \sum_{j>0} c_j \right) \mathbf{v}_0 + \sum_{j>0} c_j\mathbf{v}_j.$$

The coefficients on both sides add up to 1, so by the uniqueness assumption we must have $c_j = 0$ for all $j > 0$; but this implies that the vectors $\mathbf{v}_j - \mathbf{v}_0$ (where $j > 0$) are linearly independent. $\blacksquare$

Definition. If S is an affine subspace of the finite-dimensional vector space V and $T \subset S$ is a finite subset, then T is said to be an *affine basis* for S if T is affinely independent and affinely spans T.

There is a fundamental analog of the preceding results which relates affine bases of affine subspaces and vector space bases of vector subspaces.

THEOREM II.22. *Let V be a finite-dimensional vector space, let S be an affine subspace, and suppose that $S = \mathbf{z} + W$ for a suitable vector $\mathbf{z}$ and vector subspace V. Then the finite set $X = \{\mathbf{x}_0, \cdots, \mathbf{x}_m\} \subset S$ is an affine basis for S if and only if the set $X' = \{\mathbf{x}_1 - \mathbf{x}_0, \cdots, \mathbf{x}_n - \mathbf{x}_0\}$ is linear basis for W.*

Proof. First of all, since $\mathbf{x}_0 \in S$ we may write $S = \mathbf{x}_0 + W$ and forget about the vector $\mathbf{z}$.

Suppose that $X = \{\mathbf{x}_0, \cdots, \mathbf{x}_m\}$ is an affine basis for S, and let $\mathbf{y} \in W$. Since $\mathbf{x}_0 + \mathbf{y} \in \mathbf{x}_0 + W = S$, there exist $s_0, \cdots, s_m \in \mathbb{F}$ such that $\sum_i s_i = 1$ and $\mathbf{x}_0 + \mathbf{y} = \sum_i s_i\mathbf{x}_i$. Subtracting $\mathbf{x}_0$ from both sides and using the equation $\sum_i s_i = 1$, we see that

$$\mathbf{y} = \sum_{i>0} s_i(\mathbf{x}_i - \mathbf{x}_0)$$

and hence X' spans W. Since X' is linearly independent by Theorem 20, it follows that X' is a basis for W. $\square$

Conversely, suppose that $X' = \{\mathbf{x}_1 - \mathbf{x}_0, \cdots, \mathbf{x}_n - \mathbf{x}_0\}$ is linear basis for W. Since X' is linearly independent, by Theorem 20 we also know that X is affinely independent. To see that X affinely spans S, let $\mathbf{u} \in S$, and write $\mathbf{u} = \mathbf{x}_0 + \mathbf{v}$, where $\mathbf{v} \in S$. Since X' spans W we know that

$$\mathbf{u} = \mathbf{x}_0 + \sum_{i>0} s_i(\mathbf{x}_i - \mathbf{x}_0)$$

for appropriate scalars s_i, and if we set $s_0 = 1 - \sum_{i>0} s_i$, then we may rewrite the right hand side of the preceding equation as $\sum_{i \geq 0} s_i \mathbf{x}_i$, where by construction we have $\sum_{i \geq 0} s_i = 1$. Therefore X affinely spans S, and it follows that X must be an affine basis for X.$\blacksquare$

Definition. Suppose that we are in the setting of the theorem and X is an affine basis for S, so that each $\mathbf{y} \in S$ can be uniquely written as an affine combination $\sum_i t_i \mathbf{x}_i$, where $\sum_i t_i = 1$. Then the unique coefficients t_i are called the *barycentric coordinates* of $\mathbf{y}$ with respect to X. The physical motivation for this name is simple: Suppose that $\mathbb{F}$ is the real numbers and we place weights $w_i > 0$ at the vectors $\mathbf{v}_i$ in S such that the total weight is w units. Let $t_i = w_i/w$ be the normalized weight at $\mathbf{v}_i$; then $\sum_i t_i \mathbf{x}_i$ is the center of gravity for the resulting physical system (a version of this is true even if one allows some of the coefficients t_i to be negative).

In analogy for linear bases of vector subspaces, the number of elements in an affine basis for an affine subspace S depends on S itself. However, as illustrated by the final result of this section, there is a crucial difference in the formula relating the dimension of S to the number of elements in an affine basis.

THEOREM II.23. *If V is a finite-dimensional vector space and S is an affine subspace, then S has an affine basis. Furthermore, if we write $S = \mathbf{y} + W$ for suitable $\mathbf{y}$ and W, then every affine basis for S has exactly $\dim W + 1$ elements.*

Proof. If $X = \{\mathbf{x}_0, \cdots, \mathbf{x}_m\} \subset S$ is an affine basis for $S = \mathbf{y} + W = \mathbf{x}_0 + W$, then $X' = \{\mathbf{x}_1 - \mathbf{x}_0, \cdots, \mathbf{x}_n - \mathbf{x}_0\}$ is a linear basis for W by Theorem 22, and conversely. Therefore the existence of an affine basis for S follows from Theorem 22 and the existence of a linear basis for W. Furthermore, since every linear basis for W contains exactly $\dim W$ elements, by Theorem 22 we know that every affine basis for S contains exactly $\dim W + 1$ elements.$\blacksquare$

EXERCISES

In the exercises below, assume that all vectors lie in a fixed finite-dimensional vector space V over a field $\mathbb{F}$.

1. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in V$ (a vector space over some field) be noncollinear and for $i = 1, 2, 3$ let $\mathbf{x}_i = t_i \mathbf{a} + u_i \mathbf{b} + v_i \mathbf{c}$, where $t_i + u_i + v_i = 1$. Prove that the points $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are collinear if and only if

$$\begin{vmatrix} t_1 & u_1 & v_1 \\ t_2 & u_2 & v_2 \\ t_3 & u_3 & v_3 \end{vmatrix} = 0$$

where the left hand side is a 3×3 determinant.

2. Prove the **Theorem of Menelaus**:[6] Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in V$ (a vector space over some field) be noncollinear, and suppose we have points $\mathbf{p} \in \mathbf{bc}$, $\mathbf{q} \in \mathbf{ac}$, and $\mathbf{r} \in \mathbf{ab}$. Write these three vectors as

$$\begin{aligned} \mathbf{p} &= \mathbf{b} + t(\mathbf{c} - \mathbf{b}) = t\mathbf{c} + (1-t)\mathbf{b} \\ \mathbf{q} &= \mathbf{a} + u(\mathbf{c} - \mathbf{a}) = u\mathbf{c} + (1-u)\mathbf{a} \\ \mathbf{r} &= \mathbf{a} + v(\mathbf{b} - \mathbf{a}) = v\mathbf{b} + (1-v)\mathbf{a} \end{aligned}$$

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[6] MENELAUS OF ALEXANDRIA (*c.* 70 A. D.–*c.* 130 A. D.) worked in geometry and astronomy, and he is particularly given credit for making numerous contributions to spherical geometry and trigonometry.

where t, u, v are appropriate scalars. Then $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are collinear if and only if

$$\frac{t}{1-t} \cdot \frac{u}{1-u} \cdot \frac{v}{1-v} = -1 \ .$$

3. Prove the **Theorem of Ceva**:[7] In the setting of the preceding exercise, the lines $\mathbf{ap}$, $\mathbf{bq}$ and $\mathbf{cr}$ are concurrent (there is a point which lies on all three lines) if and only if

$$\frac{t}{1-t} \cdot \frac{u}{1-u} \cdot \frac{v}{1-v} = +1 \ .$$

4. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in V$ (as above) be noncollinear, and suppose we have points $\mathbf{y} \in \mathbf{ba}$ and $\mathbf{x} \in \mathbf{bc}$ which are distinct from $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and satisfy these three vectors as

$$\begin{aligned} \mathbf{x} &= \mathbf{b} + t(\mathbf{a} - \mathbf{b}) = t\mathbf{a} + (1-t)\mathbf{b} \\ \mathbf{y} &= \mathbf{b} + u(\mathbf{c} - \mathbf{a}) = u\mathbf{c} + (1-u)\mathbf{b} \end{aligned}$$

where t, u are appropriate scalars (neither of which is 0 or 1). Prove that the lines $\mathbf{ay}$ and $\mathbf{cx}$ have a point in common if and only if $ut \neq 1$. [*Hint:* Explain why the lines have no points in common if and only if $\mathbf{y} - \mathbf{a}$ and $\mathbf{x} - \mathbf{c}$ are linearly dependent. Write both of these vectors as linear combinations of $\mathbf{a} - \mathbf{b}$ and $\mathbf{c} - \mathbf{b}$, and show that if $\mathbf{z}$ and $\mathbf{w}$ are linearly independent, then $p\mathbf{z} + q\mathbf{w}$ and $r\mathbf{z} + s\mathbf{w}$ are linearly dependent if and only if $sp = rq$. Finally, compare the two conclusions in the preceding sentence.]

5. Let V be a finite-dimensional vector space over the field $\mathbb{F}$, let $W \subset V$ be a vector subspace, suppose that $\dim W = k$, and let $X = \{\mathbf{x}_1, \ \cdots \ , \mathbf{x}_m\}$ be a finite subset of W. Prove that W is a basis for W in the sense of linear algebra if and only if the set $X \cup \{\mathbf{0}\}$ is an affine basis for $W = \mathbf{0} + W$ if the latter is viewed as a k-plane.

6. In the setting of Exercises 2 and 3, suppose that the lines $\mathbf{ap}$, $\mathbf{bq}$ and $\mathbf{cr}$ are concurrent with $t = \frac{1}{2}$ and $v = 1-u$. Express the common point $\mathbf{g}$ of these three lines as a linear combination of $\mathbf{a}$ and $\mathbf{d}$ with the coefficients expressed in terms of u.

7. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the noncollinear points in $\mathbb{R}^2$ with Cartesian coordinates $(1,0)$, $(5,0)$ and $(5,4)$ respectively. Find the barycentric coordinates of the following points with respect to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$:

(i) The point $(2,2)$.

(ii) The point $(4,4)$.

(iii) The point $(2,-1)$.

[*Hint:* For each vector $\mathbf{x}$ as above, write $\mathbf{x} - \mathbf{a}$ as a linear combination of $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{z}$.]

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[7]G IOVANNI C EVA (1647–1734) is known for the result bearing his name, his rediscovery of Menelaus' Theorem, and his work on hydraulics.

4. Properties of coordinate affine spaces

In this section we shall generalize certain classical theorems of Euclidean geometry to affine planes over fields in which $1 + 1 \neq 0$. Similar material is often presented in many mathematics courses as *proofs of geometric theorems using vectors*. In fact, the uses of vectors in geometry go far beyond yielding alternate proofs for some basic results in classical Greek geometry; they are often the method of choice for studying all sorts of geometrical problems ranging from purely theoretical questions to carrying out the computations needed to create high quality computer graphics. We shall illustrate the uses of vector algebra in geometry further by proving some nonclassical theorems that figure significantly in the next two chapters.

Let $\mathbb{F}$ be a field in which $1 + 1 \neq 0$, and set $\frac{1}{2}$ equal to $(1 + 1)^{-1}$. Given a vector space V over $\mathbb{F}$ and two distinct vectors $\mathbf{a}, \mathbf{b} \in V$, the *midpoint* of $\mathbf{a}$ and $\mathbf{b}$ is the vector $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.

THEOREM II.24. *Let V and $\mathbb{F}$ as above, and let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be noncollinear points in V. Then the lines joining $\mathbf{x} = \mathsf{midpoint}(\mathbf{a}, \mathbf{b})$ and $\mathbf{y} = \mathsf{midpoint}(\mathbf{a}, \mathbf{c})$ is parallel to $\mathbf{bc}$.*

In ordinary Euclidean geometry one also knows that the length of the segment joining $\mathbf{x}$ to $\mathbf{y}$ is half the length of the segment joining $\mathbf{b}$ to $\mathbf{c}$ (the length of the segment is just the distance between the endpoints). We do not include such a conclusion because our setting does not include a method for defining distances (in particular, an arbitrary field has no *a priori* notion of distance).
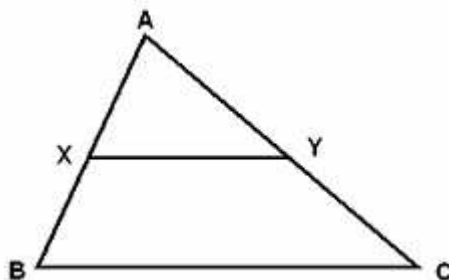
Figure II.3

Proof, Let W be the subspace spanned by $\mathbf{c} - \mathbf{b}$; it follows that $\mathbf{bc} = \mathbf{b} + W$. On the other hand, the line joining the midpoints is given by $\frac{1}{2}(\mathbf{b} + \mathbf{c}) + U$, where U is the subspace spanned by

$$\tfrac{1}{2}(\mathbf{a} + \mathbf{c}) - \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) = \tfrac{1}{2}(\mathbf{c} - \mathbf{b}) .$$

Since there is a vector $\mathbf{w}$ such that W is spanned by $\mathbf{w}$ and U is spanned by $\frac{1}{2}\mathbf{w}$, clearly $W = U$, and therefore the line joining the midpoints is parallel to $\mathbf{bc}$ by the construction employed to prove Theorem 14.∎

Definition. Let V and $\mathbb{F}$ as above, and let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be noncollinear points in V. The *affine triangle* $\triangle\,\mathbf{abc}$ is given by $\mathbf{ab} \cup \mathbf{ac} \cup \mathbf{bc}$, and the *medians* of this affine triangle are the lines joining $\mathbf{a}$ to $\mathsf{midpoint}(\mathbf{b}, \mathbf{c})$, $\mathbf{b}$ to $\mathsf{midpoint}(\mathbf{a}, \mathbf{c})$, and $\mathbf{c}$ to $\mathsf{midpoint}(\mathbf{a}, \mathbf{b})$.

THEOREM II.25. *Let V, $\mathbb{F}$, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be as above. Then the medians of the affine triangle $\triangle\,\mathbf{abc}$ are concurrent (pass through a single point) if $1 + 1 + 1 \neq 0$ and parallel in pairs if $1 + 1 + 1 = 0$.*
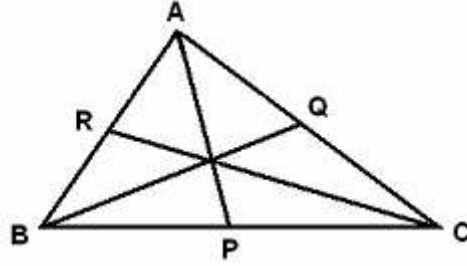
Figure II.4

Proof.   *First case.*   Suppose that $1 + 1 + 1 \neq 0$, and let $\frac{1}{3} = (1 + 1 + 1)^{-1}$. Assume that the point $\mathbf{x}$ lies on the line joining $\mathbf{a}$ to $\frac{1}{2}(\mathbf{b} + \mathbf{c})$ and also on the line joining $\mathbf{b}$ to $\frac{1}{2}(\mathbf{a} + \mathbf{c})$. Then there exist $s, t \in \mathbb{F}$ such that

$$s\mathbf{a} \; + \; (1 - s)\tfrac{1}{2}(\mathbf{b} + \mathbf{c}) \; = \; \mathbf{x} \; = \; t\mathbf{b} \; + \; (1 - t)\tfrac{1}{2}(\mathbf{a} + \mathbf{c}) \; .$$

Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are affinely independent, and in both expansions for $\mathbf{x}$ the coefficients of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ add up to 1, each expansion gives $\mathbf{x}$ as an affine combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Therefore we may equate the barycentric coordinates in the two expansions for $\mathbf{x}$. In particular, this implies $s = \frac{1}{2}(1 - t)$ and $t = \frac{1}{2}(1 - s)$. If we solve these equations, we find that $s = t = \frac{1}{3}$, so that

$$\mathbf{x} \; = \; \tfrac{1}{3}(\mathbf{a} \; + \; \mathbf{b} \; + \; \mathbf{c}) \; .$$

A routine computation shows that this point does lie on both lines.

In a similar fashion one can show that the lines joining $\mathbf{a}$ to $\frac{1}{2}(\mathbf{b} + \mathbf{c})$ and $\mathbf{c}$ to $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ also meet at the point $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$, and therefore we conclude that the latter point lies on all three medians.∎

Second case.   Suppose that $1 + 1 + 1 = 0$; in this case it follows that $\frac{1}{2} = -1$. The line joining $\mathbf{a}$ to $\frac{1}{2}(\mathbf{b} + \mathbf{c})$ is then given by $\mathbf{a} + W$, where W is spanned by

$$\tfrac{1}{2}(\mathbf{b} + \mathbf{c}) \; - \; \mathbf{a} \; = \; -(\mathbf{a} + \mathbf{b} + \mathbf{c}) \; .$$

Similarly computations show that the other two medians are given by $\mathbf{b} + W$ and $\mathbf{c} + W$. To complete the proof, we need to show that no two of these lines are equal.

However, if, say, we had $\mathbf{b} + W = \mathbf{c} + W$ then it would follow that $\mathbf{c}$, $\frac{1}{2}(\mathbf{a} + \mathbf{b})$, $\mathbf{b}$, and $\frac{1}{2}(\mathbf{a} + \mathbf{c})$ would all be collinear. Since the line joining the second and third of these points contains $\mathbf{a}$ by construction, it would then follow that $\mathbf{a} \in \mathbf{bc}$, contradicting our noncollinearity assumption. Thus $\mathbf{b} + W \neq \mathbf{c} + W$; similar considerations show that $\mathbf{a} + W \neq \mathbf{c} + W$ and $\mathbf{a} + W \neq \mathbf{b} + W$, and therefore the three medians are distinct (and by the preceding paragraph are each parallel to each other).∎

HYPOTHESIS FOR THE REMAINDER OF THIS SECTION.   *For the rest of this section, the vector space V is assumed to be* **two-dimensional.**

Definition.   Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in V$ be four ordered points, no three of which are collinear. If $\mathbf{ab} \| \mathbf{cd}$ and $\mathbf{ad} \| \mathbf{bc}$, we shall call the union

$$\mathbf{ab} \; \cup \; \mathbf{bc} \; \cup \; \mathbf{cd} \; \cup \; \mathbf{da}$$

the *affine parallelogram* determined by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, and we shall write it $\square\mathbf{abcd}$. The *diagonals* of the parallelogram are the lines $\mathbf{ac}$ and $\mathbf{bd}$.

THEOREM II.26. *Suppose that* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in V$ *as above are the vertices of an affine parallelogram. Then the diagonals* $\mathbf{ad}$ *and* $\mathbf{bc}$ *have a point in common and this point is equal to* $\mathsf{midpoint}(\mathbf{a}, \mathbf{d})$ *and* $\mathsf{midpoint}(\mathbf{b}, \mathbf{c})$.

If we take $\mathbb{F} = \mathbb{R}$ and $V = \mathbb{R}^2$, this result reduces to the classical geometric result that *the diagonals of a parallelogram bisect each other*.

Proof. The classical *parallelogram law* for vector addition states that

$$(\mathbf{c} - \mathbf{a}) \;=\; (\mathbf{d} - \mathbf{a}) \;+\; (\mathbf{b} - \mathbf{a})$$

(see the accompanying figure).
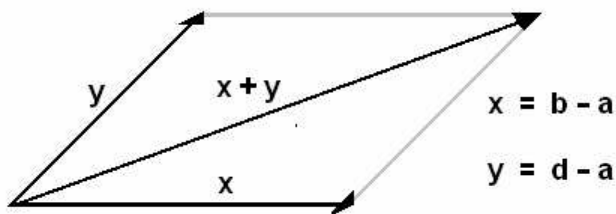
Figure II.5

In fact, it is trivial to verify that $\mathbf{d} + \mathbf{b} - \mathbf{a}$ lies on both the parallel to $\mathbf{ad}$ through $\mathbf{b}$ and the parallel to $\mathbf{ab}$ and $\mathbf{c}$, and hence this point must be $\mathbf{c}$. Therefore the common point of $\mathbf{ac}$ and $\mathbf{bd}$ satisfies the equations

$$t\mathbf{b} \;+\; (1 - t)\mathbf{d} \;=\; s\mathbf{c} \;+\; (1 - s)\mathbf{a} \;=\; s(\mathbf{d} + \mathbf{b} - \mathbf{a}) \;+\; (1 - s)\mathbf{a} \;=\; s\mathbf{d} \;+\; s\mathbf{b} + (1 - 2s)\mathbf{a} \ .$$

Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$ are affinely independent, if we equate barycentric coordinates we find that $s = t = \frac{1}{2}$. But this implies that the two lines meet at a point which is equal to both $\mathsf{midpoint}(\mathbf{a}.\mathbf{c})$ and $\mathsf{midpoint}(\mathbf{b}, \mathbf{d})$.∎

We include the following two theorems because they help motivate the construction of projective space in Chapter III.

THEOREM II.27. *Let* $\mathbf{v}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *be four points, no three of which are collinear (but coplanar). Let* $\mathbf{a}' \in \mathbf{va}$, $\mathbf{b}' \in \mathbf{vb}$, *and* $\mathbf{c}' \in \mathbf{vc}$ *be distinct from* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *such that* $\mathbf{ab} \| \mathbf{a}'\mathbf{b}'$ *and* $\mathbf{ac} \| \mathbf{a}'\mathbf{c}'$. *Then* $\mathbf{bc} \| \mathbf{b}'\mathbf{c}'$.

Figure II.6

Proof.   Since $\mathbf{a}' \in \mathbf{va}$ and $\mathbf{b}' \in \mathbf{vb}$, we may write $\mathbf{a}' - \mathbf{v} + s(\mathbf{a} - \mathbf{v})$ and $\mathbf{b}' - \mathbf{v} + t(\mathbf{b} - \mathbf{v})$ for suitable scalars s and t. Since $\mathbf{ab}||\mathbf{a}'\mathbf{b}'$, it follows that $\mathbf{b}' - \mathbf{a}' = k(\mathbf{b} - \mathbf{a})$ for some scalar k. But

$$k(\mathbf{b} - \mathbf{v}) - k(\mathbf{a} - \mathbf{v}) = k(\mathbf{b} - \mathbf{a}) = \mathbf{b}' - \mathbf{a}' = (\mathbf{b}' - \mathbf{v}) - (\mathbf{a}' - \mathbf{v}) = t(\mathbf{b} - \mathbf{v}) - s(\mathbf{a} - \mathbf{v}) .$$

Since $\mathbf{b} - \mathbf{v}$ and $\mathbf{a} - \mathbf{v}$ are linearly independent, it follows that $s = t = k$. For similar reasons we also have $\mathbf{c}' - \mathbf{v} = t(\mathbf{c} - \mathbf{v})$.

To prove $\mathbf{bc}||\mathbf{bc}'$, it suffices to note that

$$\mathbf{c}' - \mathbf{b}' = (\mathbf{c}' - \mathbf{v}) - (\mathbf{b}' - \mathbf{v}) = t(\mathbf{c} - \mathbf{v}) - t(\mathbf{b} - \mathbf{v}) = t(\mathbf{c} - \mathbf{b})$$
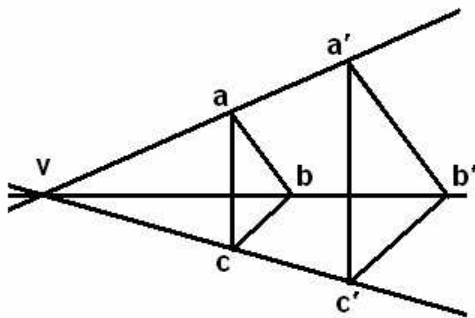
by the preceding paragraph, and hence $\mathbf{bc}||\mathbf{b}'\mathbf{c}'$ follows.∎

Here is a similar result with slightly different hypotheses:

THEOREM II.28. *Let $\mathbf{v}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be four points, no three of which are collinear (but coplanar). Let $\mathbf{a}' \in \mathbf{va}$, $\mathbf{b}' \in \mathbf{vb}$, and $\mathbf{c}' \in \mathbf{vc}$ be distinct from $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ such that $\mathbf{ab}||\mathbf{a}'\mathbf{b}'$ but $\mathbf{ac}$ and $\mathbf{a}'\mathbf{c}'$ meet in some point $\mathbf{x}$. Then $\mathbf{bc}$ and $\mathbf{b}'\mathbf{c}'$ also meet in some point $\mathbf{y}$ and we have $\mathbf{xy}||\mathbf{ab}$.*
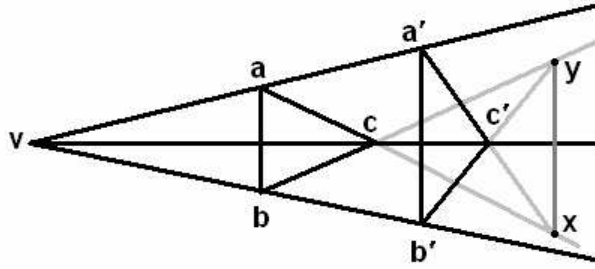
Figure II.7

Proof.   As in Theorem 27 we may write $\mathbf{a}' - \mathbf{v}' = t(\mathbf{a} - \mathbf{v})$ and $\mathbf{b}' - \mathbf{v}' = t(\mathbf{b} - \mathbf{v})$; however, $\mathbf{c}' - \mathbf{v}' = s(\mathbf{c} - \mathbf{v})$ for some $s \neq t$ (otherwise $\mathbf{ac}||\mathbf{a}'\mathbf{c}'$). Expressions for the point $\mathbf{x}$ may be computed starting with the equation

$$\mathbf{x} = r\mathbf{a} + (1 - r)\mathbf{c} = q\mathbf{a}' + (1 - q)\mathbf{c}'$$

which immediately implies that

$$\mathbf{x} - \mathbf{v} = r(\mathbf{a} - \mathbf{v}) + (1 - r)(\mathbf{c} - \mathbf{v}) = q(\mathbf{a}' - \mathbf{v}) + (1 - q)(\mathbf{c}' - \mathbf{v}) = qt(\mathbf{a} - \mathbf{v}) + (1 - q)s(\mathbf{c} - \mathbf{v}) .$$

Since $\mathbf{a} - \mathbf{v}$ and $\mathbf{c} - \mathbf{v}$ are linearly independent, we find that $r = qt$ and $1 - r = (1 - q)s$. These equations determine r completely as a function of s and t:

$$r(s, t) = \frac{t(1 - s)}{s - t}$$

A similar calculation shows that any common point to $\mathbf{bc}$ and $\mathbf{b}'\mathbf{c}'$ has the form

$$r(s, t)\mathbf{b} + \big(1 - r(s, t)\big)\mathbf{c}$$

and a reversal of the previous argument shows that this point is common to $\mathbf{bc}$ and $\mathbf{b}'\mathbf{c}'$. Therefore

$$\mathbf{y} - \mathbf{x} = r(s, t)(\mathbf{b} - \mathbf{a})$$

which shows that $\mathbf{xy}||\mathbf{ab}$.∎

There is a third result of a similar type; its proof is left as an exercise (we should note that all these results will be improved upon later).

THEOREM II.29. *In the notation of the theorems, assume that all three pairs of lines* $\{\, \mathbf{ab},\, \mathbf{a'b'} \,\}$, $\{\, \mathbf{ac},\, \mathbf{a'c'} \,\}$, *and* $\{\, \mathbf{bc},\, \mathbf{b'c'} \,\}$ *all have points of intersection. Then the three intersection points are collinear.*

EXERCISES

Definition. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in V$ be four ordered points, no three of which are collinear (with no parallelism assumptions). The union

$$\mathbf{ab}\ \cup\ \mathbf{bc}\ \cup\ \mathbf{cd}\ \cup\ \mathbf{da}$$

is called the *affine quadrilateral* determined by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, and as before we shall write it $\square\mathbf{abcd}$. The *diagonals* of the quadrilateral are the lines $\mathbf{ac}$ and $\mathbf{bd}$. The *sides* of the quadrilateral are four lines whose union forms the affine quadrilateral;

> *In the exercises below, assume that all vectors lie in the vector space* $\mathbb{F}^2$, *where* $\mathbb{F}$ *is a field in which* $1 + 1 \neq 0$.

1. Prove that an affine quadrilateral is a parallelogram if and only if its diagonals bisect each other (in the sense of Theorem 26).

2. Suppose we are given an affine parallelogram. Prove that a line joining the midpoints of a pair of parallel sides contains the intersection point of the diagonals.
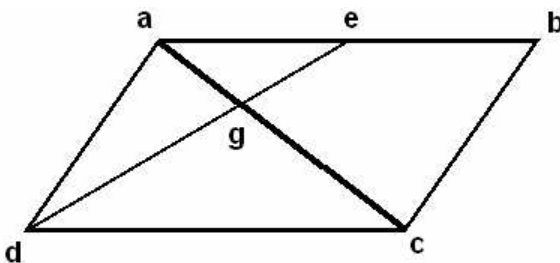
Figure II.8

3. In the figure above, assume we are given a parallelogram $\square\mathbf{abcd}$ such that

$$\mathbf{e}\ =\ \mathsf{midpoint}(\mathbf{a}, \mathbf{b})$$

and $1 + 1 + 1 \neq 0$ in $\mathbb{F}$. Prove that

$$\mathbf{g}\ =\ \tfrac{1}{3}\mathbf{c}\ +\ \tfrac{2}{3}\mathbf{a}\ =\ \tfrac{1}{3}\mathbf{c}\ +\ \tfrac{2}{3}\mathbf{e}\ .$$

Definition. An affine quadrilateral $\square\mathbf{abcd}$ is said to be an *affine trapezoid* if either $\mathbf{ab}||\mathbf{cd}$ or $\mathbf{bc}||\mathbf{ad}$ but not both (and generally the points are labeled so that the first is true). The two parallel sides are called the *bases*.

4. Suppose we are given affine trapezoid □**abcd** as above with bases **ab** and **cd**. Prove that the line joining midpoint($\mathbf{a}, \mathbf{d}$) and midpoint($\mathbf{b}, \mathbf{c}$) is parallel to the bases.

5. In the same setting as in the previous exercise, prove that the line joining midpoint($\mathbf{a}, \mathbf{c}$) and midpoint($\mathbf{b}, \mathbf{d}$) is parallel to the bases.

6. In the same setting as in the previous exercises, prove that the line joining midpoint($\mathbf{a}, \mathbf{d}$) and midpoint($\mathbf{b}, \mathbf{c}$) is equal to the line joining midpoint($\mathbf{a}, \mathbf{c}$) and midpoint($\mathbf{b}, \mathbf{d}$).

7. Prove Theorem 29. [*Hint:* This can be done using the Theorem of Menelaus.]

8. Suppose we are given an affine parallelogram **abcd**, and let **e** and **f** denote midpoint($\mathbf{a}, \mathbf{b}$) and midpoint($\mathbf{c}, \mathbf{d}$) respectively. Prove that **de** is parallel to **bf**.

9. Let L_1, L_2 and L_3 be mutually parallel lines in the vector space $\mathbb{F}^2$, and let M_1 and M_2 be two lines which are not parallel to the L_i, so that each intersection $L_i \cap M_j$ consists of a single point. Define **a**, **b**, **c**, **x**, **y**, **z** so that the following hold:

$$
\begin{aligned}
\mathbf{a} &\in M_1 \cap L_1 & \mathbf{x} &\in M_2 \cap L_1 \\
\mathbf{b} &\in M_1 \cap L_2 & \mathbf{y} &\in M_2 \cap L_2 \\
\mathbf{c} &\in M_1 \cap L_3 & \mathbf{z} &\in M_2 \cap L_3
\end{aligned}
$$

Suppose that none of these points lie on both M_1 and M_2, and write $\mathbf{c} = bfa + t(\mathbf{b} - \mathbf{a})$ and $\mathbf{z} = bfx + t(\mathbf{y} - \mathbf{x})$ for suitable scalars t, $u \in \mathbb{F}$. Prove that $t = u$. [*Hint:* The points **a**, **b** and **x** are noncollinear, and $L_1 = \mathbf{ax}$.]

10. Suppose we are in the setting of the previous excercise, but now we assume that $\mathbf{a} = \mathbf{x}$ is a common point of M_1 and M_2. Prove that $t = u$ in this case too. [*Hint:* In this case the points $\mathbf{a} = \mathbf{x}$, **b** and **y** are noncollinear, and $L_2 = \mathbf{by}$.]

> *Remark.* The preceding exercises are abstract versions of the classical geometric result that *parallel lines determine proportional segments on two transversals.*

11. Let **a**, **b** and **c** be noncollinear points, and define **d**, **e**, **f** by the equations $\mathbf{d} = \mathbf{b} + \mathbf{c} - \mathbf{a}$, $\mathbf{e} = \mathbf{a} + \mathbf{c} - \mathbf{b}$, and $\mathbf{f} = \mathbf{a} + \mathbf{b} - \mathbf{c}$. Prove the following:

(*i*) **a**, **b** and **c** are equal to midpoint($\mathbf{d}, \mathbf{e}$), midpoint($\mathbf{d}, \mathbf{f}$), and midpoint($\mathbf{e}, \mathbf{f}$) respectively.

(*ii*) We have $\mathbf{bc} \,\|\, \mathbf{ef}$, $\mathbf{ac} \,\|\, \mathbf{df}$, and $\mathbf{ab} \,\|\, \mathbf{de}$.

> In the next exercise, *assume that all vectors lie in a vector space V over a field $\mathbb{F}$ in which $1 + 1 = 0$; the most basic example is the field $\mathbb{Z}_2$ which has exactly two elements.*

12. Suppose that $1 + 1 = 0$ in $\mathbb{F}$, and aside from this we are in the setting of Theorem II.26: Specifically, let **a**, **b**, **c**, $\mathbf{d} \in V$ be four noncollinear points such that the first three are noncollinear the four points are the vertices an affine parallelogram. Prove that in this case *the diagonal lines* **ad** *and* **bc** *are* **parallel**. Is the converse also true? Give reasons for your answer.

5. Generalized geometrical incidence

> Our geometry is an abstract geometry. The reasoning could be followed by a disembodied
> spirit who had no understanding of a physical point, just as a man blind from birth could
> understand the Electromagnetic Theory of Light. — **H. G. Forder (1889–1981)**

Although we have not yet defined a geometrical incidence space of arbitrary dimension, it is clear
that the families of k-planes in $\mathbb{F}^n$ should define an n-dimensional incidence space structure on
$\mathbb{F}^n$. Given this, it is not difficult to guess what the correct definitions should be.

Definition.   A *geometrical incidence space* is a triple (S, Π, d) consisting of a set S, a family
of subsets Π the *geometrical subspaces*), and a function d from Π to the positive integers (the
dimension) satisfying the following conditions:

(G-1) : If $x_0 \; \cdots \; , x_n$ are distinct points of S such that no $P \in \Pi$ with $d(P) < n$ contains
them all, then there is a unique $P \in \Pi$ such that $d(P) = n$ and $x_i \in P$ for all i.

Notation.   We denote the geometrical subspace P in the preceding statement by $x_0 \; \cdots \; x_n$. The
condition on the x_i is expressed in the statement that *the set $\{ x_0 \; \cdots \; , x_n \}$ is (geometrically)
independent*.

(G-2) : If $P \in \Pi$ and $\{ x_0 \; \cdots \; , x_m \}$ is a set of geometrically independent points in P, then
the geometrical subspace $x_0 \; \cdots \; x_m$ is contained in P.

(G-3) : If $P \in \Pi$, then P contains at least $d(P) + 1$ points.

If $P \in \Pi$ and $d(P) = k$, then we shall say that P **is a k-plane**; the set of all k-planes is denoted
by Π_k. By convention, a 1-plane is often called a *line* and a 2-plane is often simply called a
plane. Note that the defining conditions do not give any *a priori* information about whether or
not there are any k-planes in S (however, if S contains at least two elements, it follows that Π_1
must be nonempty).

For most of the examples that we shall consider, the whole space S is one of the geometrical
subspaces. If this is the case and $\dim S = n$, then we shall say that the system (S, Π, d) is an
abstract n-dimensional geometrical incidence space. When $n = 2$ we shall also say the system is
an *abstract incidence plane*; we shall often shorten these terms by omitting the word "abstract."
— Also, if we are given a geometrical incidence space and the explicit data Π and d are either
clear from the context or are not really needed in a discussion, we shall often simply say that
"S is a geometrical incidence space."

EXAMPLE 1.   A three-dimensional incidence space (as defined above) is a geometrical incidence
space if we take $\Pi_1 = \mathcal{L}$, $\Pi_2 = \mathcal{P}$, and $\Pi_3 = \{S\}$.

EXAMPLE 2.   The **minimal** geometrical indicidence structures: Let Π be the family of all finite
subsets of a set S with at least two elements, and for each such subset Q let $d(Q) = \#(Q) - 1$,
where $\#(Q)$ is the number of elements in Q. The figure below illustrates the special case in
which S has three elements $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $d(S) = 2$.
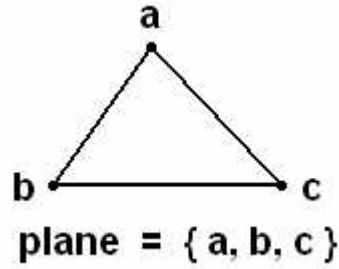
Figure II.9

EXAMPLE 3.   An important class of examples mentioned at the beginning of this section: Let V be an n-dimensional vector space over a field $\mathbb{F}$ (where $n > 0$), and define the *affine incidence structure on V* such that for each positive integer k the set Π_k is the set of k-planes considered in earlier sections of this chapter. The properties **(G-1)**–**(G-3)** may be verified as follows:

Proof of **(G-1)**.   If a set of vectors $\{\, \mathbf{v}_0, \cdots, \mathbf{v}_k\,\}$ is not geometrically independent, then the set is also affinely dependent, for if $\{\, \mathbf{v}_0, \cdots, \mathbf{v}_k\,\}$ is contained in a q-plane $\mathbf{y} + W$ for some $q < k$, then we have $\mathbf{y} + W = \mathbf{x}_0 + W$ and the k vectors $\mathbf{v}_1 - \mathbf{v}_0, \cdots, \mathbf{v}_k - \mathbf{v}_0 \in W$ must be linearly dependent because $\dim W < k$. Hence the original vectors are affinely dependent as claimed. Taking negations, we see that if $\{\, \mathbf{v}_0, \cdots, \mathbf{v}_k\,\}$ is geometrically independent, then the set is also affinely independent.

Let Q be the affine span of $\mathbf{v}_0, \cdots, \mathbf{v}_k$; then $Q = \mathbf{x}_0 + W$ where W is the linear span of $\mathbf{v}_1 - \mathbf{v}_0, \cdots, \mathbf{v}_k - \mathbf{v}_0$, and W is k-dimensional because these vectors are linearly independent. Therefore Q is a k-plane containing the vectors $\mathbf{v}_i$. Conversely, if Q' is an arbitrary k-plane containing the vectors $\mathbf{v}_i$, then we may write $Q' = \mathbf{v}_0 + U$ where U is a vector subspace of dimension k which contains the difference vectors $\mathbf{v}_1 - \mathbf{v}_0, \cdots, \mathbf{v}_k - \mathbf{v}_0$; it follows that U contains the k-dimensional vector subspace W described above, and since $\dim U = \dim W$ it follows that $W = U$, so that $Q = Q'$.∎

Proof of **(G-2)**.   Since a k-plane is closed under forming affine combinations, if $\mathbf{v}_0, \cdots, \mathbf{v}_m$ is contained in P then the affine span of $\mathbf{v}_0, \cdots, \mathbf{v}_m$ is also contained in P.∎

Proof of **(G-3)**.   Given a k-plane P, express it as $\mathbf{v}_0 + W$, where W is a k-dimensional vector subspace, and let $\mathbf{v}_1, \cdots, \mathbf{v}_k$ be a linear-algebraic basis for W. Then the set

$$\{\, \mathbf{v}_0, \ \mathbf{v}_0 + \mathbf{v}_1, \cdots, \mathbf{v}_k + \mathbf{v}_1 \,\}$$

forms an affine basis for P by Theorem 22. Hence P contains at least $(k+1)$ points.∎

We have not yet described an analog of the axiom implying that two planes in 3-space intersect in a line if they are not disjoint. Before formulating an appropriate generalization of this,[8] we derive some consequences of **(G-1)**–**(G-3)**:

THEOREM II.30. *Let $x_0, \cdots, x_m$ be geometrically independent points in a geometrical incidence space S, and suppose $y \notin \{x_0, \cdots, x_m\}$. Then the set $\{x_0, \cdots, x_m, y\}$ is geometrically independent.*

---

[8]The definition follows the proof of Theorem 35 below.

Proof.   If the points are not geometrically independent then for some $k \leq m$ there is a k-plane P which contains all of them. Since $\{x_0, \cdots, x_m\}$ is geometrically independent, it follows that $d(P) \geq m$, so that $d(P) = m$ and $P = x_0 \cdots x_m$. But this contradicts the hypothesis.∎

THEOREM II.31. *Let P be a k-plane in the geometrical incidence space S. Then there is a set of independent points $\{x_0, \cdots, x_k\}$ such that $P = x_0 \cdots x_k$.*

Proof.   Let $\mathcal{F}$ be the family of finite independent subsets of P. No element in this family contains more than $m + 1$ elements, for every subset with more points will be geometrically dependent. Let m be the largest integer for which some element of $\mathcal{F}$ has $m + 1$ elements; by the preceding sentence, we must have $m \leq k$. We need to show that $m = k$, for in this case we know that P is the unique k-plane containing the given set of geometrically independent points and hence $P = x_0 \cdots x_k$.

Assume the contrary, so that $m < k$. Let $\{x_0, \cdots, x_m\} \in \mathcal{F}$. Then the m-plane $Q = x_0 \cdots x_m$ is contained in P by **(G-2)**, and $m < k$ implies that Q is a proper subset of P. Let x_{k+1} be a point which is in P but not in Q. Then Theorem 30 implies that $\{x_0, \cdots, x_k, y\}$ is geometrically independent. But this contradicts the defining condition for m, which is that there are no geometrically independent subsets of P with $m + 2$ elements. Therefore we must have $m = k$.∎

STANDARD CONVENTIONS. Given a geometrical incidence space (S, Π, d), we shall often say that the empty set is a geometrical subspace whose dimension is -1, each one point subset of S is a geometrical subspace whose dimension is zero, and subsets of both these types are geometrically independent. We shall say that ordinary geometrical subspaces, one point subsets and the empty set are *geometrical subspaces in the extended sense.* — Properties **(G-1)** – **(G-3)** of a geometrical incidence space remain valid if we extend the definition of geometrical subspace using these conventions, and the latter allow us to state the next result more simply.

THEOREM II.32. *Every finite subset X of the geometrical incidence space S (in the extended sense) contains a maximal independent subset $Y = \{y_0, \cdots, y_k\}$. Furthermore, X is contained in $y_0 \cdots y_k$, and the latter is the unique minimal geometrical subspace containing X.*

Henceforth, unless stated otherwise the term "geometrical subspace" will be taken to mean a geometrical subspace in the extended sense.

Proof.   Let $Y \subset X$ be an independent subset with a maximum number of elements, let Q be the k-plane determined by Y, and let $w \in X$ be an arbitrary element not in Y. Since $y \notin Q$ would imply that $Y \cup \{y\}$ would be independent, it follows that $y \in Q$. Thus $X \subset Q$ as claimed. Suppose that Q' is another geometrical subspace containing X; then $Q \subset Q'$ by **(G-2)**, and hence every geometrical subspace that contains X must also contain Q.∎

When we work with vector subspaces of a vector space, it is often useful to deal with their intersections and sums. The next two results show that similar constructions hold for geometrical subspaces:

THEOREM II.33. *The intersection of a nonempty family of geometrical subspaces of S is a geometrical subspace (with the Standard Conventions for 0- and (-1)-dimensional subspaces preceding Theorem 32).*

Proof.   Clearly it suffices to consider the case where the intersection contains at least two points. Let $\{\, P_\alpha \mid \alpha \in A\,\}$ be the family of subspaces, and let $\mathcal{F}$ be the set of all finite independent subsets of $\cap_\alpha P_\alpha$. As before, the number of elements in a member of $\mathcal{F}$ is at most $d(P_\alpha) + 1$ for all α. Thus there is a member of $\mathcal{F}$ with a maximal number of elements; call this subset $\{x_0, \cdots, x_k\}$. By **(G-2)** we know that $x_0 \cdots x_k$ is contained in each P_α and hence in the intersection $\cap_\alpha P_\alpha$. If it were a proper subset and y is a point in the intersection which does not belong to $x_0 \cdots x_k$, then $\{x_0 \cdots x_k, y\}$ would be an independent subset of the intersection with more elements than $\{x_0, \cdots, x_k\}$. This contradiction means that $x_0 \cdots x_k = \cap_\alpha P_\alpha$.■

Although the union of two geometrical subspaces P and Q is usually not a geometrical subspace, the next result shows that there is always a minimal geometrical subspace containing both P and Q; this is analogous to the concept of sum for vector subspaces of a vector space.

THEOREM II.34. *If P and Q are geometrical subspaces of the geometrical incidence space S, then there is a unique minimal geometrical subspace containing them both.*

Proof.   By Theorem 31 we may write $P = x_0 \cdots x_m$ and $Q = y_0 \cdots y_n$. Let $A = \{x_0, \cdots, x_m, y_0, \cdots, y_n\}$, and let T be the smallest subspace containing A (which exists by Theorem 32). Then $P, Q \subset T$, and if T' is an arbitrary geometrical subspace containing P and Q then it also contains A, so that T' must contain T as well.■

The subspace given in the preceding result is called the **join** of P and Q, and it is denoted by $P \star Q$. Motivation for this definition is given by Exercise III.4.17 and Appendix B.

THEOREM II.35. *If P and Q are geometrical subspaces of S, then $d(P \star Q) \le d(P) + d(Q) - d(P \cap Q)$.*

It is easy to find examples of geometrical subspaces in $\mathbb{R}^n$ (even for $n = 2$ or 3) in which one has **strict** inequality. For example, suppose that L and M are parallel lines in $\mathbb{R}^3$; then the left hand side is equal to 2 but the right hand side is equal to 3 (recall that the empty set is (-1)-dimensional). Similarly, one has strict inequality in $\mathbb{R}^3$ if P is a plane and Q is a line or plane which is parallel to P.

Proof.   Let $P \cap Q = x_0 \cdots x_m$. By a generalization of the argument proving Theorem 31 (see Exercise 1 below), there exist independent points $y_0, \cdots, y_p \in P$ and $z_0, \cdots, z_q \in Q$ such that

$$P = x_0 \cdots x_m y_0 \cdots y_p, \qquad Q = x_0 \cdots x_m z_0 \cdots z_q .$$

Let $X = \{x_0, \cdots, x_m, y_1, \cdots, y_p, z_1, \cdots, z_q\}$, and let T be the unique smallest geometrical subspace containing X. It is immediate that $P \subset T$ and $Q \subset T$, so that $P \star Q \subset T$. On the other hand, if a geometric subspace B contains $P \star Q$, then it automatically contains X and hence automatically contains T. Therefore we have $T = P \star Q$.

It follows from Theorem 32 that $d(P \star Q) = \dim S \le \#(X) + 1$, and therefore we have

$$\dim(P \star Q) \le m + 1 + p + q = (m + p + 1) + (m + q + 1) - (m + 1) =$$
$$d(P) + d(Q) - d(P \cap Q)$$

which is what we wanted to prove.■

The following definition contains an abstract version of the 3-dimensional incidence assumption about two planes intersecting in a line or not at all.

Definition. A geometrical incidence space is *regular* if the following holds:

(G-4) : If P and Q are geometrical subspaces such that $P \cap Q \neq \varnothing$, then
$$d(P \star Q) \; = \; d(P) \; + \; d(Q) \; - \; d(P \cap Q) \; .$$

EXAMPLE 1. Ordinary 3-dimensional incidences as defined in Section II.1 are regular. The only nontrivial case of the formula arises when P and Q are distinct planes, so that $P \star Q = S$.

EXAMPLE 2. The minimal examples at the beginning of this section are regular. The formula in this case follows immediately from the standard *inclusion-exclusion identity* for counting the elements in finite sets:
$$\#(P \cup Q) \; = \; \#(P) \; + \; \#(Q) \; - \; \#(P \cap Q) \; .$$

EXAMPLE 3. Here is an example which is **not** regular. Let $\mathbb{F}$ be a field, take the standard notions of lines and planes for $\mathbb{F}^4$, and let $d(\mathbb{F}^4) = 3$. Then the planes V and W spanned by $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{e}_3, \mathbf{e}_4\}$ have exactly one point in common.

Logical indpendence of the regularity condition. The preceding example shows that it is **not** possible to prove the assumption in the definition of regularity from the defining assumptions for a geometrical incidence space. — For if it were possible to prove the regularity condition from the definition, then it would NOT be possible to construct an example of a geometrical incidence space that did not satisfy the regularity condition.

The preceding remark illustrates the mathematical approach to concluding that one statement (say $\mathbf{Q}$) cannot be derived as a logical consequence of other statements (say $\mathbf{P}_1, \; \cdots \; , \mathbf{P}_n$): *It is only necessary to produce* **one example** *of an object which satisfies* $\mathbf{P}_1, \; \cdots \; , \mathbf{P}_n$ *but does not satisfy* $\mathbf{Q}$.

EXAMPLE 4. The incidence structure associated to a finite-dimensional vector space V over a field $\mathbb{F}$ is regular. To prove this, we first establish the following.

THEOREM II.36. *Let P and Q be affine subspaces of the incidence space structure associated to a vector space V, and assume $P \cap Q \neq \varnothing$. Write $P = \mathbf{x} + W$ and $Q = \mathbf{x} + U$ where $\mathbf{x} \in P \cap Q$. Then $P \star Q = \mathbf{x} + (W_1 + W_2)$.*

REMARK. Since $P \cap Q = \mathbf{x} + (W_1 \cap W_2)$ is readily established (see Exercise II.2.4), Theorem 36 and the dimension formula for vector subspaces (Theorem A.9) imply the regularity of V.∎

Proof. The inclusion $P \star Q \subset \mathbf{x} + (W_1 + W_2)$ is clear since the right hand side is a geometrical subspace containing P and Q. To see the reverse inclusion, first observe that $P \star Q = \mathbf{x} + U$ for some vector subspace U; since $P, Q \subset \mathbf{x} + U$ it follows that $W_1, W_2 \subset U$ and hence $W_1 + W_2 \subset U$. The latter yields $\mathbf{x} + (W_1 + W_2) \subset P \star Q$, and therefore the two subsets are equal. ∎

Finally, we introduce an assumption reflecting the Euclidean Parallel Postulate.

Definition. Two lines in a geometrical incidence space are *parallel* if they are coplanar but disjoint. A regular incidence space is said to be affine if given a line L and a point $\mathbf{x} \notin L$, then there is a unique line M such that $\mathbf{x} \in M$ and M is parallel to L. — If S is an affine incidence space and $d(S) = n$, then we say that it is an *affine n-space*. If $n = 2$, it is also called an *affine plane*.

One can use the argument proving Theorem 14 to verify that the affine incidence space associated to a vector space is affine in the sense defined above.

<center><i>Equivalent mathematical structures</i></center>

Our discussion of geometrical incidence fits closely with the main themes of these notes, but the formulation is definitely nonstandard. Normally such incidence structures are viewed in equivalent but more abstract terms. One approach involves specifying a sequence of *dependence relations* on finite subsets of a given set S; in this formulation, there is a class of such subsets that are called *independent* and satisfy a few suitable properties. As one might expect, a finite subset of $k+1$ points is said to be independent if and only if there is no k-dimensional subspace containing them; the resulting structure is called a **matroid**. Details of abstract dependence theory and matroids are described in the paper by H. Whitney[9] listed in the bibliography, and a more recent account of matroid theory is given in the following online reference:

<center>http://home.gwu.edu/~jbonin/survey.ps</center>

The matroid approach to independence leads naturally to another interpretation in terms of partially ordering relations on sets. Every matroid has a family of subsets which satisfy the conditions for a geometrical incidence space, and the associated ordered family of subsets satisfies the conditions for a partially ordered set to be a **geometric lattice**. A classic reference for the theory of lattices is the book by G. Birkhoff[10] (*Lattice Theory*) cited in the bibliography.

<center>EXERCISES</center>

1.   Let P be a geometrical subspace of S, and suppose that $\{x_0, \cdots, x_k\}$ is an independent subset of P. Prove that there is a (possibly empty) set of points $x_{k+1}, \cdots, x_m \in P$ such that $\{x_0, \cdots, x_m\}$ is independent and $P = x_0 \cdots x_m$. [*Hint:* Imitate the proof of Theorem 31 using the family $\mathcal{G} \subset \mathcal{F}$ of all subsets containing $\{x_0, \cdots, x_k\}$.]

2.   (i) Prove that a subset of an independent set of points in a geometrical incidence space is independent.

(ii) Suppose that P and Q are geometrical subspaces of a geometrical incidence space such that $P \subset Q$. Prove that $d(P) \leq d(Q)$, with equality if and only if $P = Q$. [*Hint:* For the first part, choose $\{x_0, \cdots, x_k\}$ such that $P = x_0 \cdots x_k$ and apply Exercise 1. For the second part, suppose that $P \subset Q$ and $d(P) = d(Q) = k$. Choose $\{x_0, \cdots, x_k\}$ such that $P = x_0 \cdots x_k$ and apply **(G-1)** to conclude that $P = Q$.]

3.   (i) Let (S, Π, d) be a geometrical incidence space, let $T \subset S$ be a geometrical subspace, and let Π_T be the set of all geometrical subspaces in Π which are contained in T. Prove that (S, Π_T, d_T) is a geometrical incidence space, where d_T is the restriction or T.

(ii) Prove that a geometrical subspace of a regular geometrical incidence space is again regular. Is the analog true for affine spaces? Give reasons for your answer.

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[9] HASSLER WHITNEY (1907–1989) did his earliest work in graph theory and related areas, but he his best known for his fundamental, innovative, and highly influential research in topology.

[10] GARRETT BIRKHOFF (1911–1996) is known for his work in abstract algebra, applied mathematics and numerical computation. Several of his books have been extremely influential.

4.   Let S be a geometrical incidence space with $d(S) = n$. A *hyperplane* in S is an $(n-1)$-plane. Prove that for every $k < n$, every k-plane in S is an intersection of $(n-k)$ distinct hyperplanes. [*Hint:*   If T is a k-plane and $T = x_0 \cdots x_k$, choose $y_1, \cdots, y_{n-k}$ such that $\{x_0 \cdots x_k, y_1, \cdots, y_{n-k}\}$ is independent. Let P_i be the hyperplane determined by all these points except y_j.]

5.   (*i*) Prove that the join construction on geometrical subspaces of a geometrical incidence space satisfies the associativity condition $(P \star Q) \star R = P \star (Q \star R)$ for geometrical subspaces $P, Q, R \subset S$. [*Hint:*   Show that both of the subspaces in the equation are equal to the smallest geometrical subspace containing $P \cup Q \cup R$.]

(*ii*) Prove that the join and intersection operations for subspaces satisfy the **absorption identities** $P \star (P \cap Q) = P = P \cap (P \star Q)$.

6.   Show that the Euclidean parallelism property is not a logical consequence of the defining conditions for a regular n-dimensional geometrical incidence space. [*Hint:*   Look at the so-called trivial examples from the previous section in which Π is the collection of all finite subsets and if P is finite then $d(P) = \#(P) - 1$. Why do standard counting formulas imply that every pair of coplanar lines in this system will have a common point?]

Remark.   For affine planes and the examples considered in the previous section, the following modification of the Euclidean Parallelism Property is valid: *Given a line L and a point $\mathbf{x} \notin L$, then there is* **at most one** *line M such that $\mathbf{x} \in M$ and M is parallel to L.* It is also not possible to prove this from the conditions defining a regular incidence space, and the following *Beltrami-Klein incidence plane* is an example: Let P be the set of all points in $\mathbb{R}^2$ which are interior to the unit circle; this set is defined by the inequality $x^2 + y^2 < 1$ (see the figure below). The lines in P are taken to be the open chords of the unit circle with equation $x^2 + y^2 = 1$, or equivalently all **nonempty** intersections of ordinary lines in $\mathbb{R}^2$ with P. As the figure suggests, this example has the following property: *Given a line L and a point $\mathbf{x} \notin L$, then there are* **at least two** *lines M such that $\mathbf{x} \in M$ and M is parallel to L.* — In fact, there are *infinitely many* such lines.
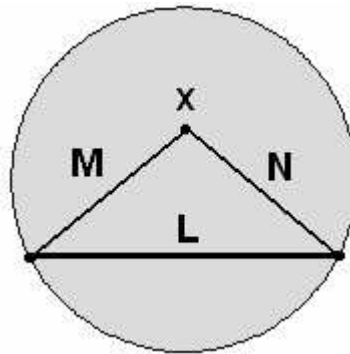
Figure II.10

The plane is the shaded region inside the circle, with the boundary excluded. The open chord L has no points in common with the open chords determined by M and N.

7.   (*Further examples as above*) Define a *finitely supportable convex body* in $\mathbb{R}^n$ to be a nonempty set D defined by finitely many strict inequalities $f_i(\mathbf{x}) > 0$, where each f_i is a linear polynomial in the coordinates of $\mathbf{x}$. Let Π_D denote the set of all nonempty intersections $P \cap D$, where P is a geometrical subspace of $\mathbb{R}^n$, and set the dimension $d_D(P \cap D$ equal to $d(P)$. Prove

that this structure makes D into a regular geometrical incidence n-space. [*Hint:* Try to do this first for $n = 2$, then for $n = 3$.]

8.   Assume we are in the setting of the preceding exercise.

(i) Let D be the open square in $\mathbb{R}^2$ defined by $x > 0$, $y > 0$, $-x > -1$ and $-y > -1$ (hence the coordinates lie between 0 and 1). Prove that D has the parallelism properties that were stated but not proved for the Beltrami-Klein plane described above.

(ii) Let D be the upper half plane in $\mathbb{R}^2$ defined by $y > 0$. Prove that for some lines L and external points $\mathbf{x}$ there is a unique parallel to L through $\mathbf{x}$, but for others there are infinitely many such parallels.

9.   Let (S, Π, d) be a geometrical incidence space, and let U, V, W be geometrical subspaces such that $U \subset W$.

(i) Prove the *semimodularity property $U \star (V \cap W) \subset (U \star V) \cap W$*. [*Hint:* Show that the left hand side is contained in $U \star V$ and W.]

(ii) Describe positive dimensional geometrical subspaces U, V, W of $\mathbb{R}^3$ (with the usual 3-dimensonal incidence structure) such that $U \star (V \cap W)$ is properly contained in $(U \star V) \cap W$. [*Hint:* Take W to be a plane containing the line U, and let V be a line which is disjoing from W.]

(iii) In the setting of the problem, suppose that S is regular and $U \cap V$ is nonempty. Prove the *modularity property $U \star (V \cap W) = (U \star V) \cap W$*. [*Hint:* By ($i$) and part ($ii$) of Exercise 2, it suffices to prove that

$$d\big(U \star (V \cap W) \big) \;=\; d\big((U \star V) \cap W \big) \, .$$

Let Δ be equal to the difference of the right hand side and the left hand side, and use **(G-4)** repeatedly to show that $\Delta = 0$.]

6. Isomorphisms and automorphisms

Although the mathematical concepts of *isomorphism* and *automorphism* are stated abstractly, they reflect basic concepts that arise very quickly in elementary geometry. Congruent triangles are fundamental examples of isomorphic objects: The statement $\Delta\,ABC \cong \Delta\,DEF$ means that the obvious 1–1 correspondence of vertices from $\{A, B, C\}$ to $\{D, E, F\}$ preserves the basic structural measurements, so that the distances between the vertices satisfy

$$d(A, B) \;=\; d(D, E), \qquad d(B, C) \;=\; d(E, F), \qquad d(A, C) \;=\; d(D, F)$$

and the (degree or radian) measurements of the vertex angles satisfy similar equations:

$$\begin{aligned}
\text{measure}(\angle\,ABC) \;&=\; \text{measure}(\angle\,DEF) \\
\text{measure}(\angle\,ACB) \;&=\; \text{measure}(\angle\,DFE) \\
\text{measure}(\angle\,CAB) \;&=\; \text{measure}(\angle\,FDE)
\end{aligned}$$

Suppose now that we have an isosceles triangle $\Delta\,ABC$ in which $d(A, B) = d(A, C)$. In such cases one has a natural symmetry of the triangle with respect to the line joining A to the midpoint of B and C, and one aspect of this symmetry is a nontrivial congruence of the isosceles triangle with itself; specifically, $\Delta\,ABC \cong \Delta\,ACB$. In mathematics, an isomorphism from an object to itself is called an automorphism. The identity map from the object to itself is a trivial example of an automorphism, and the isosceles triangle example indicates that some structures may have other automorphisms.

The word "isomorphic" means "identical shape" or "same form," and in mathematics it means that one has a rule for passing between two objects that preserves all the mathematical structure that is essential in a given context. In particular, *if two objects are isomorphic, a statement about the properties or structure of the first object is true if and only if the corresponding statement about the second object is true* (for example, in Euclidean geometry if we know that $\Delta ABC \cong \Delta DEF$, then ΔABC is equilateratl if and only if ΔDEF is equilateral). Such principles can be extremely useful, especially when one of the objects is relatively easy to work with and the other is less so; one example of this is given by Theorem II.39 below).

The precise mathematical definitions of isomorphisms and automorphisms vary, and the details depend upon the sort of objects being considered. Also, there may be several different notions of isomorphism or automorphism, depending upon the amount of structure that remains unchanged. For example, if we are working with triangles, it is often useful to consider triangles that might not be congruent but are *similar*; in other words, one still has a 1–1 correspondence correspondence of vertices from $\{A, B, C\}$ to $\{D, E, F\}$ as before such that the angle measure equations are satisfied, but now we only know that there is a positive number r (the *ratio of similitude*) such that

$$d(A, B) \;=\; r \cdot d(D, E), \quad d(B, C) \;=\; r \cdot d(E, F), \quad d(A, C) \;=\; r \cdot d(D, F) \;.$$

Of course, congruence is the special case of similarity for which $r = 1$.

Here is the appropriate concept of isomorphism for geometrical incidence spaces:

Definition.   Let (S, Π, d) and (S', Π', d') be geometrical incidence spaces. An *isomorphism* of geometrical incidence spaces from (S, Π, d) to (S', Π', d') is a 1–1 correspondence $f : S \to S'$ such that if $P \subset S$, then $P \in \Pi$ if and only if its image $f[P]$ belongs to Π', and in this case $d(P) = d'(\,f[P]\,)$. In other words, P is a k-plane in S if and only if $f[P]$ is a k-plane in S'.[11]

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[11]As noted in the Prerequisites, we use $f[P]$ to denote the image of P under the mapping f.

It is standard to write $A \cong B$ when two mathematical systems are isomorphic, and we shall do so throughout these notes.

The first theorem of this section implies that isomorphism of geometrical incidence spaces is an equivalence relation.

THEOREM II.37. (*i*) *If* $f : S \to S'$ *is an isomorphism of geometrical incidence spaces, then so is its inverse* $f^{-1} : S' \to S$.

(*ii*) *If* $f : S \to S'$ *and* $g : S' \to S''$ *are isomorphisms of geometrical incidence spaces, then so is their composite* $g \circ f : S \to S''$.

Proof.   (*i*) If f is 1–1 and onto, then it has an inverse map f^{-1} which is also 1–1 and onto. If $Q \subset S$ is a k-plane, then the identity $Q = f\left[\,f^{-1}[Q]\,\right]$ implies that $f^{-1}[Q]$ is a k-plane in S. Similarly, if $f^{-1}[Q]$ is a k-plane in S then so is $Q = f\left[\,f^{-1}[Q]\,\right]$.

(*ii*) If f and g are 1–1 onto, then so is $g \circ f$. If $P \subset S$ is a k-plane, then so is $f[P]$ because f is an isomorphism of geometrical incidence spaces, and since g is also an isomorphism of geometrical incidence spaces then
$$g \circ f[P] \;=\; g\left[\,f[P]\,\right]$$
is also a k-plane. Conversely, if the latter is a k-plane, then so is $f[P]$ since g is an isomorphism of geometrical incidence spaces, and therefore P is too because f is an isomorphism of geometrical incidence spaces.∎

The next result illustrates earlier assertions that isomorphisms preserve the basic properties of mathematical structures like abstract geometrical incidence spaces.

THEOREM II.38. *Let* $f : S \to S'$ *is an isomorphism of geometrical incidence spaces, and let* $X = \{\, \mathbf{x}_0, \, \cdots \, \mathbf{x}_m \,\}$ *be a finite subset of* S. *Then* X *is a geometrically independent subset of* S *if and only if* $f[X]$ *is a geometrically independent subset of* X'.

Proof.   Suppose that X is independent, and assume that $f[X]$ is not. Then there is some k-plane $Q \subset S'$ such that $d(P) < q$ and $f[X] \subset Q$. Let $P = f^{-1}[Q]$. By the definition of isomorphism, it follows that P is also a k-plane, where $k < q$, and $X = f^{-1}\left[\,f[X]\,\right]$ is contained in P. Therefore X is not independent, which contradicts our choice of X. Thus it follows that $f[X]$ is geometrically independent if X is. The converse statement follows by applying the preceding argument to the inverse isomorphism f^{-1}.∎

If $\mathcal{G}$ is a family of geometrical incidence spaces, then a *classifying family* for $\mathcal{G}$ is a subfamily $\mathcal{C}$ such that each object in $\mathcal{G}$ is isomorphic to a unique space in $\mathcal{C}$. The standard coordinate affine spaces $\mathbb{F}^n$ (where $\mathbb{F}$ is some field) comes relatively closed to being a classifying family for n-dimensional affine incidence spaces for $n \geq 3$. It is only necessary to add standard coordinate affine n-spaces over *skew-fields* (see the second paragraph of Appendix A). The only difference between fields and skew-fields is that multiplication is not necessarily commutative in the latter; the standard nontrivial example is given by the **quaternions**, which are discussed in Appendix A. — If $\mathbb{F}$ is a skew-field, let $\mathbb{F}^n$ be the **right** vector space of ordered n-tuples as defined in Appendix A. As indicated there, all of the linear algebra used to prove that $\mathbb{F}^n$ is an affine n-space goes through if $\mathbb{F}$ is a skew-field. In particular, the right vector space $\mathbb{F}^n$ is an affine n-space. The classification of n-dimensional affine spaces (where $n \geq 3$) is then expressible as follows:

THEOREM II.39. *Let $(S, etc.)$ be an affine n-space, where $n \geq 3$. Then there is a skew-field $\mathbb{F}$ such that S is isomorphic to $\mathbb{F}^n$ as a geometrical incidence space. Furthermore, if $\mathbb{E}$ ind $\mathbb{F}$ are skew-fields such that S is geometrically isomorphic to $\mathbb{E}^n$ and $\mathbb{F}^n$, then $\mathbb{E}$ and $\mathbb{F}$ are algebraically isomorphic.*

The proof of this result involves several concepts we have not yet introduced and is postponed to Remark 3 following the proof of Theorem IV.19. Specifically, it will be a fairly simple application of the projective coordinatization theorem (Theorem IV.18) to the synthetic projective extension of S (which is defined in the Addendum to Chapter 3).

Automorphisms and symmetry

A standard dictionary definition of the word *symmetry* is "regularity in form and arrangement," and the significance of automorphisms in mathematics is that they often yield important patterns of regularity in a mathematical system. This is apparent in our previous example involving isosceles triangles. Of course, the amount of regularity can vary; for example, an equilateral triangle has a more regular structure than, say, an isosceles right triangle. Generally speaking, the more regular an object is, the more automorphisms it has.

Definition. An *automorphism* of a geometrical incidence space (S, Π, d) and (S', Π', d') is a 1–1 onto map from S to itself which is an isomorphism of geometrical incidence spaces. — The identity map of a mathematical object is normally an example of an automorphism, and it is a routine exercise to check that for each geometrical incidence space (S, Π, d), the identity map of S defines an automorphism of (S, Π, d).

The following result is an immediate consequence of Theorem 38 and the preceding sentence.

THEOREM II.40. *The set of all automorphisms of a geometrical incidence space (S, Π, d) forms a group with respect to composition of mappings.*∎

Notation. This group is called the *geometric symmetry group* of (S, Π, d) and is denoted by $\mathbf{\Sigma}(S, \Pi, d)$.

EXAMPLE 1. Let (S, Π, d) be the affine incidence space associated to a vector space V. If T is an invertible linear self-map of V, then T also defines an incidence space automorphism. This is true because the condition $\text{Kernel}(T) = \{\mathbf{0}\}$ implies that T maps each k-dimensional vector subspace W to another subspace $T[W]$ of the same dimension (see Theorem A.14.(iv)). Furthermore, if S is an arbitrary subset and $\mathbf{x} \in V$, then it is easy to check that $T[\mathbf{x} + S] = T(\mathbf{x}) + T[S]$ (this is left to the reader as an exercise). .

EXAMPLE 2. If V is as above and $\mathbf{x} \in V$, define the mapping $T_{\mathbf{x}} : V \to V$ (*translation by* $\mathbf{x}$) to be the mapping $T_{\mathbf{x}}(\mathbf{v}) = \mathbf{x} + \mathbf{v}$. Then $T_{\mathbf{x}}$ is clearly 1–1 and onto. Furthermore, it defines a geometrical space automorphism because $T_{\mathbf{x}}[\mathbf{y} + W] = (\mathbf{x} + \mathbf{y}) + W$ shows that P is a k-plane in V if and only if $T_{\mathbf{x}}[P]$ is (again, filling in the details is left to the reader).∎

The preceding observations show that affine incidence spaces associated to vector spaces generally have many geometric symmetries. Define the *affine group* of V — denoted by $\text{AFF}(V)$ — to be all symmetries expressible as a composite $T \circ S$, where T is a translation and S is an invertible linear transformation. We shall say that the elements of $\text{AFF}(V)$ are *affine transformations*. The

terminology suggests that the affine transformations form a subgroup, and we shall verify this in the next result.

THEOREM II.41. *The set* $\mathrm{AFF}(V)$ *is a subgroup of the group of all geometric symmetries of* V. *It contains the groups of linear automorphisms and translations as subgroups.*

Proof. First of all, the identity map is a linear transformation and it is also translation by $\mathbf{0}$, so clearly the identity belongs to $\mathrm{AFF}(V)$. If A is an arbitrary affine transformation, then for each $\mathbf{x} \in V$ we have $A(\mathbf{x}) = S(\mathbf{x}) + \mathbf{y}$, where $\mathbf{y} \in V$ and S is an invertible transformation. If A' is another such transformation, write $A'(\mathbf{x}) = S'(\mathbf{x}) + \mathbf{y}'$ similarly. Then we have

$$A' \circ A(\mathbf{x}) \;=\; A'\big(S(\mathbf{x}) + \mathbf{y}\big) \;=$$

$$S'\big((S(\mathbf{x}) + \mathbf{y})\big) + \mathbf{y}' \;=\; S'S(\mathbf{x}) + \big(S'(\mathbf{y}) + \mathbf{y}'\big)$$

showing that $A' \circ A$ is also affine; hence $\mathrm{AFF}(V)$ is closed under multiplication. To see that this family is closed under taking inverses, let A be as above, and consider the transformation

$$B(\mathbf{x}) \;=\; S^{-1}(\mathbf{x}) \;-\; S^{-1}(\mathbf{y}) \; .$$

By the above formula, $B \circ A$ and $A \circ B$ are both equal to the identity, so that A^{-1} is the affine transformation B. The second statement in the theorem follows from Examples 1 and 2 above.■

In general, $\mathrm{AFF}(V)$ is **not** the full group of geometric symmetries. For example, if $\mathbb{F} = \mathbb{C}$ (the complex numbers), the cooreinatewise conjugation map

$$\chi : \mathbb{C}^n \;\longrightarrow\; \mathbb{C}^n$$

taking $(z_1, \cdots, z_n)$ to $(\overline{z_1}, \cdots, \overline{z_n})$ is a geometrical symmetry of $\mathbb{C}^n$ which does not lie in $\mathrm{AFF}(\mathbb{C}^n)$.[12] However, if $\mathbb{F}$ is the integers mod p (where p is prime), the rational numbers, or the real numbers, then $\mathrm{AFF}(V)$ does turn out to be the entire group of geometric symmetries of $\mathbb{F}^n$.

Historical note.   Although the concept of geometric symmetry was not formulated explicitly until the early 19th century, special symmetries of $\mathbb{R}^n$ (for at least $n = 2, 3$) known as *rigid motions* or *isometries* are implicit in classical Greek geometry, particularly in attempts at "proof by superposition" (a rigid motion is defined to be a 1–1 correspondence T from $\mathbb{R}^n$ to itself that preserves distances (*i.e.*, $d(\mathbf{x}, \mathbf{y}) = d\big(T(\mathbf{x}), T(\mathbf{y})\big)$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ — such maps also preserve angles).[13] More precisely, superposition phrases of the form, "Place figure A so that points B coincide with points C," may be interpreted as saying, "Find a rigid motion of $\mathbb{R}^n$ that maps points B to points C." Indeed, it seems that the inability of classical geometry to justify the notion of superposition resulted from the lack of a precise definition for rigid motions.

EXERCISES

1.   Give a detailed verification of the assertion $T[\mathbf{x} + S] = T(\mathbf{x}) + T[S]$ which appears in Example 1 above.

---

[12]See Exercise V.2.5 for a full explanation.

[13]The fact that rigid motions are geometrical symmetries follows because they all lie in the group $\mathrm{AFF}(V)$ defined previously. A proof is given in the Addendum to Appendix A; see Theorem A.23.

2. If V and W are vector spaces over a field $\mathbb{F}$, a map $T : V \to W$ is called *affine* if it has the form $T(\mathbf{v}) = S(\mathbf{v}) + \mathbf{w}_0$, where S is linear and $\mathbf{w}_0 \in W$. Prove that

$$T\left(t\mathbf{x} + (1-t)\mathbf{y} \right) = tT(\mathbf{x}) + (1-t)T(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in V$ and $t \in \mathbb{F}$.

3. Suppose that $\{\mathbf{v}_0, \cdots, \mathbf{v}_n\}$ and $\{\mathbf{w}_0, \cdots, \mathbf{w}_n\}$ are affine bases for the vector space V. Prove that there is a unique $T \in \mathrm{AFF}(V)$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 0, \cdots, n$. [*Hint:* The sets $\{\mathbf{v}_1 - \mathbf{v}_0, \cdots, \mathbf{v}_n - \mathbf{v}_0\}$ and $\{\mathbf{w}_1 - \mathbf{w}_0, \cdots, \mathbf{w}_n - \mathbf{w}_0\}$ are bases. Apply Theorem A.13(v).]

4. Suppose $\mathbb{F} \neq \mathbb{Z}_2$ is a field, V is a vector space over $\mathbb{F}$, and $T : V \to V$ is a 1–1 onto map satisfying the condition in Exercise 2:

$$T\left(t\mathbf{x} + (1-t)\mathbf{y} \right) = tT(\mathbf{x}) + (1-t)T(\mathbf{y})$$

Prove that T is an affine transformation. [*Hint:* Set $S(\mathbf{v}) = T(\mathbf{v}) - T(\mathbf{0})$ and prove S is linear. Observe that S maps $\mathbf{0}$ to itself. — If the problem seems too hard as stated, try to prove if for fields in which $1 + 1 \neq 0$.]

5. Let $f : (S, \Pi, d) \to (S', \Pi', d')$ be an isomorphism of geometrical incidence spaces, and let P and Q be nonempty geometrical subspaces of S in the extended sense (*i.e.*, we include the possibility that either consists of one point). If "$\star$" is the join construction described in the notes, prove that $f[P \star Q] = f[P] \star f[Q]$. [*Hint:* Recall that the join of two geometrical subspaces is the unique smallest geometrical subspace containing both of them.]

6. Let $f : (S, \Pi, d) \to (S', \Pi', d')$ be an isomorphism of geometrical incidence spaces.

(i) Prove that (S, Π, d) is regular if and only if (S', Π', d') is.

(ii) Prove that (S, Π, d) is an affine n-space if and only if (S', Π', d') is.

(iii) Let $m > 2$ be an integer. Prove that every line in (S, Π, d) contains exactly m points if and only if the same is true for (S', Π', d'). Explain why a similar conclusion holds if we substitute "infinitely many" for "exactly m" in the statement.

7. Let V be a vector space over a field of characteristic zero (*i.e.*, no iterated finite sum $1 + \cdots + 1$ is equal to zero), and let $T \in \mathrm{AFF}(V)$ be an element such that some finite power $T^n = T \circ \cdots \circ T$ is the identity. Prove that there is some point $\mathbf{x} \in V$ such that $T(\mathbf{x}) = \mathbf{x}$ (*i.e.*, a **fixed point**). [*Hints:* Let $\mathbf{v} \in V$ be arbitrary, and consider the vector

$$\frac{1}{n} \sum_{i=0}^{n-1} T^i(\mathbf{v}) \ .$$

Note that the transformation T may have more than one fixed point; for example, if we take the reflection map T on $\mathbb{R}^2$ which sends (x, y) to $(x, -y)$, then T^2 is the identity and T sends every point on the x-axis to itself.]

CHAPTER III

CONSTRUCTION OF PROJECTIVE SPACE

In this chapter we describe the classical process of adding points at infinity to an affine plane or a 3-dimensional affine space. The objects obtained in this manner are called **projective planes** or **projective spaces**, and predictably they are some of the main objects studied studied in projective geometry.

1. Ideal points and lines

> Extending the space ... [is a] fruitful method for extracting understandable results from the bewildering chaos of special cases. — **J. Dieudonné (1906–1992)**[1]

In calculus — particularly in the study of limits — it is frequently convenient to add one or two *numbers at infinity* to the real number system.[2] Among the reasons for this are the following:

- (i) It allows one to formulate otherwise complicated notions more understandable (for example, infinite limits).
- (ii) It emphasizes the similarities between the infinite limit concept and the ordinary limit concept.
- (iii) It allows one to perform formal manipulations with limits much more easily.

For example, suppose we add a single point at infinity (called ∞ as usual) to the real numbers. If f is a real-valued rational function of the form $f(t) = p(t)/q(t)$, where p and q are polynomials with no common factors and q is not identically zero, then strictly speaking f is not definable at the roots of q. However, an inspection of the graph of f suggests defining its value at these points to be ∞, and if this is done the function is also continuous at the roots of q (in an appropriate sense). Proceeding further along these lines, one can even define $f(\infty)$ in such a way that f is continuous at ∞; the limit value may be a finite number or ∞, depending upon whether or not the degree of p is less than or equal to the degree of q (in which case the limit value is finite; see Exercise 6 for more on this) or the degree of p is greater than the degree of q (in which case the limit value is infinite).

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[1] JEAN DIEUDONNÉ is known for his work in algebra, his participation in major projects which provided systematic accounts of the foundations for several branches of mathematics, and his later writings on the history of mathematics, mainly covering the past two centuries.

[2] Of course, if this is done then one must also recognize that the *numbers at infinity* do not necessarily have all the useful properties of ordinary real numbers. The existence of such difficulties has been recognized since ancient times, and in particular some of these issues are implicit in the celebrated paradoxes which are attributed to ZENO OF ELEA (*c.* 490 B.C.E. – *c.* 425 B.C.E.).

The discussion above illustrates the ideas presented in the following quotation from previously cited the book by R. Winger.[3]

> Such exceptions are not uncommon in mathematics but they can frequently be avoided by aid of appropriate expedients. Often it suffices to modify definitions or merely adopt contentions of language. But sometimes new postulates or assumptions are required. This in algebra we might say that the quadratic equation $x^2 - 2ax + a^2 = 0$ has only one root a. For the sake of uniformity however it is customary to say that the equation has *two equal roots*. Here a change of language is all that is needed. On the other hand if the equation $x^2 + x + 1 = 0$ is to have any root it is necessary to extend the domain of numbers to include the imaginary numbers. With these conventions — that a repeated root counts for two and that imaginary roots are to be accepted equally with real — we can say *every* quadratic equation has two roots.
>
> Again we might say that a circle cuts a line of its plane in two points, one point or no point. But with the proper modifications we can make the geometry conform to the algebra. Thus a tangent is considered as meeting the curve in "two coincident points." But in order that the statement shall be true universally it is necessary to introduce a new class of points, the "imaginaries." Imaginary points correspond to the imaginary numbers of algebra [*i.e.*, their coordinates are given by complex numbers]. If in solving the equations of line and circle the roots turn out to be imaginary, the points of intersection are said to be imaginary. "No point" is now replaced by "two imaginary points" when without exception sl a line cuts a circle in two points — real and distinct, coincident or imaginary. The new statement not only serves every purpose of the old but is really more descriptive of the true relation of line and circle.

The preceding quotation illustrates that "imaginary numbers" in algebra have geometric analogs which can be viewed as "imaginary points." Similarly, *the previously discussed algebraic and analytic notions of* **"numbers at infinity"** *have geometric analogs which can be viewed as* **"points at infinity,"** and these play a fundamental role in projective geometry.

One intuitive motivation for considering such points at infinity arose in connection with the mathematical theory of *perspective drawing* which was developed in the 15th and 16th centuries. In modern and informal terms, the goal of this theory is to describe the photographic images of physical objects on a planar surface such as a projection screen, and artist's canvas, or a sheet of paper. Everyday experience with viewing photographs shows that some properties of objects are faithfully reflected by a photographic image while others are not. For example, distances often change drastically and two physical objects with the same measurements usually project to images with quite different measurements. On the other hand, the photographic images of a physical line will be contained in a line on the image plane.

We shall be particularly interested in a curious phenomenon involving parallel lines.[4] If one sees enough examples, it becomes apparent that the images of parallel lines are not necessarily parallel, and if the images are not parallel then the images of all lines parallel to the two given lines appear to meet at some point on the horizon; furthermore, all these horizon points lie appear to lie on a single line which defines the horizon (see Figure III.1[5] below). It is possible to

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[3]Winger, *Introduction to Projective Geometry*, pp. 31–32.

[4]This was known in ancient times and is mentioned in the work of Marcus Vitruvius Pollio (*c.* 80 B. C. E. – 25 B. C. E.) titled *De Architectura*.

[5]Source: `http://www.math.nus.edu.sg/aslaksen/projects/perspective/alberti.htm`

give a mathematical explanation for these empirical observations, but we shall not do so here. A more detailed discussion appears in Section IV.2 of the online reference

$$\texttt{http://math.ucr.edu/}\sim\texttt{res/math133/geometrynotes4a.pdf}$$

that was cited in the Preface.
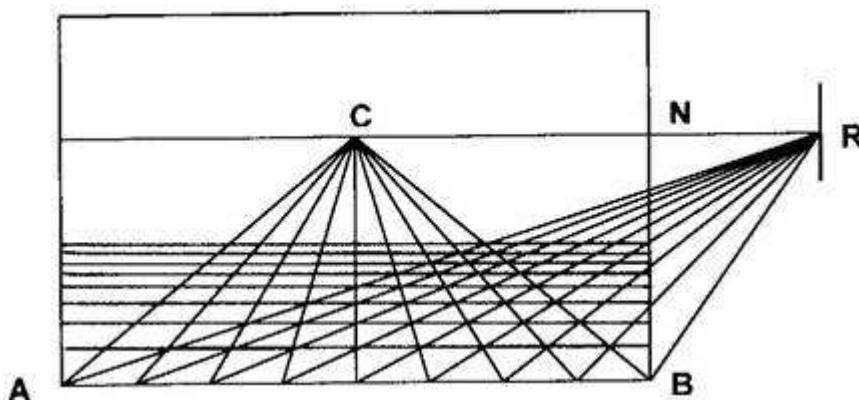
Figure III.1

Such pictures lead to speculation whether we should think of lines as having *points at infinity*, such that every line has exactly one such point, two parallel lines have the same point at infinity, and the points at infinity for lines on a given plane all lie on a *line at infinity*. These ideas emerged near the beginning of the 17th century, and in particular they were developed and extended in the writings of G. Desargues[6] (1591–1661) and J. Kepler[7] (1571–1630).

The main purpose of this section is to provide a mathematical setting for Desargues' and Kepler's intuitive ideas. In particular, the following continuation of the previous quotation contains the main motivation for the construction of projective space (and much of projective geometry):

> To say that two parallel lines do not meet is like saying that certain lines have no point of intersection with a circle. There we found that the exception could be removed by introducing imaginary intersections. In an exactly analogous fashion we may introduce a second new class of points into geometry, *points at infinity*, which will serve for the "intersections of parallel lines."

The formal process for adding points at infinity to the Euclidean plane is best described as follows:

Definition. Let $(P, \mathcal{L})$ be an affine plane, and let L be a line in P. The L-**direction** in P (or the **direction** of P) consists of L and all lines parallel to L.

Note. We are using the notation for incidence plane described in Section II.1 rather than the more general notation in Section II.5. The translation is straightforward: If $(P, \mathcal{L})$ is one description and (P, Π, d) is the other, then $d(P) = 2$, $\Pi = \mathcal{L} \cup \{P\}$ and $d = 1$ on $\mathcal{L}$.

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[6]GIRARD DESARGUES is mainly known for transforming the theory of geometrical perspective into the subject now called projective geometry, and he also did work on conic sections and other geometrical problems motivated by practical considerations.

[7]JOHANNES KEPLER is best known for discovering the basic laws of planetary motion, but he also made some important contributions to mathematics.

THEOREM III.1. *Two directions in P are either disjoint or identical.*

Proof. Consider the binary relation $\sim$ on lines defined by $L \sim M$ if and only if $L = M$ or $L||M$. By Theorem II.7 this is an equivalence relation. The L-direction is merely the equivalence class of L with respect to this relation and will be denoted by $[L]$.∎

Definition. If P is an affine plane, the *projective extension* pf P, denoted by $P^{\wedge}$, consists of P together with all directions in P. An element of $P^{\wedge}$ is an ordinary point if it is a point in P and an *ideal point* or *point at infinity* if it is a direction in P. It follows that a point in is either ordinary or ideal but not both.[8]

If L is a line in P, then its *projective extension* to $P^{\wedge}$ consists of all points of L together with the direction $[L]$, and it is denoted by $L^{\wedge}$. The *ideal line* in $P^{\wedge}$ consists of all ideal points in $P^{\wedge}$ and is denoted by L_{∞} (the line of points at infinity or the *line at infinity*).

EXAMPLES INVOLVING USES OF POINTS AT INFINITY. The most immediate reason for introducing points at infinity is that they can be used to simplify some of the statements which appeared in earlier chapters. For example, the statement

> The lines L, M and N are either concurrent (pass through a single point) or are parallel in pairs.

in the conclusion of Theorem II.25 translates to

> The projective extensions of the lines L, M and N contain a common point (*i.e.*, the extended lines $L^{\wedge}$, $M^{\wedge}$ and $N^{\wedge}$ are concurrent).

Furthermore, the statement

> There is a point C which lies on line AB or else there are two lines L, and M such that $L||AB$ and $M||AB$,

which is the conclusions of Theorem II.28 and II.29, translates to

> There are three lines AB, L and M such that either (i) L and M meet at a point $C \in AB$, or else the lines L and M are both parallel to and their common point at infinity lies on the projective extension of AB (*i.e.*, if the projective extensions of L and M meet at the point C, then C lies on the projective extension of the line AB).

Since the hypotheses of Theorems II.27–29 are very similar (the only difference being parallelism assumptions about these lines) the second translated statement illustrates a corresponding similarity in the conclusions that one might suspect (or at least hope for). In fact, the conclusion of Theorem II.27 (that one has three pairs of parallel lines) fits into the same general pattern, for in this case the conclusion reduces to the collinearity of the directions $A^{\wedge}$, $B^{\wedge}$ and $C^{\wedge}$, where these directions contain the pairs $\{\, L, L' \,\}$, $\{\, M, M' \,\}$, and $\{\, N, N' \,\}$ respectively.

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[8]A rigorous proof of this fact requires some technical propositions from set theory. The details of this justification are not important for the rest of these notes, but for the sake of completeness here is the proof: Suppose that $\mathbf{x} = [L]$ for suitable $\mathbf{x}$ and L. Then there is a line M such that $\mathbf{x} \in \mathbf{M}$ and L is parallel to M or equal to M. This would imply that $\mathbf{x} \in M$ and $[M] = [L] = \mathbf{x}$, so we would have objects a and b such that $a \in b$ and $b \in a$; the standard mathematical foundations for set theory contain an assumption (the *Axiom of Foundation/Regularity*) which implies that such situations never arise. For further information, see Proposition 4.2 in the following online notes: `http://math.ucr.edu/~res/math144/setsnotes3.pdf`

One conceivable objection to *ideal points* or *points at infinity* is the impossibility of visualizing such entities. The mathematical answer to such objection is contained in the following quotation from an article by O. Veblen (1880–1960):[9]

> Ordinary points are just as much idealized as are the points at infinity. No one has ever seen an actual point [with no physical width, length or thickness] or realized it by an experiment of any sort. Like the point at infinity it is an ideal creation which is useful for some of the purposes of science.

Here is a slightly different response:[10]

> [With the introduction of points at infinity,] The propositions of projective geometry acquire a simplicity and a generality that they could not otherwise have. Moreover, the elements at infinity give to projective geometry a degree of unification that greatly facilitates the thinking in this domain and offers a suggestive imagery that is very helpful ... On the other hand, projective geometry stands ready to abandon these ... whenever that seems desirable, and to express the corresponding propositions in terms of direction of a line ... to the great benefit of ... geometry. ...

> The extra point which projective geometry claims to add to the Euclidean [or affine] line is [merely] the way in which projective geometry accounts for the property of a straight line which Euclidean [or affine] geometry recognizes as the "direction" of the line. The difference between the Euclidean [or affine] line and the projective line is purely verbal. The geometric content is the same. ... Such a change in nomenclature does not constitute an [actual] increase in the geometric content.

To summarize the preceding quotations, sometimes it is convenient to work in a setting where one has points at infinity, and in other cases it is more convenient to work in a setting which does not include such objects. This is very closely reflects the standard usage for numbers at infinity. They are introduced when they are useful — with proper attention paid to the differences between them and ordinary numbers — and not introduced when it is more convenient to work without them.

EXERCISES

In the exercises below, assume that P is an affine plane and that $x^{\wedge}$ (where $x = P$ or a line in P) is defined as in the notes.

1. Prove that every pair of lines in $P^{\wedge}$ (as defined above) has a common point in $P^{\wedge}$. [*Hint:* There are three cases, depending upon whether one has extensions of two ordinary lines that have a common point in P, extensions of two ordinary lines that are parallel in P, or the extension of one line together with the line at infinity.]

2. Suppose that $L_1, L_2, L_3, M_1, M_2, M_3$ are six distinct lines in an affine plane P. Write out explicitly what it means (in affine terms) for the three points determined by $L_i^{\wedge} \cap M_i^{\wedge}$ (where $i = 1, 2, 3$) to be collinear in $P^{\wedge}$. Assume that the three intersection points are distinct.

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[9]Encyclopædia Britannica, 14th Edition (1956),Vol. 18, p.173 (article on Projective Geometry). OSWALD VEBLEN made several important contributions to geometry and topology.

[10]N. Altshiller Court (1881–1968), *Mathematics in Fun and in Earnest*, pp. 110, 112

3.   Suppose that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ are six distinct points in an affine plane P. Write out explicitly what it means (in affine terms) for the three extended lines $(\mathbf{x}_i\mathbf{y}_i)^{\wedge}$ (where $i = 1, 2, 3$) to be concurrent in $P^{\wedge}$. Assume that the three ordinary lines $\mathbf{x}_i\mathbf{y}_i$ are distinct.

4.   Prove that every line in $P^{\wedge}$ contains at least three points. There are two cases, depending upon whether the line is the extension of an ordinary line or the line at infinity, and the latter requires more work than the former.

5.   Suppose that we are given four distinct lines L_1, L_2, L_3, L_4 in P such that no three of the projective extensions $L_i^{\wedge}$ contain a common point in $P^{\wedge}$. Translate this into a statement about L_1, L_2, L_3, L_4 which does not involve ideal points or projective extensions.

6.   Suppose that f is real-valued rational function of the form $f(t) = p(t)/q(t)$, where p and q are nonzero polynomials, and write

$$p(t) = a_n t^n + \cdots + a_0 , \qquad q(t) = b_m t^m + \cdots + b_0$$

where $a_n \neq 0$, $b_m \neq 0$, and the degrees m and n satsify $m \leq n$. How should one define $f(\infty)$? There are two cases, depending on whether $m < n$ or $m = n$.

7.   In the setting of the previous exercise, explain why the value of $f + g$ at ∞ is equal to $f(\infty) + g(\infty)$ if f and g are rational functions such that $f(\infty)$ and $g(\infty)$ are finite, and the value is equal to ∞ if one of the values $f(\infty)$, $g(\infty)$ is finite and the other is infinite. Give examples of f and g such that $f(\infty) = g(\infty) = \infty$ for which (a) the value of $f + g$ at ∞ is equal to ∞, (b) the value of $f + g$ at ∞ is equal to 0, (c) the value of $f + g$ at ∞ is equal to 1. — The point of this exercise is that sometimes familiar statements generalize when one adds elements at infinity, but in other instances such familiar statements definitely cannot be generalized.

2. Homogeneous coordinates

If $\mathbb{F}$ is a field and $\mathbb{F}^2$ is the coordinate affine plane described in Chapter II, then the construction in the previous section can of course be applied to $\mathbb{F}^2$. The purpose of this section is to introduce coordinates for the projective extension $(\mathbb{F}^2)^\wedge$ that are consistent with the usual coordinates for ordinary points. Since every point in $\mathbb{F}^2$ is specified by two scalars (the first and second coordinates), we shall not be able to describe a point entirely in terms of two coordinates. Instead, it will be necessary to use **three** coordinates to specify a point in $(\mathbb{F}^2)^\wedge$, with the understanding that **different triples of scalars may represent the same point.** This can be inconvenient sometimes, but it is an issue that already arises in elementary analytic geometry; specifically, when we try to specify a point in the ordinary plane $\mathbb{R}^2$ by polar coordinates, it is always necessary to remember that (r, θ), $(-r, \theta + \pi)$ and $(r, \theta + 2\pi)$ represent the same point in $\mathbb{R}^2$ if $r \neq 0$ (and of course all pairs of the form $(0, \theta)$ represent the origin!).

To be more specific about the meaning of compatibility, we would like our coordinates for $(\mathbb{F}^2)^\wedge$ to have the following properties:

(i) The ordinary coordinates for a point $(x, y) \in \mathbb{F}^2$ should be easily retrievable from the systems of coordinates we shall introduce on $(\mathbb{F}^2)^\wedge$, and vice versa.

(ii) If we are given a line L in $\mathbb{F}^2$ defined by an equation of the form $ax + by + c = 0$ where a and b are not both zero, the coefficients a and b should be easily retrievable from the coordinates of the associated point at infinity $L^\wedge$, and vice versa.

Of course, in the second part the coefficients a, b, c are not uniquely defined, and any nonzero multiple of these equations yields an equivalent equation for the line. This sort of ambiguity up to multiplication by some common nonzero factor is the key idea behind the definition of **homogeneous coordinates** for points in $(\mathbb{F}^2)^\wedge$.

Suppose first that L is a line through $\mathbf{0}$ in $\mathbb{F}^2$. Then points on this line have the form (tx_1, tx_2), where $\mathbf{x} = (x_1, x_2)$ is a fixed nonzero vector. With some imagination, one might speculate about trying to define the coordinates for the point at infinity on L by something like $(\infty x_1, \infty x_2)$ or equivalently by

$$\left(\frac{x_1}{0}, \frac{x_2}{0} \right) .$$

Of course, we cannot naïvely do this in a logically sound manner (for example, if $\mathbb{F} = \mathbb{R}$ then $1/0 = 2/0 = \infty$ and $0/0$ is indeterminate), but we can express the concept using an ordered triple

$$(v_1, v_2, 0)$$

which is meant to suggest that we would divide the first two coordinates by zero if this made sense. As already noted, if we use such notation then we must also be ready to agree that every ordered triple of the form $(tv_1, tv_2, 0)$, where $t \neq 0$, is also a valid description fot the original line L. We can formalize this by saying that every such triple is a set of *homogeneous coordinates* for the point at infinity $L^\wedge$.

For the sake of uniformity, we would also like to describe coordinates for ordinary points as ordered triples (y_1, y_2, y_3) such that if $t \neq 0$ and (y_1, y_2, y_3) is a valid set of coordinates for a point, then so is (ty_1, ty_2, ty_3). We do this by agreeing that if $\mathbf{x} \in \mathbb{F}^2$, then *every ordered triple of the form (tx_1, tx_2, t), where $t \neq 0$, is a set of* homogeneous coordinates *for* $\mathbf{x}$.

The next result states that the preceding definitions of homogeneous coordinates for points of $(\mathbb{F}^2)^\wedge$ have the desired compatibility properties and that every ordered triple $(y_1, y_2, y_3) \neq (0,0,0)$ is a valid set of coordinates for some point in $(\mathbb{F}^2)^\wedge$.

THEOREM III.2. *Every nonzero element of $\mathbb{F}^3$ is a set of homogeneous coordinates for some point in $(\mathbb{F}^2)^\wedge$. Two nonzero elements are homogeneous coordinates for the same point if and only if each is a nonzero multiple of the other.*

Proof.   Let $(y_1, y_2, y_3) \neq (0,0,0)$ in $\mathbb{F}^3$. If $y_3 \neq 0$ then (y_1, y_2, y_3) is a set of homogeneous coordinates for the ordinary point

$$\left(\frac{y_1}{y_3}, \frac{y_2}{y_3} \right) \ .$$

On the other hand, if $y_3 = 0$ then either $y_1 \neq 0$ or $y_2 \neq 0$ and the point has the form $(y_1, y_2, 0)$; the latter is a set of homogeneous coordinates for the point at infinity on the line joining the two distinct ordinary points $\mathbf{0}$ and (y_1, y_2); these points are distinct because at least one of the y_i is nonzero.

By definition, if $t \neq 0$ then (y_1, y_2, y_3) and (ty_1, ty_2, ty_3) are sets of homogeneous coordinates for the same point, so it is only necessary to prove the converse statement. Therefore suppose that (x_1, x_2, x_3) and (y_1, y_2, y_3) are homogeneous coordinates which represent the same point. There are two cases, depending upon whether or not this point is an ordinary point or an ideal point.

Suppose the point under consideration is the ordinary point $(z_1, z_2) \in \mathbb{F}^2$. Then there exist nonzero constants a and b such that

$$(x_1, x_2, x_3) \ = \ (az_1, az_2, a) \qquad (y_1, y_2, y_3) \ = \ (bz_1, bz_2, b) \ .$$

It follows immediately that $(y_1, y_2, y_3) \ = \ ba^{-1} (x_1, x_2, x_3)$.

Suppose now that the point under consideration is an ideal point, and choose $\mathbf{v} \in \mathbb{F}^2$ so that $\mathbf{v} \neq \mathbf{0}$ and the ideal point is contained in the line $\mathbf{0v}$ (the existence of such a line is guaranteed by the Euclidean Parallelism Property, which holds in $\mathbb{F}^2$). Let $\mathbf{v} = (v_1, v_2)$, so that the ideal point has homogeneous coordinates $(v_1, v_2, 0)$. In this case there exist nonzero constants a and b such that

$$(x_1, x_2, x_3) \ = \ (av_1, av_2, 0) \qquad (y_1, y_2, y_3) \ = \ (bv_1, bv_2, 0) \ .$$

It follows immediately that $y_3 = x_3 = 0$ and hence the vectors $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ satisfy $\mathbf{x} = a\mathbf{v}$ and $\mathbf{y} = b\mathbf{v}$; the latter implies that $\mathbf{y} = ba^{-1}\mathbf{x}$ and hence again in this case we conclude that (x_1, x_2, x_3) and (y_1, y_2, y_3) are nonzero multiples of each other.∎

THEOREM III.3. *There is a $1-1$ correspondence between the points of the projective extension $(\mathbb{F}^2)^\wedge$ and the 1-dimensional vector subspaces of $\mathbb{F}^3$ such that a point $\mathbf{x}$ in the former corresponds to a 1-dimensional subspace V if and only if the set of all possible homogeneous coordinates for $\mathbf{x}$ is the set of nonzero vectors in V.*

Proof.   Given $\mathbf{x} \in (\mathbb{F}^2)^\wedge$, let $V(\mathbf{x})$ denote the vector space spanned by all the homogeneous coordinates for $\mathbf{x}$. By the preceding result, we know that $V(\mathbf{x})$ is a 1-dimensional vector subspace because it contains a nonzero vector and every pair of nonzero vectors in $V(\mathbf{x})$ are nonzero scalar multiples of each other. Theorem 2 also implies that the correspondence sending $\mathbf{x}$ to $V(\mathbf{x})$ is

onto (every 1-dimensional vector space is spanned by some nonzero vector). Thus it remains to show that the correspondence is 1–1.

Suppose that $\mathbf{x}$ and $\mathbf{y}$ are such that $V(\mathbf{x}) = V(\mathbf{y})$. There are three possible cases, depending upon whether both points are ordinary points, both points are ideal points, or one point is ordinary and the other is ideal.

CASE 1. Suppose both points are ordinary. Let $\mathbf{v} = (v_1,\, v_2)$ where $\mathbf{v} = \mathbf{x}$ or $\mathbf{y}$. If $V(\mathbf{x}) = V(\mathbf{y})$, then we know that $(x_1,\, x_2,\, 1)$ and $(y_1,\, y_2,\, 1)$ span the same 1-dimensional subspace and hence are nonzero multiples of each other. However, if c is a nonzero scalar such that $(x_1,\, x_2,\, 1) = c\,(y_1,\, y_2,\, 1)$, then it follows immediately that $c = 1$, which implies that $\mathbf{x} = \mathbf{y}$.

CASE 2. Suppose both points are ideal. Suppose that the points in question are the ideal points on the lines joining the origin in $\mathbb{F}^2$ to the nonzero points $\mathbf{u} = (u_1,\, u_2)$ and $\mathbf{v} = (v_1,\, v_2)$. It then follows that the vectors $(u_1,\, u_2,\, 0)$ and $(v_1,\, v_2,\, 0)$ span the same 1-dimensional subspace and hence are nonzero multiples of each other. But this implies that $\mathbf{u}$ and $\mathbf{v}$ are also nonzero multiples of each other, which in turn means that the lines joining $\mathbf{0}$ to $\mathbf{u}$ and $\mathbf{v}$ are equal and hence have the same ideal point. Therefore the map taking $\mathbf{x}$ to $V(\mathbf{x})$ is 1–1 on the set of ideal points.

CASE 3. Suppose one point is ordinary and the other is ideal. If the ordinary point has homogeneous coordinates given by $(x_1,\, x_2,\, 1)$ and the ideal point has homogeneous coordinates given by $(y_1,\, y_2,\, 0)$, then once again these two vectors must be nonzero multiples of each other. Since the third coordinate of the first vector is equal to 1, this is impossible, and thus we see that ordinary points and ideal points cannot determine the same 1-dimensional subspace of $\mathbb{F}^3$.∎

EXERCISES

1. Suppose that the line L in $\mathbb{F}^2$ is defined by the equation

$$ax \;+\; by \;=\; c$$

where not both of a and b are zero. Show that homogeneous coordinates for the point $L^{\wedge}$ are given by $(-b,\, a,\, 0)$. [*Hint:* What is the equation of the line through $(0,0)$ which is parallel to L?]

2. Suppose that $\mathbb{F} = \mathbb{R}$ (the real numbers), and $\mathbf{x} = (x_1,\, x_2) \in \mathbb{R}^2$ is nonzero. Let $(y_1,\, y_2,\, y_3)$ be a set of homogeneous coordinates for $\mathbf{x}$ such that $y_1^2 + y_2^2 = 1$. Prove that $|y_3|$ is the reciprocal of the distance from $\mathbf{x}$ to the origin. [*Hint:* Start by explaining why $x_1^2 + x_2^2 > 0$.]

> *Note.* Exercise 2 reflects one reason why ideal points are also known as *points at infinity*. Their last coordinates always vanish, so a formal candidate for their difference to the origin would be the reciprocal of 0, which we often think of as ∞, at least informally.

3. Equations of lines

In Theorem I.7 we noted that lines in $\mathbb{F}^2$ are precisely the subsets defined by linear equations of the form $ax + by + c = 0$, where a and b are not both zero. An analogous characterization for lines in $(\mathbb{F}^2)^{\wedge}$ will be given in this section.

THEOREM III.4. *Let L be a line in $\mathbb{F}^2$ defined by the equation $ax + by + c = 0$ as above, and let $L^{\wedge}$ be the extension of L to a line in $(\mathbb{F}^2)^{\wedge}$. Then $L^{\wedge}$ consists of all points $\mathbf{x}$ which are representable by homogeneous coordinates (u_1, u_2, u_3) satisfying*

$$a\,u_1 + b\,u_2 + c\,u_3 = 0 \, .$$

Remark. If one set of homogeneous coordinates for a point $\mathbf{x}$ satisfies the equation above, then so does every other set, for every other set has the form (ku_1, ku_2, ku_3) for some $k \neq 0$ and hence $a\,u_1 + b\,u_2 + c\,u_3 = 0$ implies

$$a(ku_1) + b(ku_2) + c(ku_3) = k(a\,u_1 + b\,u_2 + c\,u_3) = k \cdot 0 = 0 \, .$$

Proof. If (x, y) is an ordinary point if $(\mathbb{F}^2)^{\wedge}$, then it belongs to L and has homogeneous coordinates $(x, y, 1)$. Therefore every ordinary point of L has homogeneous coordinates which satisfy the homogeneous linear equation in three variables that is displayed in the theorem. Furthermore, by Exercise 1 for the preceding section, the ideal point of $(\mathbb{F}^2)^{\wedge}$ has homogeneous coordinates given by $(-b, a, 0)$, and hence the ideal point also has homogeneous coordinates which satisfy the homogeneous linear equation $a\,u_1 + b\,u_2 + c\,u_3 = 0.\square$

Conversely, suppose that (x, y) is an ordinary point not on the line L. Then (x, y) has homogeneous coordinates $(x, y, 1)$, and these coordinates do not satisfy the equation $a\,u_1 + b\,u_2 + c\,u_3 = 0$. By the remark preceding the proof, it follows that an ordinary point lies on L (equivalently, $(\mathbb{F}^2)^{\wedge}$) if and only if it has homogeneous coordinates which satisfy the equation $a\,u_1 + b\,u_2 + c\,u_3 = 0$.

Finally suppose $\mathbf{z}$ is an ideal point whose homogeneous coordinates satisfy $a\,z_1 + b\,z_2 + c\,z_3 = 0$. Since $\mathbf{z}$ is an ideal point, we also know that $z_3 = 0$. Therefore we must have $a\,z_1 + b\,z_2 = 0$. However, the solution space for this nontrivial equation is 1-dimensional by Theorem A.10, and hence the nonzero vector (z_1, z_2) must be a nonzero multiple of $(-b, a)$. Therefore Exercise 1 of the preceding section implies that $\mathbf{z}$ must be the ideal point of $(\mathbb{F}^2)^{\wedge}$, and this completes the proof of the theorem.■

There is a similar characterization of the line at infinity.

THEOREM III.5. *A point of $(\mathbb{F}^2)^{\wedge}$ is an ideal point if and only if it has homogeneous coordinates which satisfy the equation $u_3 = 0$.*

Proof. Suppose that we are given an ideal point. By construction it has homogeneous coordinates which satisfy $u_3 = 0$. Conversely, if we are given an ordinary point $\mathbf{x}$, then as noted before we know that $\mathbf{x}$ has homogeneous coordinates of the form $(x_1, x_2, 1)$, and hence **every** set of homogeneous coordinates (u_1, u_2, u_3) must satisfy $u_3 \neq 0$. Thus the ideal points are characterized by the vanishing of the third homogeneous coordinate.■

The previous two theorems yield the desired characterization of lines in $(\mathbb{F}^2)^\wedge$.

THEOREM III.6. *A set of points X in $(\mathbb{F}^2)^\wedge$ is line if and only if there exist a, b, $c \in \mathbb{F}$ not all zero such that $\mathbf{y} \in L$ if and only if $\mathbf{y}$ has homogeneous coordinates (y_1, y_2, y_3) satisfying $a\,y_1 \ + \ b\,y_2 \ + \ c\,y_3 \ = \ 0$.*

Proof.   By the preceding two results, every line in $(\mathbb{F}^2)^\wedge$ is defined by a nontrivial linear homogeneous equation of the type described. Conversely, suppose that $(a, b, c) \neq (0, 0, 0)$. If we have $a = b = 0$, then the linear homogeneous equation is equivalent to the equation $y_3 = 0$, which defines the line at infinity. On the other hand, if at least one of $\{a,\ b\}$ is nonzero, then by Theorem 4 we know that the equation defines the extension $L^\wedge$ of the line L in $\mathbb{F}^2$ with equation $ax + by + c = 0$.■

EXERCISES

1.   Find the equations of the lines joining points in $(\mathbb{R}^2)^\wedge$ with the following homogeneous coordinates:

(i) $(1, 3, 0)$ and $(0, 5, -1)$

(ii) $(2, 5, -3)$ and $(3, -2, 0)$

(iii) $(2, 1, 0)$ and $(-1, 1, 0)$

(iv) $(4, -6, 3)$ and $(4, -6, 1)$

2.   Find the homogeneous coordinates of the intersection points of the following lines in $(\mathbb{R}^2)^\wedge$:

(i) $x_1 \ + \ x_3 \ = \ 0$ and $2x_1 \ + \ 3x_2 \ + \ x_3 \ = \ 0$.

(ii) $2x_1 \ + \ 3x_2 \ - \ 4x_3 \ = \ 0$ and $x_1 \ - \ x_2 \ + \ 3x_3 \ = \ 0$.

(iii) $2x_1 \ + \ x_2 \ = \ 0$ and $-x_1 \ + \ x_2 \ = \ 0$.

(iv) $4x_1 \ - \ 6x_2 \ + \ 3x_3 \ = \ 0$ and $4x_1 \ - \ 6x_2 \ + \ x_3 \ = \ 0$.

3.   For $i = 1, 2, 3$ let L_i be the line in $(\mathbb{R}^2)^\wedge$ defined by the equation $a_i x_1 + b_i x_2 + c_i x_3 \ = \ 0$, and assume these lines are distinct. Prove that the lines L_i have a point in common if and only if the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

is not invertible (equivalently, if and only if the determinant of the matrix is zero).

4. Higher-dimensional generalizations

In Chapter II we generalized the geometrical properties of coordinate affine planes and 3-spaces to arbitrary dimensions. The purpose of this section is to show that preceding construction for adding ideal points to $\mathbb{F}^2$ can be extended to the affine spaces associated to arbitrary n-dimensional vector spaces over a given field $\mathbb{F}$, where n is an arbitrary positive integer.

NOTATION. The projective extension $(\mathbb{F}^2)^{\wedge}$ that was defined and studied in the preceding three sections will be called the *coordinate projective plane over* $\mathbb{F}$ and will be denoted by the symbol $\mathbb{F}\mathbb{P}^2$; this conforms to standard mathematical terminology.

The results of the preceding two sections imply the following two basic properties of $\mathbb{F}\mathbb{P}^2$:

  (1) The *points* of $\mathbb{F}\mathbb{P}^2$ correspond to *one-dimensional* vector subspaces of $\mathbb{F}^3$.
  (2) The *points* of $\mathbb{F}\mathbb{P}^2$ correspond to *two-dimensional* vector subspaces of $\mathbb{F}^3$, and a point $\mathbf{x}$ lies on a line L if and only if the one-dimensional subspace V associated to $\mathbf{x}$ is contained in the two-dimensional subspace W associated to L.

The first of these is just a restatement of Theorem 3. The second statement follows because a subset of $\mathbb{F}^3$ is a 2-dimensional vector subspace if and only if it is the set of solutions to a nontrivial linear homogeneous equation in three variables; this is essentially a special case of the characterization of $(n-1)$-dimensional vector subspaces of an n-dimensional vector space in the exercises for Section A.4 (in Appendix A).

Motivated by this identification, if V is a finite-dimensional left or right[11] vector space over a skew-field $\mathbb{F}$, we shall define the *projective space with coefficients in a* **nonzero** *vector space* V as follows: Its *points* are the elements of the set $\mathcal{S}_1(V)$ of 1-dimensional vector subspaces of V. Note that if W is a nonzero vector subspace of V, then $\mathcal{S}_1(W)$ is contained in $\mathcal{S}_1(V)$. The *geometric subspaces* of $\mathcal{S}_1(V)$ are given by all subsets of the form $\mathcal{S}_1(W)$, where W is a vector subspace of dimension ≥ 2 and the geometric *dimension* of $\mathcal{S}_1(W)$ is equal to $\dim W - 1$. — The shift of dimensions is consistent with our previous construction of homogeneous coordinates for $\mathbb{F}\mathbb{P}^2$; in particular, if $\dim V = n$, then the dimension of $\mathcal{S}_1(V)$ is equal to $n - 1$.

If we adopt the conventions of Section II.5 for geometrical subspaces of dimension 0 and -1 (one point subsets are zero-dimensional and the empty set is (-1)-dimensional, then the equation

$$d\left(\mathcal{S}_1(W)\right) \; = \; \dim W \; - \; 1$$

also holds in these extended cases because $\dim X = 1$ implies $cS_1(X) = \{\, X \,\}$ and also

$$cS_1\left(\{\mathbf{0}\}\right) \; = \; \varnothing$$

(since the zero subspace has no 1-dimensional subspaces).

NOTATION. If $\mathbf{x} \in \mathcal{S}_1(V)$, then symbols like $\widetilde{\mathbf{x}}$ will denote nonzero vectors in V which belong to the 1-dimensional subspace $\mathbf{x}$, and such a vector $\widetilde{\mathbf{x}}$ will be called a set of *homogeneous coordinates* for $\mathbf{x}$.

If $V = \mathbb{F}^{n+1}$, then $\mathcal{S}_1(V)$ will be callled the *standard (coordinate) projective n-space over* $\mathbb{F}$ and it is usually denoted by $\mathbb{F}\mathbb{P}^n$. As with affine spaces, the projective spaces whose geometrical

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[11]The geometric significance of doing everything for both left and right vector spaces will be apparent in Chapter V, but if one is only interested in cases where $\mathbb{F}$ is a field then the distinction is unnecessary.

usefulness is most evident are the projective spaces over the real numbers $\mathbb{R}$; these are the so-called *real projective spaces*, and in some cases they are also simply called *projective spaces of n dimensions*. The corresponding objects over the complex numbers $\mathbb{C}$ (the *complex projective spaces*) are nearly as important; the Fundamental Theorem of Algebra suggests one reason for this (however, there are also others). Other types of projective spaces are useful in various contexts which are beyond the scope of these notes.

THEOREM III.7. *If V is an n-dimensional vector space (where $n \geq 3$), then $\mathcal{S}_1(V)$ is a regular $(n-1)$-dimensional geometrical incidence space with respect to the notions of geometrical subspace and dimension that are defined above.*

Proof. We shall first verify the conditions **(G-1)**–**(G-3)** in order, and in the next theorem we shall prove a strengthened version of **(G-4)** in the next theorem.

PROOF OF **(G-1).** Let $X_0, \cdots, X_k$ be 1-dimensional subspaces of V that are not contained in any subspace $\mathcal{S}_1(W)$ of dimension less than k; *i.e.*, there is no vector subspace $W \subset V$ such that $\dim W \leq k$ and $X_i \subset W$ for all i. By the sum formula for dimensions of vector subspaces, we know that $\dim(X_0 + \cdots + X_k) \leq k+1$; furthermore, if strict inequality holds, then $X_0 + \cdots + X_k$ is a vector space of dimension $\leq k$ containing each X_i, and hence we know that all the subspaces X_i belong to some geometric subspace $\mathcal{S}_1(W)$ of dimension less than k. Since we are assuming this does not happen, it follows that $\dim(X_0 + \cdots + X_k) = k+1$, and accordingly the k-plane

$$\mathcal{S}_1(X_0 + \cdots + X_k)$$

contains all the 1-dimensional subspaces X_i. To prove the uniqueness part of **(G-1)**, suppose that $\mathcal{S}_1(W)$ is a k-plane such that $X_i \subset W$ for all i. It follows immediately that $X_0 + \cdots + X_k \subset W$ and

$$k+1 \;=\; \dim(X_0 + \cdots + X_k) \;\leq\; \dim W \;=\; k+1$$

and hence $(X_0 + \cdots + X_k) = W$ by Theorem A.8 so that $\mathcal{S}_1(X_0 + \cdots + X_k)$ is the unique k-plane containing all the X_i.∎

PROOF OF **(G-2).** Suppose that $X_i \in \mathcal{S}_1(W)$ for $0 \leq i \leq m$, and assume that the set $\{X_0, \cdots, X_m\}$ is independent. By the previous proof, the unique m-plane containing the X_i is $\mathcal{S}_1(X_0 + \cdots + X_m)$; since $X_i \subset W$ by our hypotheses, it follows that $(X_0 + \cdots + X_m) \subset W$, and therefore we also have $\mathcal{S}_1(X_0 + \cdots + X_m) \subset \mathcal{S}_1(W)$.∎

PROOF OF **(G-3).** Suppose that $\mathcal{S}_1(W)$ is a k-plane, so that $\dim W = k+1$. Let $\mathbf{w}_0, \cdots, \mathbf{w}_k$ be a basis for W, and let for each i such that $0 \leq i \leq k$ let X_i be the 1-dimensional vector subspace spanned by $\mathbf{w}_i$. Then $\{X_0, \cdots, X_k\}$ is a set of $k+1$ distinct points in $\mathcal{S}_1(W)$.∎

The next result will show that a strengthened form of **(G-4)** holds for $\mathcal{S}_1(V)$.

THEOREM III.8. *If P and Q are geometrical subspaces of $\mathcal{S}_1(V)$ then*

$$d(P) \;+\; d(Q) \;=\; d(P \star Q) \;+\; d(P \cap Q) \;.$$

The difference between this statement and **(G-4)** is that the latter assumes $P \cap Q \neq \varnothing$, but the theorem contains no such assumption.

Proof. We shall first derive the following equations:

(i) $\mathcal{S}_1(W_1 \cap W_2) = \mathcal{S}_1(W_1) \cap \mathcal{S}_1(W_2)$

(ii) $\mathcal{S}_1(W_1 + W_2) = \mathcal{S}_1(W_1) \star \mathcal{S}_1(W_2)$

Derivation of (i). X is a 1-dimensional subspace of $W_1 \cap W_2$ if and only if X is a 1-dimensional subspace of both W_1 and W_2.

Derivation of (ii).   If W is a vector subspace of U, then $\mathcal{S}_1(W) \subset \mathcal{S}_1(U)$. therefore $\mathcal{S}_1(W_i) \subset \mathcal{S}_1(W_1 + W_2)$ for $i = 1, 2$. Therefore we also have $\mathcal{S}_1(W_1 + W_2) \subset \mathcal{S}_1(W_1) \star \mathcal{S}_1(W_2)$. To prove the reverse inclusion, choose U such that $\mathcal{S}_1(U) = \mathcal{S}_1(W_1) \star \mathcal{S}_1(W_2)$. Then $W_i \subset U$ for $I = 1, 2$ follows immediately, so that $W_1 + W - 2 \subset U$. Consequently we have

$$\mathcal{S}_1(W_1 + W_2) \subset \mathcal{S}_1(U) \subset \mathcal{S}_1(W_1 + W_2)$$

which immediately yields $\mathcal{S}_1(W_1 + W_2) = \mathcal{S}_1(U) = \mathcal{S}_1(W_1 + W_2)$.

To prove the theorem, note that

$$d\left(\mathcal{S}_1(W_1) \star \mathcal{S}_1(W_2)\right) = d\left(\mathcal{S}_1(W_1 + W_2)\right) =$$
$$\dim(W_1 + W_2) - 1 = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2 - 1 =$$
$$(\dim W_1 - 1) + (\dim W_2 - 1) - (\dim(W_1 \cap W_2) - 1) =$$
$$d\left(\mathcal{S}_1(W_1)\right) + d\left(\mathcal{S}_1(W_2)\right) - d\left(\mathcal{S}_1(W_1 \cap W_2)\right) . \blacksquare$$

EXAMPLE 1.   If $\mathcal{S}_1(V)$ is 2-dimensional, then Theorem 9 states that every two lines in $\mathcal{S}_1(V)$ have a common point because $d(L_1 \star L_2) \le 2 = d\left(\mathcal{S}_1(V)\right)$ implies

$$d(L_1 \cap L_2) = d(L_1) + d(L_2) - d(L_1 \star L_2) \ge 1 + 1 - 2 = 0 .$$

EXAMPLE 2.   Similarly, if $\mathcal{S}_1(V)$ is 3-dimensional, then every pair of planes has a line in common because $d(P_1 \star P_2) \le 3 = d\left(\mathcal{S}_1(V)\right)$ implies

$$d(P_1 \cap P_2) = d(P_1) + d(P_2) - d(P_1 \star P_2) \ge 2 + 2 - 3 = 1 .$$

The next result will play a significant role in Chapter IV.

THEOREM III.9. *If V as above is at least 2-dimensional, then every line in $\mathcal{S}_1(V)$ contains at least three points.*

Proof.   Let W be a 2-dimensional subspace of V, and let $\mathbf{w}_1$ and $\mathbf{w}_2$ form a basis for W. Then $\mathcal{S}_1(W)$ contains the three distinct points $\mathsf{Span}(\mathbf{w}_1)$, $\mathsf{Span}(\mathbf{w}_2)$, and $\mathsf{Span}(\mathbf{w}_1 + \mathbf{w}_2)$.$\blacksquare$

Projective extension of the affine space V

Given an n-dimensional vector space V over a field $\mathbb{F}$, we shall now construct a projective extension of the affine space structure on V which generalizes the previous construction of $\mathbb{FP}^2$ from $\mathbb{F}^2$. The central object in this construction is a 1–1 mapping J_V from V to $\mathbb{P}(V) = \mathcal{S}_1(V \times \mathbb{F})$ which sends $\mathbf{v} \in V$ to the 1-dimensional vector subspace of $V \times \mathbb{F}$ spanned by $(\mathbf{v}, 1)$. Notice that $V \times \mathbb{F}$ is a vector space with addition and scalar multiplication defined coordinatewise. As in the previous construction, a point is said to be *ordinary* if it lies in the image of J_V, and the remaining points, which are represented by homogeneous coordinates of the form $(\mathbf{v}, 0)$, where $\mathbf{v} \ne \mathbf{0}$, are called *ideal points*. Once again, the ideal point with homogeneous coordinates $(\mathbf{v}, 0)$

is the point at infinity on the line joining $\mathbf{0}$ to $\mathbf{v}$. We may summarize the preceding discussion as follows:

THEOREM III.10. *A point in* $\mathbb{P}(V) = \mathcal{S}_1(V \times \mathbb{F})$ *is either ordinary or ideal but not both.*∎

Finally, we include an analog of Theorem II.17:

THEOREM III.11. *A set $H \subset \mathbb{FP}^n$ is an $(n-1)$-plane if and only if there exist $\mathbf{a}_1, \cdots, a_{n+1} \in \mathbb{F}$ not all zero such that $\mathbf{x} \in H$ if and only if there exist homogeneous coordinates $(x_0, \cdots, x_n)$ for $\mathbf{x}$ such that $\sum_i a_i x_i = 0$.*

Proof. The theorem follows because a subset of $\mathbb{F}^{n+1}$ is an n-dimensional vector subspace if and only if it is the set of all points which solve a nontrivial linear homogeneous equation of the type described in the theorem (compare Theorem A.10).∎

Several additional and important properties of the 1–1 mapping

$$\mathrm{J}_V : V \longrightarrow \mathbb{P}(V) = \mathcal{S}_1(V \times \mathbb{F})$$

are developed in the exercises, and the latter are particularly worthy of attention.

EXERCISES

1. Translate the following statements about $\mathbb{F}^3$ into the language of ideal points.

(a) Through a given point there is a unique plane parallel to a given plane.

(b) Two lines which are parallel to a third line are parallel to each other.

(c) If a line is parallel to each of two intersecting planes it is parallel to their line of intersection, and conversely.

(d) If a line L is parallel to a plane P any plane containing L cuts P in a line parallel to L.

(e) Through a given line one plane, and only one, can be passed parallel to a given skew line.

(f) Through a given point one plane, and only one, can be passed that is parallel to each of two skew lines.

(g) All the lines through a point and parallel to a given plane lie on a plane parallel to the first plane.

(h) If a plane contains one of two parallel lines but not the other, it is parallel to the other line.

(i) The intersection of a plane with two parallel planes is a pair of parallel lines.

2. Let V be a vector space over $\mathbb{F}$ of dimension ≥ 3, let $\mathbf{x}_0, \cdots, \mathbf{x}_m$ be distinct points in $\mathcal{S}_1(V)$, and for each i let $\widetilde{\mathbf{x}}_i$ be a set of homogeneous coordinates for $\mathbf{x}_i$. Prove that the set $\{\mathbf{x}_0, \cdots, \mathbf{x}_m\} \subset \mathcal{S}_1(V)$ is *geometrically* independent if and only if the set $\{\widetilde{\mathbf{x}_0}, \cdots, \widetilde{\mathbf{x}_m}\} \subset V$ is *linearly* independent.

3. Determine which of the following vectors in $\mathbb{R}^3$ correspond to homogeneous coordinates of collinear points in $\mathbb{RP}^2$:

(i) (5, 2, 4)   (5, -2, 1)   (15, 2, 9)

(ii) (3, 1, -2)   (8, -3, 4)   (5, -2, 6)

(iii) (1, 5, 1)   (1, 1, -1)   (3, 4, 1)

(iv) (1, 2, 3)   (3, 0, 3)   (-2. 3, -1)

4.   Determine which of the following vectors in $\mathbb{R}^4$ correspond to homogeneous coordinates of collinear or coplanar points in $\mathbb{RP}^3$:

(i) (1, 2, 1, 3)   (2, 1, 3, 3)   (1, 0, 3, 0)   (2, 1, 1, 5)

(ii) (1, 1, 2, 1)   (0, 1, 1, 2)   (-1, 1, 2, 0)   (2, 0, 0, -3)

(iii) (1, 1, 1, 0)   (1, 1, 0, 1)   (1, 0, 1, 1)   (0, 1, 1, 1)

(iv) (1, 2, 1, 0)   (1, 0, -1, -1)   (0, 2, 2, 1)   (1, 2, 0, -1)

5.   Let $\mathbb{F}$ be a field with q elements (for example, if p is a prime then the field $\mathbb{Z}_p$ has p elements). Prove that $\mathbb{FP}^n$ has $1 + q + \cdots + q^n$ points. [*Hint:* Let $\pi : \mathbb{F}^{n+1} - \{\mathbf{0}\} \to \mathbb{FP}^n$ be the map taking a nonzero vector to the point for which it is a set of homogeneous coordinates. Explain why there are $(q-1)$ possible choices of homogeneous coordinates for every point. Using simple counting considerations, show that the number k of points in $\mathbb{FP}^n$ times the number of choices for homogeneous coordinates is equal to the number of nonzero elements in $\mathbb{F}^{n+1}$, which is $q^{n+1} - 1$. This yields an equation for k; solve this equation.]

6.   For each of the pairs of planes in $\mathbb{R}^3$ given below, the intersection is a line L in $\mathbb{R}^3$. Find homogeneous coordinates for the ideal point of L.

(i) $3x + 3y + z = 2$ and $3x - 2y = -13$

(i) $1x + 2y + 3z = 4$ and $2x + 7y + z = 8$

7.   Prove Theorem 11.

> HYPOTHESIS AND NOTATION. For the rest of these exercises, assume that V is a finite-dimensional (left or right) vector space over $\mathbb{F}$. If $\mathbf{x} \in V$ is nonzero, we shall denote the 1-dimensional subspace spanned by $\mathbf{x}$ by $\mathbb{F} \cdot \mathbf{x}$ or more simply by $\mathbb{F}\mathbf{x}$.

8.   Using Theorem 9, answer the following questions and prove that your answer is correct:

(i) In $\mathcal{S}_1(V)$, what is the intersection of a line with a hyperplane that does not contain it?

(ii) In $\mathcal{S}_1(V)$, what is the smallest number of hyperplanes that do not contain a common point?

9.   Prove that the projective extension map

$$J_V : V \quad \longrightarrow \quad \mathbb{P}(V) = \mathcal{S}_1(V \times \mathbb{F})$$

is 1–1 but not onto.

10.   Let Q be a k-plane in V (where $k \leq \dim V$). Prove that $J[Q]$ is contained in a unique k-plane $Q' \subset \mathbb{P}(V)$. [*Hint:* If $\{\mathbf{x}_0, \cdots, \mathbf{x}_k\}$ is an affine basis for Q, let W be the affine span of the vectors $(\mathbf{x}_i, 1)$ and consider $\mathbb{P}(W)$.]

11.   In the preceding exercise, prove that the ideal points of $\mathbb{P}(W)$ form a $(k-1)$-plane in $\mathbb{P}(V \times \mathbb{F})$. In particular, every line contains a unique ideal point and every plane contains a unique line of ideal points.

12.   (*Partial converse to Exercise* 10). Let W be a $(k+1)$-dimensional subspace of $V \times \mathbb{F}$ such that

$$\dim W \;-\; \dim\big(W \cap (V \times \{\mathbf{0}\})\big) \;=\; 1 \;.$$

Prove that $J^{-1}\big[\mathbb{P}(W)\big]$ is a k-plane of V.

13.   Let V and W be vector spaces over $\mathbb{F}$, and let $T : V \to W$ be a linear transformation that is one-to-one.

(*i*) Let $\mathbf{v} \neq \mathbf{0}$ in V. Show that the mapping $\mathcal{S}_1(T) : \mathcal{S}_1(V) \to \mathcal{S}_1(W)$ taking $\mathbb{F} \cdot \mathbf{v}$ to $\mathbb{F} \cdot T(\mathbf{v})$ is well-defined (*i.e.*, if $\mathbf{v}$ and $\mathbf{x}$ are nonzero vectors that are nonzero multiples of each other than so are their images under T).

(*ii*) If $S : W \to X$ is also a linear transformation, show that $\mathcal{S}_1(S \circ T) \;=\; \mathcal{S}_1(S) \circ \mathcal{S}_1(T)$. Also show that if T is an identity mapping then so is $\mathcal{S}_1(T)$.

(*iii*) If T is invertible, prove that $\mathcal{S}_1(T)$ is also invertible and that $\mathcal{S}_1(T)^{-1} \;=\; \mathcal{S}_1\left(T^{-1}\right)$.

14.   Suppose that V is an n-dimensional vector space and let $\{\, \mathbf{x}_1, \; \cdots \; , \mathbf{x}_n \,\}$ and $\{\, \mathbf{y}_1, \; \cdots \; , \mathbf{y}_n \,\}$ be geometrically independent subsets of $\mathcal{S}_1(V)$. Find an invertible linear transformation T such that $\mathcal{S}_1(T)$ maps $\mathbf{x}_i$ to $\mathbf{y}_i$ for $1 \leq i \leq n$.

15.   Let T_0 be an element of the affine group $\mathrm{AFF}(V)$. Prove that there is an invertible linear transformation $T : V \times \mathbb{F} \to V \times \mathbb{F}$ such that

$$\mathcal{S}_1(T) \circ \mathrm{J}_V \;=\; \mathrm{J}_V \circ T_0 \;.$$

[*Hint:*   Let $\{\, \mathbf{v}_1, \; \cdots \; , \mathbf{v}_n \,\}$ be a linear basis for V and choose T so that $T(\mathbf{v}_i, 0) = (\, T_0(\mathbf{v}_i), 0\,)$ for all i and $T(\mathbf{0}, 1) = (\mathbf{0}, 1)$. Show that $T(\mathbf{x}, 1) = (\varphi(\mathbf{x}), 1)$ for all $\mathbf{x} \in V$ using the expansion of $\mathbf{x}$ as an affine combination of the basis vectors $\{\, \mathbf{v}_1, \; \cdots \; , \mathbf{v}_n \,\}$ and the zero vector.]

16.   Suppose that $T : V \times \mathbb{F} \to V \times \mathbb{F}$ is an invertible linear transformation which maps $V \times \{\mathbf{0}\}$ to itself. Prove that $\mathcal{S}_1(T)$ maps $\mathrm{J}_V[V]$ to itself and the induced self-map of V is in $\mathrm{AFF}(V)$.

17.   Let S and T be geometrical subspaces of $\mathcal{S}_1(V)$. Prove that the join $S \star T$ is the set of all points $\mathbf{z}$ such that $\mathbf{z} \in S \cup T$ or $\mathbf{z} \in \mathbf{xy}$, where $\mathbf{x} \in S$ and $\mathbf{y} \in T$. In other words, the join $S \star T$ is the set of all points on lines **joining** points of S and T. [*Hint:*   If $S = \mathcal{S}_1(W)$ and $T = \mathcal{S}_1(U)$, then $S \star T \;=\; \mathcal{S}_1(W + U)$.]

> *Note.*   A corresponding description of the join in **affine** geometry is given in Appendix B.

Addendum. Synthetic construction of projective space

The preceding three sections of this chapter described the analytic approach to generalizing the synthetic construction of Section III.1. In this addendum to Section III.4 we shall discuss the synthetic approach to construction of a projective space from an affine space. Since some of the arguments are lengthy and we shall not use this material subsequently except in a few peripheral exercises and remarks, many of the details have been omitted; the latter are generally straightforward (but often tedious), and thus they are left to the reader.

We shall need the following results on affine incidence spaces of arbitrary dimension. The proofs are quite similar to their 3-dimensional special cases in Section II.1.

THEOREM III.12. *Suppose that L, M and N are lines in an affine incidence n-space ($n \geq 2$) and $L \neq N$. If $L||M$ and $M||N$, then $L||N$.* ∎

THEOREM III.13. *Let S be an an affine n-space let H be a hyperplane in S, and let $\mathbf{x}$ be a point of S which does not lie in H. Then there is a unique hyperplane H' in S such that $\mathbf{x} \in H'$ and $H \cap H' = \varnothing$ (in other words, a **parallel hyperplane** to H which passes through $\mathbf{x}$).* ∎

Motivated by the 2-dimensional case, define a *direction* in an affine n-space S to be an equivalence class of parallel lines, and set $S^{\wedge}$ equal to S together with all the directions in S. Denote the set of all directions by S_{∞}. We define two types of geometrical subspaces of $S^{\wedge}$ as follows:

(A) *Extensions of subspaces of S.* If P is a geometrical subspace of S, set $P^{\wedge}$ equal to P together with all directions $L^{\wedge}$ containing a representative L_0 which lies in P. The **dimension** of $P^{\wedge}$ is defined to be equal to the dimension of P.

(B) *Ideal geometrical subspaces* or *subspaces at infinity.* If P is a geometrical subspace of S, then its set of ideal points $P_{\infty} = P^{\wedge} \cap S_{\infty}$ is a geometrical subspace whose dimension is equal to $d(P) - 1$. [In particular, S_{∞} is a hyperplane in S, and it is called the *ideal hyperplane* or the *hyperplane (of points) at infinity.*

We shall take the above as the **definition** of the *synthetic projective extension $S^{\wedge}$ of an affine n-space S*, and $\mathrm{X}_S : S \to S^{\wedge}$ will denote the inclusion of S in its projective extension.

The following result expresses the equivalences between the synthetic and analytic approaches to projective extensions of coordinate affine spaces.

THEOREM III.14. *Let V be a finite-dimensional vector space over $\mathbb{F}$ such that $\dim V \geq 2$. Then there is a $1 - 1$ correspondence*

$$h_V : V^{\wedge} \longrightarrow \mathcal{S}_1(V \times \mathbb{F})$$

with the following properties:

(i) $\mathrm{J}_V = h_V \circ \mathrm{X}_V$

(ii) A subset $Q \subset V^{\wedge}$ is a k-plane in $V^{\wedge}$ if and only if $h_V[Q]$ is a k-plane in $\mathcal{S}_1(V \times \mathbb{F})$.

Proof. We shall break the argument down into a sequence of steps.

STEP 1. *Construction of a $1-1$ correspondence between $V^{\wedge}$ and $\mathcal{S}_1(V \times \mathbb{F})$.*

If $\mathbf{v} \in V$ and $X(\mathbf{v}) = \mathbf{v}'$, define $h_V(\mathbf{v}') = \mathbb{F} \cdot (\mathbf{v}, 1)$; this construction is well-defined since the mapping X_V is 1–1. Given an ideal point $L^{\wedge}$ associated to some line L, let $\mathbf{u} \neq \mathbf{0}$ span the unique 1-dimensional vector subspace of V which is parallel or equal to L, and let

$$h_V\left(L^{\wedge}\right) = \mathbb{F} \cdot (\mathbf{u}, 0) \; .$$

This is well-defined because $L||M$ implies that $\mathbf{0u}$ is parallel or equal to M (by Theorem 13), and the right hand side of the formula remains the same if we replace $\mathbf{u}$ by any other nonzero point $\mathbf{u}'$ of $\mathbf{0u}$ (since $\mathbf{u}'$ and $\mathbf{u}$ are nonzero multiples of each other). By definition we have $J_V = h_V \circ X_V$.

We shall now show that h_V is 1–1. (*a*) If $\mathbf{a}, \mathbf{b} \in V$, are distinct vectors, then $X_V(\mathbf{a}) \neq X_V(\mathbf{b})$. If the images of these points under h_V are equal, then we have

$$J_X(\mathbf{a}) = h_V \circ X_V(\mathbf{a}) = h_V \circ X_V(\mathbf{b}) = J_V(\mathbf{b})$$

which is a contradiction because J_V is known to be 1–1. Therefore the map h_V is 1–1 on the image of X_V. (*b*) Suppose now that $L^{\wedge}$ and $M^{\wedge}$ are ideal points such that

$$h_V\left(L^{\wedge}\right) = h_V\left(M^{\wedge}\right) \; .$$

This means that there is a single line $\mathbf{0u}$ in V which is parallel or equal to each of L and M. which in turn means that $L^{\wedge} = M^{\wedge}$. (*c*) Suppose now that we have an ordinary point $X_V(\mathbf{a})$ and an ideal point $L^{\wedge}$ which have the same image under h_V. If this is true then $\mathbb{F} \cdot (\mathbf{a}, 1)$ is equal to $\mathbb{F} \cdot (\mathbf{u}, 0)$, where $\mathbf{u}$ is given as before. Since every nonzero vector in the first subspace has a nonzero last coordinate and every vector in the second subspace has a zero last coordinate, it is clear that the two subspaces cannot be equal, and therefore an ordinary point and an ideal point cannot have the same images under h_V. — This completes the proof that h_V is 1–1.

We shall now show that h_V is onto. Let $\mathbf{0} \neq (\mathbf{v}, c) \in V \times \mathbb{F}$. If $c \neq 0$, then we have

$$\mathbb{F} \cdot (\mathbf{v}, c) = \mathbb{F} \cdot (c^{-1}\mathbf{v}, 1) = J_V(c^{-1}\mathbf{v}) = h_V \circ X X_V(c^{-1}\mathbf{v}) \; .$$

On the other hand, if $c = 0$ then $\mathbf{v} \neq \mathbf{0}$ and

$$\mathbb{F} \cdot (\mathbf{v}, 0) = h_V\left((\mathbf{0v})^{\wedge}\right) \; .$$

STEP 2. *Under the above correspondence, a subset $Q \subset V^{\wedge}$ is a hyperplane if $h_V[Q]$ is a hyperplane in $\mathcal{S}_1(V \times \mathbb{F})$.* — We shall only sketch the argument, leaving verification of the details to the reader.

This uses a result from the Exercises for Section 4 of Appendix A: *If X is m-dimensional vector space over $\mathbb{F}$, then $Y \subset X$ is an $(m-1)$-dimensional vector subspaces if and only if there is a nonzero linear transformation (or* functional*) $g : X \to \mathbb{F}$ such that $\mathbf{x} \in Y$ if and only if $g(\mathbf{x}) = 0$.* — The linear functional g is not unique, for if Y is the zero set for g and c is a nonzero constant, then Y is also the zero set for $c \cdot g$.

By the previously mentioned exercises and the discussion following Theorem A.12, every linear functional on $V \times \mathbb{F}$ has the form $g(\mathbf{v}, t) = g_0(\mathbf{v}) + a \cdot t$, where g_0 is a linear functional on V and $a \in \mathbb{F}$. — Extensions of ordinary hyperplanes in V are defined by expressions of this form for which $g_0 \neq 0$, and the ideal hyperplane in V is defined by linear functionals of this form in which $g_0 = 0$ and $b \neq 0$.

Suppose we are given a hyperplane $H^{\wedge}$ in $V^{\wedge}$ which is the extension of a hyperplane H in V. Then H is defined by an equation of the form $f(\mathbf{v}) = b$ for some nonzero linear functional f and scalar b, and $h_V[H^{\wedge}]$ is equal to $\mathcal{S}_1(W)$, where W is the zero set of the linear functional $g(\mathbf{v}, t) = f(\mathbf{v}) - b \cdot t$. Since we also know that h_V maps the ideal hyperplane in $V^{\wedge}$ to the zero set of the functional $g(\mathbf{v}, t) = t$, it follows that the image of a hyperplane in $V^{\wedge}$ is a hyperplane in $\mathcal{S}_1(V \times \mathbb{F})$.$\square$

Conversely, suppose that $\mathcal{S}_1(W)$ is a hyperplane in $\mathcal{S}_1(V \times \mathbb{F})$, and suppose that the n-dimensional vector subspace W is defined by the nonzero linear functional. Write $g(\mathbf{v}, t) = g_0(\mathbf{v}) + a \cdot t$ as above, where either $g_0 \neq 0$ or $a \neq 0$. In the first of these cases $\mathcal{S}_1(W)$ is the image of the extended ordinary hyperplane defined by the equation $g_0(\mathbf{v}) = -a$, and in the second case $\mathcal{S}_1(W)$ is the image of the ideal hyperplane.

STEP 3.   *Let k be an integer satisfying $1 \leq k \leq n-2$. In both $V^{\wedge}$ and $\mathbb{FP}^n$, a k-dimensional geometrical subspace Q is the intersection of an $(k+1)$-dimensional subspace Q' with a hyperplane H not containing it.*

The proofs of these statements are variants of Exercise II.5.3 which shows that every geometrical subspace is an intersection of hyperplanes. There is an analogous result for vector subspaces of an m-dimensional vector spaces: Every k-dimensional subspace is an intersection of $(m - k)$ distinct hyperplanes. Once again, the argument breaks down into various cases.

STEP 4.   *By downward induction on k such that $1 \leq k \leq (n-1)$, one shows that Q is a k-plane in both $V^{\wedge}$ if and only if $h_V[Q]$ is a k-plane in $\mathbb{FP}^n$.*

Once again, we shall only sketch the argument: The case $k = (n - 1)$ is covered by Step 2, and the recursive step — showing that the validity of the result for $(k + 1)$ implies its validity for k — follows from the description of k-planes in Step 3.$\blacksquare$

<center>*Abstract projective extensions*</center>

Definition.   Let S be a geometrical incidence space of dimension $n \geq 3$. An *abstract projective extension* of S is a 1–1 map $\varphi : S \to \mathbb{FP}^n$, where $\mathbb{F}$ is some skew-field, such that if Q is a k-plane in S then there is a unique k-plane $Q' \subset \mathbb{FP}^n$ such that $Q = \varphi^{-1}[Q']$.

The results of this chapter prove the existence of projective extensions for the coordinate affine n-spaces (in fact, for *all* affine n-spaces by Theorem II.38). On the other hand, it is not difficult to see that projective extensions exist for many other geometrical incidence spaces (*e.g.*, this holds for the examples in Exercises II.5.7 and II.5.8). In fact, if one has a 3-dimensional (regular) geometrical incidence space which also has a notion of betweenness satisfying Hilbert's Axioms of Order, then the existence of an abstract projective extension (with a suitable analog of betweenness as defined in Section VI.4 below) is given by results due to A. N. Whitehead.[12] Extremely general, and purely incidence-theoretic, conditions for the existence of projective

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[12]See Chapter III of the book, *The Axioms of Descriptive Geometry*, in the bibliography. — ALFRED NORTH WHITEHEAD (1861–1947) was an extremely well-known philosopher who worked extensively on the logical foundations of mathematics during the period from the late 1880s until about 1913, at which time he shifted his attention to other areas of philosophy. Whitehead is particularly known for his study of the foundations of mathematics with the well known philosopher Bertrand Russell (1872–1970), which is largely contained in a massive and ambitious three volume work called *Principia Mathematica*.

extensions have been established by S. Gorn.[13] A discussion of the classical Non-Euclidean geometries using their projective extensions appears in Chapters 12 and 13 of the book by Fishback listed in the bibliography.

THE 2-DIMENSIONAL CASE. Clearly there is no problem in formulating a definition of abstract projective extension for incidence planes, and the constructions of this chapter shows that every *affine* coordinate incidence plane admits a projective extension. In fact, if one defines an abstract notion of projective plane as in Section IV.1 below, then the constructions of Section III.1 show that for every abstract affine plane P there admits a map $\varphi : P \to P^*$ such that P^* is an abstract projective plane and (as before) if L is a line in P then there is a unique line $L' \subset \mathbb{FP}^n$ such that $L = \varphi^{-1}[L']$. However, two major differences are that (i) in contrast to the situation in higher dimensions, we cannot always take P^* to be a coordinate projective plane $\mathbb{FP}^2$, (ii) the proofs of the results of Whitehead and Gorn on projective extensions for non-affine incidence structures do **not** extend to the 2-dimensional case. We shall discuss these points further in the next chapter.

EXERCISES

1.   Assuming that S is an affine n-space, prove that $S^{\wedge}$ is a regular geometrical incidence space, and in fact the dimensions of subspaces satisfy the strong form of the regularity condition

$$d(T_1 \star T_2) \;=\; d(T_1) \;+\; d(T_2) \;-\; d(T_1 \cap T_2)$$

for all geometrical subspaces T_1 and T_2.

2.   Suppose that $f : S \to S'$ is a geometric symmetry of an affine n-space. Prove that f extends to a unique geometric symmetry $f^{\wedge}$ of $S^{\wedge}$. [*Hint:*   If $L||L'$, then $f[L] \; || \; f[L']$; verify this and use it to define $f^{\wedge}$.]

3.   In the notation of the preceding exercise, prove that $(\mathrm{id}_S)^{\wedge}$ is the identity on $S^{\wedge}$ and if $g : S \to S$ is another symmetry then $(g \circ f)^{\wedge} \;=\; g^{\wedge} \circ f^{\wedge}$. Finally, show that

$$(f^{-1})^{\wedge} \;=\; \left(f^{\wedge} \right)^{-1} \;.$$

[*Hint:*   In each desired identity, explain why both sides of the equation extend the same geometric symmetry of S.]

4.   Prove Theorems 13 and 14, and fill in the details for the proof of Theorem 15.

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[13]A reference to the original 1940 research article appears in the bibliography. — SAUL GORN (1913–1992) began his professional career as a mathematician, but his interests moved to computer science with the emergence of that subject during the 1940s, and he played a significant role in the establishment of computer science as an independent branch of the mathematical sciences. As a researcher, he is best known for his theory of mechanical languages based upon work of twentieth century philosophers like L. Wittgenstein (1889–1951) on human linguistics.

CHAPTER IV

SYNTHETIC PROJECTIVE GEOMETRY

The purpose of this chapter is to begin the study of projective spaces, mainly from the synthetic point of view but with considerable attention to coordinate projective geometry.

1. Axioms for projective geometry

The basic incidence properties of coordinate projective spaces are expressible as follows:

Definition. A geometrical incidence space (S, Π, d) is **projective** if the following hold:

(P-1) : Every line contains at least three points.

(P-2) : If P and Q are geometrical subspaces of S then
$$d(P \star Q) \; = \; d(P) \; + \; d(Q) \; - \; d(P \cap Q) \, .$$

In particular, **(P-2)** is a strong version of the regularity condition **(G-4)** introduced in Section II.5. The above properties were established for $\mathbb{FP}^n$ ($n \geq 2$) in Theorems III.10 and III.9 respectively. It is useful to assume condition **(P-1)** for several reasons; for example, lines in Euclidean geometry have *infinitely many* points, and **(P-1)** implies a high degree of regularity on the incidence structure that is not present in general (compare Exercise 2 below and Theorem IV.11). — In this connection, note that Example 2 in Section II.5 satisfies **(P-2)** and *every* line in this example contains *exactly* two points.

Elementary properties of projective spaces

The following is a simple consequence of the definitions.

THEOREM IV.1. *If S is a geometrical subspace of a geometrical incidence space S', then S is a geometrical incidence space with respect to the subspace incidence structure of Exercise* II.5.3.■

If P is a projective incidence space and $d(P) = n \geq 1$, then P is called a *projective n-space*; if $n = 2$ or 1, then one also says that P is a *projective plane* or *projective line*, respectively.

THEOREM IV.2. *If P is a projective plane and L and M are distinct lines in P, then $L \cap M$ consists of a single point.*■

THEOREM IV.3. *If S is a projective 3-space and P and Q are distinct planes in S, then $P \cap Q$ is a line.*■

These follow from **(P-2)** exactly as Examples 1 and 2 in Section III.4 follow from Theorem III.9.

We conclude this section with another simple but important result:

THEOREM IV.4. *In the definition of a projective space, property* **(P-1)** *is equivalent to the following (provided the space is not a line):*

(P-1′) : Every plane contains a subset of four points, no three of which are collinear.

Proof.   Suppose that **(P-1)** holds. Let P be a plane, and let X, Y and Z be noncollinear points in P. Then the lines $L = XY$, $M = XZ$, and $N = YZ$ are distinct and contained in P. Let W be a third point of L, and let V be a third point of M.
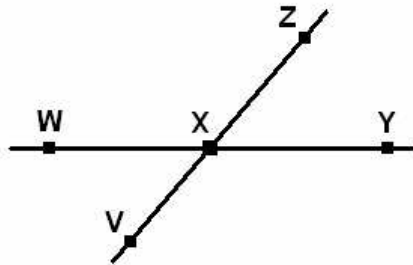
Figure IV.1

Since L and M are distinct and meet at X, it follows that the points V, W, Y, Z must be distinct (if any two are equal then we would have $L = M$; note that there are six cases to check, with one for each pair of letters taken from W, X, Y, Z). Similarly, if any three of these four points were collinear then we would have $L = M$, and therefore no three of the points can be collinear (there are four separate cases that must be checked; these are left to the reader).□

Conversely, suppose that **(P-1′)** holds. Let L be a line, and let P be a plane containing L. By our assumptions, there are four points A, B, C, $D \in P$ such that no three are collinear.
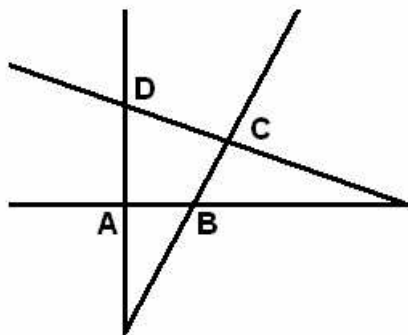
Figure IV.2

Let $M_1 = AB$, $M_2 = BC$, $M_3 = CD$, and $M_4 = AD$. Then the lines M_i (where $1 \leq i \leq 4$) are distinct and coplanar, and no three of them are concurrent (for example, $M_1 \cap M_2 \neq M_3 \cap M_4$, and similarly for the others). It is immediate that M_1 contains at least three distinct points;

namely, the points A and B plus the point where M_1 meets M_3 (these three points are distinct because no three of the lines M_i are concurrent). Similarly, each of the lines M_2, M_3 and M_4 must contain at least three points.

If L is one of the four lines described above, then we are done. Suppose now that L is **not** equal to any of these lines, and let P_i be the point where L meets M_i. If at least three of the points P_1, P_2, P_3, P_4 are distinct, then we have our three distinct points on L. Since no three of the lines M_i are concurrent, it follows that no three of the points P_i can be equal, and therefore if there are not three distinct points among the P_i then there must be two distinct points, with each P_i equal to a unique P_j for $j \neq i$. Renaming the M_i if necessary by a suitable reordering of $\{1, 2, 3, 4\}$, we may assume that the equal pairs are given by $P_1 = P_3$ and $P_2 = P_4$. The drawing below illustrates how such a situation can actually arise.
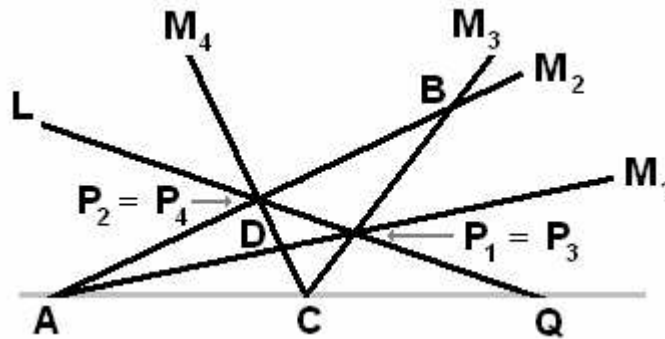
Figure IV.3

We know that $P_1 = P_3$ and $P_2 = P_4$ are two distinct points of L, and Figure IV.3 suggests that the point Q where AC meets L is a third point of L. To **prove** this, we claim it will suffice to verify the following statements motivated by Figure IV.3:

  (*i*) The point A does not lie on L.

  (*ii*) The line AC is distinct from M_1 and M_2.

Given these properties, it follows immediately that the three lines AC, M_1 and M_2 — which all pass through the point A which does not lie on L — must meet L in three distinct points (see Exercise 4 below).

Assertion (*i*) follows because $A \in L$ implies

$$A \; \in \; M_2 \cap L \; = \; M_4 \, \cap \, M_2 \, \cap \, L$$

and since $A \in M_1 \cap M_2$ this means that M_1, M_2, and M_4 are concurrent at A. However, we know this is false, so we must have $A \in L$. To prove assertion (*ii*), note that if $AC = M_1 = AB$, then A, B, C are collinear, and the same conclusion will hold if $AC = M_2 = BC$. Since the points A, B, C are noncollinear by construction, it follows that (*ii*) must also hold, and as noted above this completes the proof that L has at least three points.∎

EXERCISES

1. Let (S, Π, d) be an n-dimensional projective incidence space ($n \geq 2$), let P be a plane in S, and let $X \in P$. Prove that there are at least three distinct lines in P which contain X.

2. Let $n \geq 3$ be an integer, let P be the set $\{0, 1, \cdots, n\}$, and take the family of subsets $\mathcal{L}$ whose elements are $\{1, \cdots, n\}$ and all subsets of the form $\{0, k\}$, where $k > 0$. Show that $(P, \mathcal{L})$ is a regular incidence plane which satisfies **(P-2)** but not **(P-1)**. [*Hint:* In this case **(P-2)** is equivalent to the conclusion of Theorem IV.2.]

3. This is a generalization of the previous exercise. Let S be a geometrical incidence space of dimension $n \geq 2$, and let ∞_S be an object **not** belonging to S (the axioms for set theory give us explicit choices, but the method of construction is unimportant). Define the **cone** on S to be $S^\bullet = S \cup \{\infty_S\}$, and define a subset Q of $S^\bullet$ to be a k-planes of $S^\bullet$ if and only if **either** Q is a k-plane of S **or** $Q = Q_0 \cup \{\infty_S\}$, where Q_0 is a $(k-1)$-plane in S (as usual, a 0-plane is a set consisting of exactly one element). Prove that $S^\bullet$ with these definitions of k-planes is a geometrical incidence $(n+1)$-space, and that $S^\bullet$ satisfies **(P-2)** if and only if S does. Explain why $S^\bullet$ does not satisfy **(P-1)** and hence is not projective.

4. Let $(P, \mathcal{L})$ be an incidence plane, let L be a line in P, let X be a point in P which does not lie on L, and assume that $M_1, \cdots M_k$ are lines which pass through X and meet L in points $Y_1, \cdots Y_k$ respectively. Prove that the points $Y_1, \cdots Y_k$ are distinct if and only if the lines $M_1, \cdots M_k$ are distinct.

5. Let (S, Π, d) be a regular incidence space of dimension ≥ 3, and assume that every plane in S is projective (so it follows that **(P-1)** holds). Prove that S is projective. [*Hint:* Since S is regular, condition **(P-2)** can only fail to be true for geometrical subspaces Q and R such that $Q \cap R = \varnothing$. If $d(R) = 0$, so that R consists of a single point, then condition **(P-2)** holds by Theorem II.30. Assume by induction that **(P-2)** holds whenever $d(R_0) \leq k - 1$, and suppose that $d(R) = k$. Let $R_0 \subset R$ be $(k-1)$-dimensional, and choose $\mathbf{y} \in R$ such that $\mathbf{y} \notin R_0$. Show that $Q \star R_0 \subset Q \star R$, and using this prove that $d(Q \star R)$ is equal to $d(Q) + k$ or $d(Q) + k + 1$. The latter is the conclusion we want, so assume it is false. Given $\mathbf{x} \in Q$, let $\mathbf{x} R$ denote the join of $\{\mathbf{x}\}$ and R, and define $\mathbf{y} Q$ similarly. Show that $\mathbf{x} R \cap Q$ is a line that we shall call L, and also show that $R \cap \mathbf{y} Q$ is a line that we shall call M. Since $L \subset Q$ and $M \subset R$, it follows that $L \cap M = \varnothing$. Finally, show that $\mathbf{x} R \cap \mathbf{y} Q$ is a plane, and this plane contains both L and M. Since we are assuming all planes in S are projective, it follows that $L \cap M$ is nonempty, contradicting our previous conclusion about this intersection. Why does this imply that $d(Q \star R) = d(Q) + k + 1$?]

6. Prove that the geometric subspaces of a projective incidence space satisfy the modularity property stated in Exercise II.5.8(iii). [*Hint:* Imitate the proof of that exercise using **(P-2)** instead of **(G-4)**.]

2. Desargues' Theorem

In this section we shall prove a synthetic version of a fundamental result of plane geometry due to G. Desargues (1591–1661). The formulation and proof of Desargues' Theorem show that projective geometry provides an effective framework for proving nontrivial geometrical theorems.

THEOREM IV.5. (Desargues' Theorem) *Let P be a projective incidence space of dimension at least three, and let $\{A, B, C\}$ and $\{A', B', C'\}$ be triples of noncollinear points such that the lines AA', BB' and CC' are concurrent at some point X which does not belong to either of $\{A, B, C\}$ and $\{A', B', C'\}$. Then the points*

$$
\begin{aligned}
D &\in BC \cap B'C' \\
E &\in AC \cap A'C' \\
F &\in AB \cap A'B'
\end{aligned}
$$

are collinear.
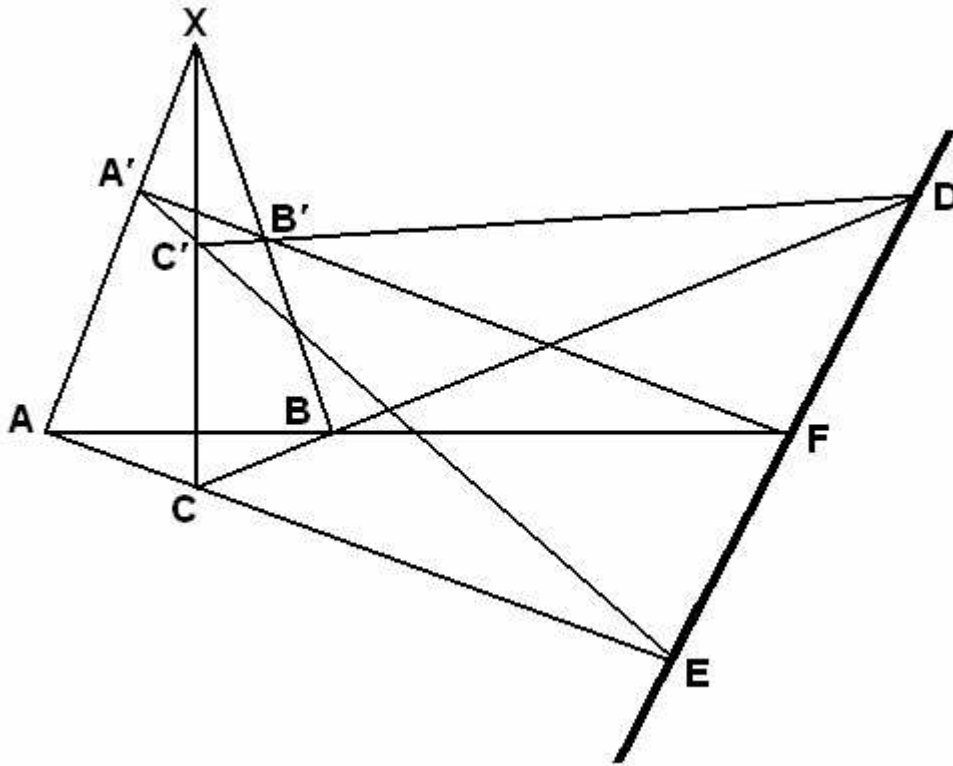
Figure IV.4

Proof. The proof splits into two cases, depending upon whether or not the sets $\{A, B, C\}$ and $\{A', B', C'\}$ are coplanar. One feature of the proof that may seem counter-intuitive is that the noncoplanar case is the easier one. In fact, we shall derive the coplanar case using the validity of the result in the noncoplanar case.

CASE 1.   *Suppose that the planes determined by $\{A, B, C\}$ and $\{A', B', C'\}$ are distinct.*   Then the point X which lies on AA', BB' and CC' cannot lie in either plane. On the other hand, it follows that the points $\{A', B', C'\}$ lie in the 3-space S determined by $\{X, A, B, C\}$, and therefore we also know that the planes ABC and $A'B'C'$ are contained in S. These two planes are distinct (otherwise the two triples of noncollinear points would be the same), and hence their intersection is a line. By definition, all three of the points D, E, F all lie in the intersection of the two planes, and therefore they all lie on the two planes' line of intersection.

CASE 2.   *Suppose that the planes determined by $\{A, B, C\}$ and $\{A', B', C'\}$ are identical.*   The idea is to realize the given configuration as the photographic projection of a similar noncoplanar configuration on the common plane. Since photographic projections onto planes preserve collinearity, this such a realization will imply that the original three points D, E, F are all collinear.

Under the hypothesis of Case 2, all the points under consideration lie on a single plane we shall call P. Let Y be a point not on P, and let $Z \in AY$ be another. Consider the line $A'Z$; since A' and Z both lie on the plane AXY, the whole line $A'Z$ lies in AXY. Thus $A'Z$ and XY meet in a point we shall call Q.
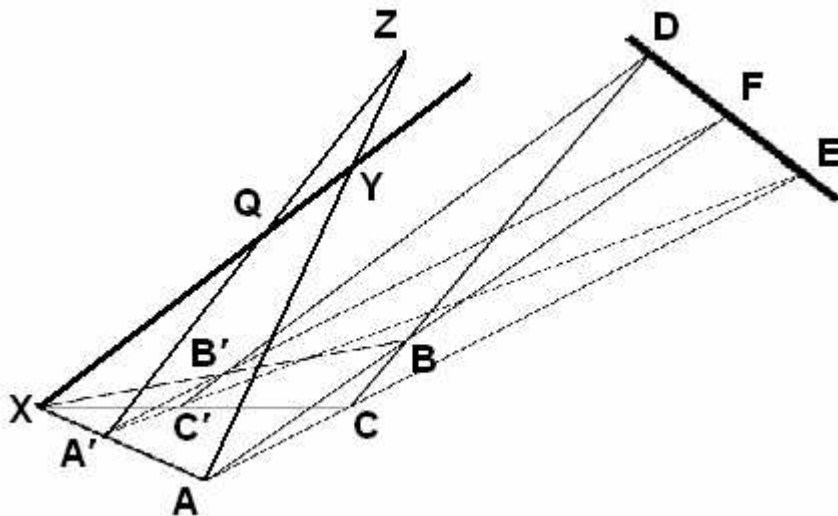
Figure IV.5

Consider the following three noncoplanar triangle pairs:

  (i) $C'QB'$ and CYB.

  (ii) $C'QA'$ and CYA.

  (iii) $B'QA'$ and BYA.

Since AA', BB' and CC' all meet at X, the nonplanar case of the theorem applies in all thre cases. Let $G \in BY \cap B'Q$ and $H \in CY \cap C'Q$, and note that $Z \in AR \cap A'Q$. Then the truth of the theorem in the noncoplanar case implies that each of the triples

$$\{D, H, G\}, \qquad \{F, Y, G\}, \qquad \{E, H, Z\}$$

is collinear.

Let P' be the plane DFZ. Then $G \in P'$ by the collinearity of the second triple, and hence $H \in P'$ by the collinearity of the first triple. Since $Z, H \in P'$, we have $E \in P'$ by the collinearity of the third triple. All of this implies that

$$E \ \in \ P' \ \cap \ P \ = \ DF$$

which shows that the set $\{D, E, F\}$ is collinear.∎

Definition. A projective plane P is said to be *Desarguian* if Theorem 5 is always valid in P.

By Theorem 5, every projective plane that is isomorphic to a plane in a projective space of higher dimension is Desarguian. In particular, if $\mathbb{F}$ is a skew-field, then $\mathbb{FP}^2$ is Desarguian because $\mathbb{FP}^2$ is isomorphic to the plane in $\mathbb{FP}^3$ consisting of all points having homogeneous coordinates in $\mathbb{F}^4$ of the form $(x_1, x_2, x_3, 0)$. In Section 4 we shall note that, conversely, every Desarguian plane is isomorphic to a plane in a projective 3-space.

An example of a non-Desarguian projective plane (the *Moulton plane*)[1] can be given by taking the real projective plane $\mathbb{RP}^2$ as the underlying set of points, and modifying the definition of lines as follows: The new lines will include the line at infinity, all lines which have slope ≤ 0 or are parallel to the y-axis, and the broken lines defined by the equations

$$
\begin{aligned}
y &= m(x - a), & x &\leq a \ (i.e., \ y \ \leq 0) \\
y &= \tfrac{1}{2}m(x - a), & x &\geq a \ (i.e., \ y \ \geq 0)
\end{aligned}
$$

where $m > 0$. As the points at infinite of the latter lines we take those belonging to $y = m(x-a)$. A straightforward argument shows that the axioms for a projective plane are satisfied (see Exercise 4 below). However, as Figure IV.6 suggests, Desargues' Theorem is false in this plane.
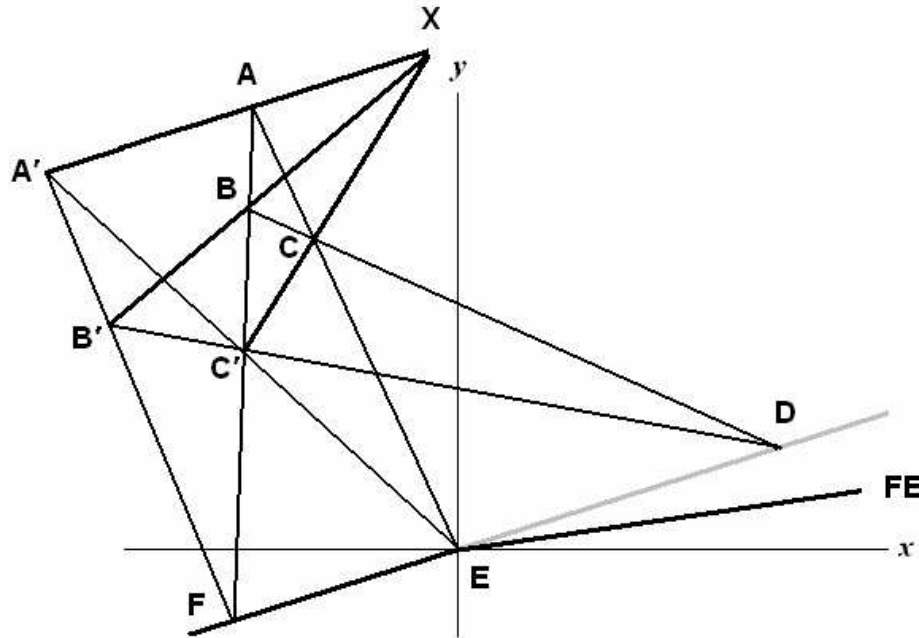
Figure IV.6

A different and more abstract type of non-Desarguian projective planes (the *free projective plane* on a suitable configuation of points and lines) is given at the end of Chapter 2 in Hartshorne's book, and an example of a **finite** non-Desarguian projective plane is given on pages 158–159 of that reference.[2]

Informally speaking, Desargues' Theorem fails to hold for some projective planes because there is not enough room in two-dimensional spaces to apply standard techniques. Perhaps surprisingly, there are several other significant examples of geometrical problems in which higher dimensional cases (say $n \geq N$, where N depends upon the problem) are simpler to handle than lower dimensional ones (see the paper of Gorn for another example).

Finally, we note that Theorems II.27–29 are basically special cases of Desargues' Theorem.

EXERCISES

1. Explain why the results mentioned above are essentially special cases of Desargues' Theorem.

2. Is it possible to "plant ten trees in ten rows of three?" Explain your answer using Desargues' Theorem.[3]

3. Explain why each of the following pairs of triangles in Euclidean 3-space satisfies the hypotheses of Desargues' Theorem.

  (i) Two coplanar triangles such that the lines joining the corresponding vertices are parallel.

  (ii) A triangle and the triangle formed by joining the midpoints of its sides.

  (iii) Two congruent triangles in distinct planes whose corresponding sides are parallel.

4. Prove that the Moulton plane (defined in the notes) is a projective plane.

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[2]A reformatted version is available at `http://filebox.vt.edu/users/jabrunso/Math/Hartshorne.pdf`. The page numbers in this document do not match those of the original book; in particular, the finite example appears on page 79 of the electronic document.

[3]For more information on the relevance of projective geometry to counting and arrangement problems, see Section IV.3.

3. Duality

> By the principle of duality ... geometry is at one stroke [nearly] doubled in extent with no
> expenditure of extra labor, — **Eric Temple Bell (1883–1960)**, *Men of Mathematics*[4]

Consider the following fundamental properties of projective planes:

(1) Given two distinct points, there is a unique line containing both of them.

(1*) Given two distinct lines, there is a unique point contained in both of them.

(2) Every line contains at least three distinct points.

(2*) Every point is contained in at least three distinct lines.

The important point to notice is that Statement (n^*) is obtained from Statement (n) by inter-changing the following words and phrases:

(i) *point* $\longleftrightarrow$ *line*

(ii) *is contained in* $\longleftrightarrow$ *contains*

Furthermore, Statement (n) is obtained from Statement (n^*) by exactly the same process. Since the four properties $(1) - (1^*)$ and $(2) - (2^*)$ completely characterize projective planes (see Exercise 1 below), one would expect that points and lines in projective planes behave somewhat symmetrically with respect to each other.

This can be made mathematically precise in the following manner: Given a projective plane $(P, \mathcal{L})$, we define a **dual plane** $(P^*, \mathcal{L}^*)$ such that P^* is the set $\mathcal{L}$ of lines in P and $\mathcal{L}^*$ is in 1–1 correspondence with P. Specifically, for each $x \in P$ we define the *pencil of lines with vertex x* to be the set

$$\mathfrak{p}(x) \; = \; \{\, L \, \in \, \mathcal{L} \, = \, P^* \mid x \, \in \, L \,\} \, .$$

In other words, $L \in \mathfrak{p}(x)$ if and only if $X \in L$. Let P^{**} denote the set of all pencils associated to the projective plane whose points are given by P and whose lines are given by P^*.

THEOREM IV.6. *If (P, P^*) satisfies properties $(1) - (1^*)$ and $(2) - (2^*)$, then (P^*, P^{**}) also does.*

Proof.   There are four things to check:

(a) Given two lines, there is a unique pencil containing both of them.

(a^*) Given two pencils, there is a unique line contained in both of them.

(b) Every pencil contains at least three lines.

(b^*) Every line is contained in at least three pencils.

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[4]See the bibliography for more information and comments on this book. ERIC TEMPLE BELL is known for his books on the history of mathematics and his research in several areas of number theory.

However, it is clear that (a) and (a^*) are rephrasings of (1^*) and (1) respectively, and likewise (b) and (b^*) are rephrasings of (2^*) and (2) respectively. Thus $(1) - (1^*)$ and $(2) - (2^*)$ for (P^*, P^{**}) are logically equivalent to $(1) - (1^*)$ and $(2) - (2^*)$ for (P, P^*).∎

As indicated above, we call (P^*, P^{**}) the *dual projective plane* to (P, P^*).

By Theorem 6 we can similarly define P^{***} to be the set of all pencils in P^{**}, and it follows that (P^{**}, P^{***}) is also a projective plane. However, repetition of the pencil construction does not give us anything new because of the following result:

THEOREM IV.7. *Let (P, P^*) be a projective plane, and let $\mathfrak{p}_0 : P \to P^{**}$ be the map sending a point x to the pencil $\mathfrak{p}(x)$ of lines through x. Then $\mathfrak{p}_0$ defines an isomorphism of incidence planes from (P, P^*) to the double dual projective plane (P^{**}, P^{***}).*

Proof. By construction the map $\mathfrak{p}_0$ is onto. It is also 1–1 because $\mathfrak{p}(x) = \mathfrak{p}(y)$ implies every line passing through x also passes through y. This is impossible unless $x = y$.

This it remains to show that L is a line in P if and only if $\mathfrak{p}(L)$ is a line in P^{**}. But lines in P^{**} have the form

$$\mathfrak{p}(L) = \{ w \in P^{**} \mid L \in \mathfrak{p}(x) \} .$$

where L is a line in P. Since $L \in \mathfrak{p}(x)$ if and only if $x \in L$, it follows that $\mathfrak{p}(x) \in \mathfrak{p}(L)$ if and only if $x \in L$. Hence $\mathfrak{p}(L)$ is the image of L under the map $\mathfrak{p}_0$, and thus the latter is an isomorphism of (projective) geometrical incidence planes.∎

The preceding theorems yield the following important phenomenon[5] which was described in the first paragraph of this section; it was discovered independently by J.-V. Poncelet (1788–1867) and J. Gergonne (1771–1859)[6].

METATHEOREM IV.8. (Principle of Duality) *A theorem about projective planes remains true if one interchanges the words* point *and* line *and also the phrases* contains *and* is contained in.∎

The justification for the Duality Principle is simple. The statement obtained by making the indicated changes is equivalent to *a statement about duals of projective planes* which corresponds to the *original statement for projective planes*. Since duals of projective planes are also projective planes, the modified statement must also hold.

Definition. Let **A** be a statement about projective planes. The *dual statement* is the one obtained by making the changes indicated in Metatheorem 8, and it is denoted by $\mathfrak{D}(\mathbf{A})$ or $\mathbf{A}^*$.

EXAMPLE 1. The phrase *three points are collinear* (contained in a common single line) dualizes to *three lines are concurrent* (containing a common single point).

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[5]This is called a **Metatheorem** because it is really a statement *about* mathematics rather than a theorem *within* mathematics itself. In other words it is a *theorem about theorems*.

[6]JEAN-VICTOR PONCELET is known for his work in geometry and engineering. His work on projective geometry began while he was a prisoner in Russia during the Napoleonic Wars. JOSEPH DIAZ GERGONNE is known for his military service for the First Republic in France, his founding of an important mathematical journal during the second decade of the 19th century, and highly noteworthy results and insights covering several areas of geometry.

EXAMPLE 2. The property **(P-1′)**, which assumes the existence of four points, not three of which are collinear, dualizes to the statement, *There exist four lines, no three of which are concurrent.* This statement was shown to follow from **(P-1′)** in the course of proving Theorem 4.

By the metatheorem, a statement $\mathbf{A}^*$ is true for all projective planes if $\mathbf{A}^*$ is. On the other hand, Theorem 7 implies that $\mathbf{A}^{**}$ is logically equivalent to $\mathbf{A}$. This is helpful in restating the Principle of Duality in a somewhat more useful form:

Modified Principle of Duality. *Suppose that $\mathbf{A}_1, \cdots, \mathbf{A}_n$ are statements about projective planes. Then $\mathbf{A}_n$ is true in all projective planes satisfying $\mathbf{A}_1, \cdots, \mathbf{A}_{n-1}$ if and only if $\mathbf{A}_n^*$ is true in all projective planes satisfying $\mathbf{A}_1^*, \cdots, \mathbf{A}_{n-1}^*$.*∎

In practice, one often knows that the statements $\mathbf{A}_1, \cdots, \mathbf{A}_{n-1}$ imply their own duals. Under these circumstances, the Modified Principle of Duality shows that $\mathbf{A}_n^*$ is true if $\mathbf{A}_n$ is true, and conversely. The next two theorems give significant examples of such statements; in both cases we have $n = 2$.

THEOREM IV.9. *If a projective plane contains only finitely many points, then it also contains only finitely many lines.*

Proof. Suppose the plane contains n elements. Then there are 2^n subsets of the plane. But lines are subsets of the plane, and hence there are at most 2^n lines.∎

In Theorems 11-15 we shall prove very strong duality results for projective planes with only finitely many points. Recall that examples of such systems are given by the coordinate projective planes $\mathbb{Z}_p\mathbb{P}^2$, where p is a prime.

We shall now give a considerably less trivial example involving duality.

THEOREM IV.10. *If a projective plane (P, P^*) is Desarguian, then the dual of Desargues' Theorem is also true in (P, P^*).*

Proof. The first step in the proof is to describe the dual result in terms of (P, P^*).

The dualization of two abstract triples of noncollinear points is two distinct triples of nonconcurrent lines, which we denote by $\{\alpha, \beta, \gamma\}$ and $\{\alpha', \beta', \gamma'\}$. Next, the concurrency hypothesis for AA', BB' and CC' dualizes to a hypothesis that $E \in \alpha \cap \alpha'$, $F \in \beta \cap \beta'$, and $D \in \gamma \cap \gamma'$ are collinear. Finally, the collinearity conclusion in Desargues' Theorem dualizes to a statement that three lines are concurrent. Specifically, if we set

$$
\begin{array}{ll}
A \in \beta \cap \gamma & A' \in \beta' \cap \gamma' \\
B \in \alpha \cap \gamma & B' \in \alpha' \cap \gamma' \\
C \in \alpha \cap \beta & C' \in \alpha' \cap \beta'
\end{array}
$$

then we wish to show that AA', BB' and CC' are concurrent. As the drawing below illustrates, the data for the dual theorem are similar to the data for the original theorem, the key difference being that Desargues' Theorem assumes concurrency of the lines AA', BB' and CC', using this to prove the collinearity of D, E and F, while the dual theorem assumes collinearity of the three points and aims to prove concurrency of the three lines.
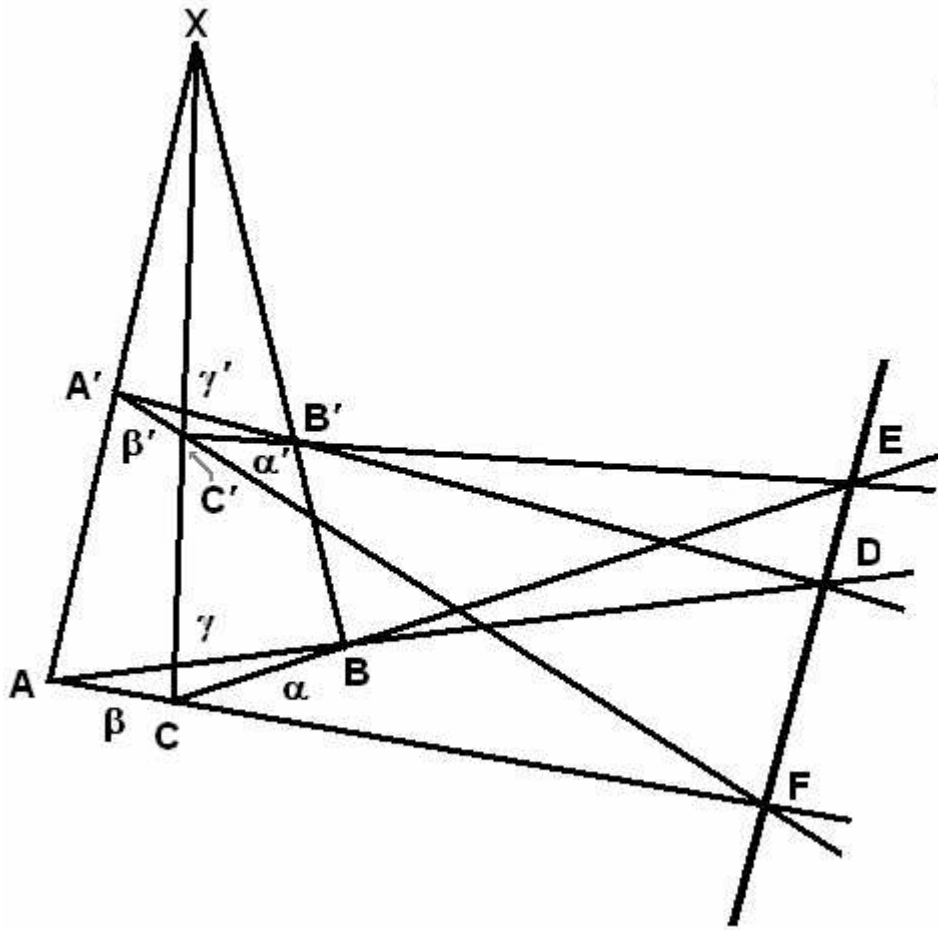
Figure IV.7

By construction, the lines $B'C'$, BC and DF all meet at E. Therefore Desargues' Theorem applies to the triples $\{C', C, F\}$ and $\{B', B, D\}$, and hence we may conclude that the three points

$$X \in BB' \cap CC'$$
$$G \in BD \cap CF$$
$$H \in B'D \cap C'F$$

are collinear. Using this, we obtain the following additional conclusions:

(1) B, $D \in \gamma$, and therefore $\gamma = BD$.

(2) B', $D \in \gamma'$, and therefore $\gamma' = B'D$.

(3) C, $F \in \beta$, and therefore $\beta = CF$.

(4) C', $F \in \beta'$, and therefore $\beta' = C'F$.

Therefore $G \in \gamma \cap \beta$ and $H \in \gamma' \cap \beta'$. Since the common points of these lines are A and A' by definition, we see that X, A and A' are collinear. In other words, we have $X \in AA' \cap BB' \cap CC'$, and hence the three lines are concurrent, which is what we wanted to prove.■

Duality and finite projective planes

We shall illustrate the usefulness of duality by proving a few simple but far-reaching results on projective planes which contain only finitely many points. We have already noted that for each prime p there is a corresponding projective plane $\mathbb{Z}_p\mathbb{P}^2$. As noted below, these are more than abstract curiosities, and they play an important role in combinatorial theory (the study of counting principles) and its applications to experimental design and error-correcting codes.

By Theorem 9, it follows that a projective plane is finite if and only if its dual plane is finite. In fact, one can draw much stronger conclusions.

THEOREM IV.11. *Let P be a finite projective plane. Then all lines in P contain exactly the same number of points.*

Proof. Let L and M be the lines. Since there exist four points, no three of which are collinear, there must be a point **p** which belongs to neither L nor M.
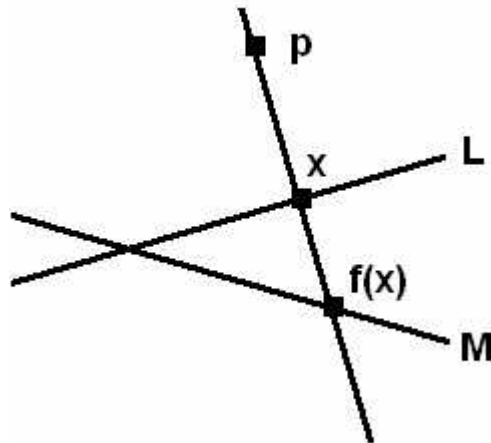
Figure IV.8

Define a map $f : L \to M$ by sending $\mathbf{x} \in L$ to the point $f(\mathbf{x})$ where **px** meets M. It is a straightforward exercise to verify that f is 1–1 and onto (see Exercise 6 below).∎

Dualizing the preceding, we obtain the following conclusion.

THEOREM IV.12. *Let P be a finite projective plane. Then all points in P are contained in exactly the same number of lines.*∎

Observe that the next result is **self-dual**; in other words, the dual statement is logically equivalent to the original one.

THEOREM IV.13. *Let P be a finite projective plane. Then the number of points on each line is equal to the number of lines containing each point.*

Proof. Let L be a line in P and let $\mathbf{x} \notin L$. Define a map from lines through $\mathbf{x}$ to points on L by sending a line M with $\mathbf{x} \in M$ to its unique intersection point with L. It is again a routine exercise to show this map is 1–1 and onto (again see Exercise 6 below).∎

Definition. The *order* of a finite projective plane is the positive integer $n \geq 2$ such that every line contains $n + 1$ points and every point lies on $n + 1$ lines. The reason for the subtracting one from the common number is as follows: If $\mathbb{F}$ is a finite field with q elements, then the order of $\mathbb{F}\mathbb{P}^2$ will also be equal to q.

The results above yield the following interesting and significant restriction on the number of points in a finite projective plane:

THEOREM IV.14. *Let P be a finite projective plane of order n. Then P contains exactly n^2+n+1 points.*

In particular, for most positive integers m it is not possible to construct a projective plane with exactly m points. More will be said about the possibilities for m below.

Proof. We know that $n + 1$ is the number or points on every line and the number of lines through every point. Let $\mathbf{x} \in P$. If we count all pairs $(\mathbf{y}, L)$ such that L is a line through $\mathbf{x}$ and $\mathbf{y} \in L$, then we see that there are exactly $(n + 1)^2$ of them. In counting the pairs, some points such as $\mathbf{x}$ may appear more than once. However, $\mathbf{x}$ is the only point which does so, for $\mathbf{y} \neq \mathbf{x}$ implies there is only one line containing both points. Furthermore, by the preceding result we know that $\mathbf{x}$ appears exactly $n + 1$ times. Therefore the correct number of points in P is given by subtracting n (not $n + 1$) from the number of ordered pairs, and it follows that P contains exactly

$$(n+1)^2 \; - \; n \; = \; n^2 \; + \; n \; + \; 1$$

distinct points.∎

The next theorem follows immediately by duality.

THEOREM IV.15. *Let P be a finite projective plane of order n. Then P contains exactly n^2+n+1 lines.*∎

Further remarks on finite projective planes

We shall now consider two issues raised in the preceding discussion:

(1) The possible orders of finite projective planes.

(2) The mathematical and nonmathematical uses of finite projective planes.

ORDERS OF FINITE PROJECTIVE PLANES. The theory of finite fields is completely understood and is presented in nearly every graduate level algebra textbook (for example, see Section V.5 of the book by Hungerford in the bibliography). For our purposes it will suffice to note that for each prime number p and each positive integer n, there is a field with exactly $q = p^n$ elements. It follows that *every prime power is the order of some projective plane.*

The possible existence of projective planes with other orders is an open question. However, many possible orders are excluded by the following result, which is known as the **Bruck–Ryser–Chowla Theorem:**[7] *Suppose that the positive integer n has the form $4k + 1$ or $4k + 2$ for some positive integer k. Then n is the sum of two (integral) squares.*

References for this result include the books by Albert and Sandler, Hall (the book on group theory), and Ryser in the bibliography as well as the original paper by Bruck-Ryser (also in the bibliography) and the following online reference[8]:

$$\texttt{http://www.math.unh.edu/}\sim\texttt{dvf/532/7proj-plane.pdf}$$

Here is a list of the integers between 2 and 100 which cannot be orders of finite projective planes by the Bruck-Ryser-Chowla Theorem:

6	14	21	22	30	33	38	42
45	46	54	57	62	66	69	70
74	77	78	82	86	93	94	97

The smallest positive integer ≥ 2 which is not a prime power and not excluded by the Bruck-Ryser-Chowla Theorem is 10. In the late nineteen eighties a substantial argument — which used sophisticated methods together with involved massive amounts of computer calculations — showed that no projective planes of order 10 exist. Two papers on this work are listed in the bibliography; the article by one author (C. W. H. Lam) in the *American Mathematical Monthly* was written to explain the research on this problem to a reasonably broad general audience of mathematicians and students.

By the preceding discussion, the existence of projective planes of order n is understood for $n \leq 11$, and the first open case is $n = 12$. Very little is known about this case.

PROJECTIVE GEOMETRY AND FINITE CONFIGURATIONS. Of course, one can view existence problems about finite projective planes as extremely challenging puzzles similar to Magic Squares (including the *Sudoku* puzzles that have recently become extremely popular), but one important reason for studying them is their relevance to questions of independent interest.

One somewhat whimsical "application" (*plant ten trees in ten rows of three*) was mentioned in Exercise IV.2.2, where the point was that such configurations exist by Desargues' Theorem. In fact, the study of projective spaces — especially finite ones – turns out to have many useful consequences in the study of finite *tactical configurations* or *block designs*, which is part of *combinatorial theory* or *combinatorics*. Specifically, finite projective planes are of interest as examples of *Latin squares*, or square matrices whose entries are in a finite set such that each element appears in every row and every column exactly once. As noted earlier, such objects play a significant role in the areas of statistics involving the design of experiments and in the theory of error-correcting codes. Although further comments are well outside the scope of these notes, the references in the bibliography by Buekenhout, Crapo and Rota, Hall (the book on combinatorial

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[7]Sarvadaman D. S. Chowla (1907–1995) was a British-Indian-American matheamtician who made numerous contributions to number theory and related fields.

[8]To update this document, the conjecture of E. Catalan (1814–1894) has recently been shown to be true bu P. Mihăilescu (Eugène Catalan is known for his contributions to number theory and related areas). The proof of Catalan's Conjecture is at a very advanced level, but for the sake of completeness here is a reference: P. Mihăilescu, *Primary cyclotomic units and a proof of Catalan's Conjecture*, [Crelle] Journal für die reine und angewandte Mathematik **572** (2004), 167–195.

theory), Kárteszi, Lindner and Rodgers, and (the previously cited book by) Ryser are all sources for further information. There are also several online web sites dedicated to questions about finite geometry.

Duality in higher dimensions

The concept of duality extends to all projective n-spaces (where $n \geq 2$), but the duality is more complicated for $n \geq 3$ than it is in the 2-dimensional case. Specifically, if S is a projective n-space, then points of the dual space S^* are given by the *hyperplanes* of S. for each k-plane P in S, there is an associated *(linear) bundle of hyperplanes with center P*, which we shall denote by $\mathfrak{b}(P)$, and the dimension of $\mathfrak{b}(P)$ is set equal to $n - k - 1$. We shall also denote the set of all such bundles by Π^* the associated dimension function by d^*. The following result is the appropriate generalization of Theorems 6 and 7 which allows extension of the principle of duality to higher dimensions.

THEOREM IV.16. *If (S, Π, d) is a projective n-space for some $n \geq 2$, then so is the dual object (S^*, Π^*, d^*). Furthermore, the map $\mathcal{E}$ sending $\mathbf{x} \in S$ to the bundle of hyperplanes $\mathfrak{b}(\mathbf{x})$ with center x defines an isomorphism of geometrical incidence spaces.*

We shall not give a direct proof of this result for two reasons.

1. Although the proof is totally elementary, it is a rather long and boring sequence of routine verifications.

2. The result follows from a coordinatization theorem in the next section (Theorem 18) and the results of Section VI.1, at least if $n \geq 3$ (and we have already done the case $n = 2$).

An excellent direct proof of Theorem 16 is given in Sections 4.3 and 4.4 of Murtha and Willard, *Linear Algebra and Geometry* (see the bibliography for further information).

EXERCISES

1. Prove that properties $(1) - (1^*)$ and $(2) - (2^*)$ completely characterize projective planes. In other words, if $(P, \mathcal{L})$ is a pair consisting of a set P and a nonempty collection of proper subsets $\mathcal{L}$ satisfying these, then there is a geometrical incidence space structure $(P, \mathcal{L}, d)$ such that the incidence space is a projective plane and $\mathcal{L}$ is the family of lines in Π.

2. Write out the plane duals of the following phrases, and sketch both the given data and their plane duals.

  (i) Two lines, and a point on neither line.

  (ii) Three collinear points and a fourth point not on the line of the other three.

  (iii) Two triples of collinear points not on the same line.

  (iv) Three nonconcurrent lines, and three points such that each point lies on exactly one of the lines.

3. Draw the plane duals of the illustrated finite sets of points which are marked heavily in the two drawings below.
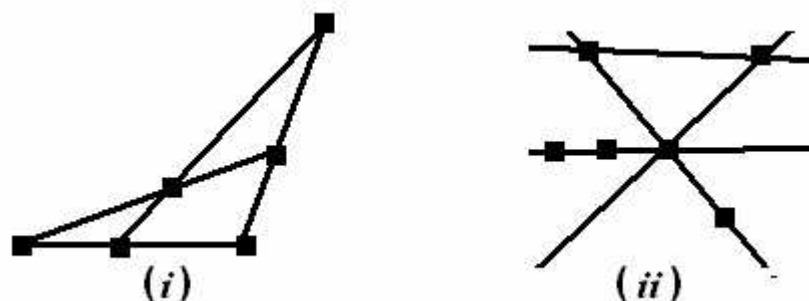
Figure IV.9

4. Suppose that $f : S \to T$ is an incidence space isomorphism from one projective n-space ($n \geq 2$) to another. Prove that f induces a 1–1 correspondence $f^* : S^* \to T^*$ taking a hyperplane $H \subset S$ to the image hyperplane $f^*(H) = f[H] \subset T$. Prove that this correspondence has the property that B is an r-dimensional bundle of hyperplanes (in S^*) if and only if $f^*[B]$ is such a subset of T^* (hence it is an isomorphism of geometrical incidence spaces, assuming Theorem 16).

5. Prove that the construction in the preceding exercise sending f to f^* has the following properties:

(i) If $g : T \to U$ is another isomorphism of projective n-spaces, then $(g \circ f)^* = g^* \circ f^*$.

(ii) For all choices of S the map $(\mathrm{id}_S)^*$ is equal to the identity on S^*.

(iii) For all f we have $\left(f^{-1} \right)^* = \left(f^* \right)^{-1}$.

6. Complete the proof of Theorem 11.

7. Let $(P, \mathcal{L})$ be a finite affine plane. Prove that there is a positive integer n such that

(i) every line in P contains exactly n points,

(ii) every point in P lies on exactly n lines,

(iii) the plane P contains exactly n^2 points.

NOTE. Another problem involving the ideas of this section and the previous one is stated and solved in the following online document:

http://math.ucr.edu/~res/progeom/treasure-problem.pdf

4. Conditions for coordinatization

Since the results and techniques of linear algebra are applicable to coordinate projective n-spaces $\mathcal{S}_1(V)$ (where $\dim V = n + 1$), these are the most conveniently studied of all projective spaces. Thus it is desirable to know when a projective n-space is isomorphic to one having the form $\mathcal{S}_1(V)$, where $\dim V = n + 1$. The following remarkable theorem shows that relatively weak hypotheses suffice for the existence of such an isomorphism.

THEOREM IV.17. *Let P be a projective n-space in which Desargues' Theorem is valid (for example, $n \geq 3$ or P is a Desarguian plane). Then there is a skew-field $\mathbb{F}$ such that P is isomorphic to $\mathbb{FP}^n$ (where we view $\mathbb{F}^{n+1}$ as a right vector space over $\mathbb{F}$). If $\mathbb{E}$ is another skew-field such that P is isomorphic to $\mathbb{EP}^n$, then $\mathbb{E}$ and $\mathbb{F}$ are isomorphic as skew-fields.*

A well-illustrated proof of Theorem 17 from first principles in the case $n = 2$ appears on pages 175–193 of the book by Fishback listed in the bibliography. Other versions of the proof appear in several other references from the bibliography. Very abstract approaches to the theorem when $n = 2$ appear in Chapter III of the book by Bumcrot and also in the book by Artzy. The proof in Chapter 6 of Hartshorne's book combines some of the best features of the other proofs. There is also a proof of Theorem 17 for arbitrary values of $n \geq 2$ in Chapter VI from Volume I of Hodge and Pedoe. Yet another reference is Sections 4.6 and 4.7 of Murtha and Willard. For more information on the uniqueness statement, see Theorem V.10.

In Section II.6 we noted that the geometric properties of isomorphic geometrical incidence spaces are the same, and that this can be extremely useful if one of the isomorphic objects is more convenient to work with than the other. If we combine this observation with Theorem 17, we obtain an important far-reaching principle in projective geometry which will play a fundamental role in the remaining units of these notes:

> *In order to prove a statement about projective spaces of dimension ≥ 3 and Desarguian projective planes,* **it suffices to prove the statement for the algebraically defined coordinate projective spaces** $\mathbb{FP}^n$, *where $n \geq 2$.*∎

Theorem 17 also yields a classification for projective n-spaces ($n \geq 3$) that is parallel to Theorem II.38 (see Remark 3 below for further discussion). Because of its importance, we state this classification separately.

THEOREM IV.18. *Let P be a projective n-space, where $n \geq 3$. Then there is a skew-field $\mathbb{F}$, unique up to algebraic isomorphism, such that P and $\mathbb{FP}^n$ are isomorphic as geometrical incidence spaces.*∎

Theorem 17 also implies the converse to a remark following the definition of a Desarguian projective plane in Section IV.2.

THEOREM IV.19. *If a projective plane is Desarguian, then it is isomorphic to a plane in a projective 3-space.*

Proof.   By Theorem 17, the plane is isomorphic to $\mathbb{FP}^2$ for some skew-field $\mathbb{F}$. But $\mathbb{FP}^2$ is isomorphic to the plane in $\mathbb{FP}^3$ defined by $x_4 = 0$.■

REMARK 1.   Suppose that P is the Desarguian plane $\mathbb{FP}^2$. By Theorem 10, the dual projective plane P^* is also Desarguian and hence is isomorphic to some plane $\mathbb{EP}^2$. As one might expect, the skew-fields $\mathbb{F}$ and $\mathbb{E}$ are closely related. In fact, by Theorem V.1 the dual plane P^* is isomorphic to $\mathcal{S}_1(\mathbb{F}_3)$, where $\mathbb{F}_3$ is a *left* vector space over $\mathbb{F}$, and left vector spaces over $\mathbb{F}$ correespond to right vector spaces over the **opposite skew-field** $\mathbb{F}^{\mathsf{OP}}$, whose elements are the same as $\mathbb{F}$ and whose multiplication is given by reversing the multiplication in $\mathbb{F}$. More precisely, one defines a new product $\otimes$ in $\mathbb{F}$ via $a \otimes b = b \cdot a$, and if V is a left vector space over $\mathbb{F}$ define vector space operations via the vector addition on $\mathbb{F}$ and the right scalar product $\mathbf{x} \otimes a = a \cdot \mathbf{x}$ (here the dot represents the original multiplication). It is a routine exercise to check that $\mathbb{F}^{\mathsf{OP}}$ is a skew-field and $\otimes$ makes left $\mathbb{F}$-vector spaces into right vector spaces over $\mathbb{F}^{\mathsf{OP}}$. Thus, since $\mathbb{F}_3$ is a 3-dimensional right $\mathbb{F}^{\mathsf{OP}}$-vector space, it follows that $\mathbb{E}$ must be isomorphic to $\mathbb{F}^{\mathsf{OP}}$.

REMARK 2.   If P is the coordinate projective plane $\mathbb{FP}^2$ and multiplication in $\mathbb{F}$ is commutative, then the preceding remark implies that P and P^* are isomorphic because $\mathbb{F}$ and $\mathbb{F}^{\mathsf{OP}}$ are identical in such cases. However, the reader should be warned that the isomorphisms from P to P^* are much less "natural" than the standard isomorphisms from P to P^{**} (this is illustrated by Exercise V.1.5), and if $\mathbb{F}$ is not commutative then $\mathbb{F}$ and $\mathbb{F}^{\mathsf{OP}}$ are not necessarily isomorphic (however, there are examples such that $\mathbb{F}$ is noncommutative but $\mathbb{F} \cong \mathbb{F}^{\mathsf{OP}}$. More information on the noncommutative case appears in Appendix C.

EXAMPLE 3.   Theorem II.18 follows from the coordinatization result (Theorem 18) which is stated above. For if S is an affine n-space, let $\overline{S}$ be its synthetic projective extension as defined in the Addendum to Section III.4. By Theorem III.16, we know that $\overline{S}$ is an n-dimensional projective incidence space, so that $\overline{S}$ is isomorphic to $\mathbb{FP}^n$ for some skew-field $\mathbb{F}$ and $S \subset \overline{S}$ is the complement of some hyperplane. Since there is an element of the geometric symmetry group of $\mathbb{FP}^n$ taking this hyperplane to the one defined by $x_{n+1} = 0$ (see Exercise III.4.14), we can assume that S corresponds to the image of $\mathbb{F}^n$ in $\mathbb{FP}^n$. Since k-planes in S are given by intersections of k-planes in $\overline{S}$ with S and similarly the k-planes in $\mathbb{F}^n$ are given by intersections of k-planes in $\mathbb{FP}^n$ with the image of $\mathbb{F}$, it follows that the induced 1–1 correspondence between S and $\mathbb{F}^n$ is a geometrical incidence space isomorphism.

Concluding remarks

The material below is background information that will not be used subsequently.

A general coordinatization theory for projective planes that are not necessarily Desarguian exists; the corresponding algebraic systems are generalizations of skew-fields known as *planar ternary rings*; one important difference between such structures and skew-fields is that the multiplication in planar ternary rings is not necessarily associative. Among the more accessible references for this material are the previously cited book by Albert and Sandler, Chapter 4 of the book by Artzy, Chapters III and VI of the book by Bumcrot, Chapter 20 of the book by Hall on group theory, and the 2007 survey article by C. Weibel (see the bibliography for more details).

One important example of a planar ternary ring which is not a skew-field (the **Cayley numbers** or **octonions**) is described on pages 195–196 of Artzy.[9] As one might expect, the Cayley numbers are associated to a projective plane called the *Cayley projective plane*. However, the coordinatization theorem implies that are no Cayley projective spaces of higher dimension.

Although the multiplication rule for Cayley numbers is not associative, it satisfies a weaker verstion of associativity

$$(xx)y \; = \; x(xy) \; , \qquad (yx)x \; = \; y(xx) \qquad \text{for all} \quad x, \, y \; .$$

Ternary rings which satisfy this condition are said to be *alternative*, and a projective plane with alternative ternary rings is said to be a *Moufang plane*.[10] Brief summaries of facts about alternative rings appears in the online references

http://en.wikipedia.org/wiki/Alternative_algebra

http://planetmath.org/encyclopedia/ArtinsTheoremOnAlternativeAlgebras.html

http://mathworld.wolfram.com/DivisionAlgebra.html

and more detailed information may be found in Section 20.6 of the book by Hall (on group theory) and Chapter III of the book by Schafer which are listed in the bibliography. Still further information is given in the papers by Baez, Bruck and Kleinfeld, Kleinfeld and Palais which are also listed in the bibliography.

Finally, in Section A.5 of Appendix A we state a key identity which relates the quaternion product in $\mathbb{R}^4$ with the usual cross product of vectors in $\mathbb{R}^3$. One can use the multiplication of Cayley numbers to define a vector product on $\mathbb{R}^7$ with properties similar to those of the 3-dimensional cross product, but by combining ideas in the references listed above with some fundamental results of algebraic topology from the 1950s, one can prove that *dimension 7 is the only other dimension in which a reasonable analog of the 3-dimensional cross product exists.* Further information appears at the end of Section I.2 in the following online reference:

http://math.ucr.edu/~res/geometrynotes1.pdf

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[9]A brief and accurate online summary is given in http://en.wikipedia.org/wiki/Octonion. The reliability issue for Wikipedia articles is discussed in a footnote for Appendix C, and it also applies here.

[10]RUTH MOUFANG (1905–1977) is known for work on the interrelationships between algebraic systems and the foundations of projective geometry; she was the first woman to serve as a professor in a German university and the first German woman with a doctorate to be employed as a mathematician in industry.

CHAPTER V

PLANE PROJECTIVE GEOMETRY

In this chapter we shall present the classical results of plane projective geometry. For the most part, we shall be working with coordinate projective planes and using homogeneous coordinates, but at certain points we shall also use synthetic methods, especially when it is more convenient to do so. Our treatment will make extensive use of concepts from linear algebra. Since one major geometric result (Pappus' Theorem) is closely connected to the algebraic commutativity of multiplication in a skew-field, we shall be fairly specific about using left or right vector spaces in most sections of this chapter.

1. Homogeneous line coordinates

If $\mathbb{F}$ is a skew-field, it will be convenient to let view $\mathbb{FP}^n$ as the set of all 1-dimensional vector subspaces of the $(n+1)$-dimensional right vector space $\mathbb{F}^{n+1,1}$ of $(n+1) \times 1$ column matrices over $\mathbb{F}$ with the obvious entrywise right multiplication:

$$
\begin{pmatrix} x_1 \\ \cdots \\ \cdots \\ x_{n+1} \end{pmatrix} \cdot c = \begin{pmatrix} x_1\,c \\ \cdots \\ \cdots \\ x_{n+1}\,c \end{pmatrix}
$$

It follows from Theorem III.12 that a line in $\mathbb{FP}^2$ is definable by an equation of the form $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$, where u_1, u_2, u_3 are not all zero. Furthermore, two triples of coefficients (u_1, u_2, u_3) and (v_1, v_2, v_3) define the same line if and only if there is a nonzero $k \in \mathbb{F}$ such that $U_i = k\,v_i$ for $i = 1, 2, 3$. Thus we see that *a line in $\mathbb{FP}^2$ is completely determined by a one-dimensional subspace of the* left *vector space of 1×3 row matrices.* — Therefore the dual projective plane to $\mathbb{FP}^2$ is in 1–1 correspondence with the 1-dimensional subspaces of $\mathbb{F}^{1,3}$, where the latter is the left vector space of 1×3 matrices. Under this correspondence the lines in the dual of $\mathbb{FP}^2$ correspond to lines in $\mathcal{S}_1(\mathbb{F}^{1,3})$. For a line in the dual is the set of lines through a given point, and by a reversal of the previous argument a line in $\mathcal{S}_1(\mathbb{F}^{1,3})$ is just the set of elements whose homogeneous coordinates (u_1, u_2, u_3) satisfy a linear homogeneous equation of the form $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$, where x_1, x_2, x_3 are not all zero. Much as before, the triples (x_1, x_2, x_3) and (y_1, y_2, y_3) define the same set if and only if $y_i = x_i\,m$ for some nonzero constant $m \in \mathbb{F}$. We can summarize the discussion above as follows:

THEOREM V.1. *Let $\mathbb{F}$ be a skew-field, and identify $\mathbb{FP}^2$ with $\mathcal{S}_1(\mathbb{F}^{3,1})$ as above. Then the dual plane $\left(\mathbb{FP}^2\right)^*$ is isomorphic to $\mathcal{S}_1(\mathbb{F}^{1,3})$ such that if the point $\mathbf{x}$ corresponds to the 1-dimensional right vector subspace $\xi \cdot \mathbb{F}$ and the line L corresponds to the 1-dimensional left vector subspace $\mathbb{F} \cdot \lambda$, then $\mathbf{x} \in L$ if and only if $\lambda \cdot \xi = 0$.*∎

The dot indicates matrix multiplication operation

$$\mathbb{F}^{1,3} \times \mathbb{F}^{3,1} \ \longrightarrow \ \mathbb{F}^{1,1} \ \cong \mathbb{F} \ .$$

As before, $\mathbb{F} \cdot \lambda$ denotes all left scalar multiples of λ, and we similarly let $\xi \cdot \mathbb{F}$ denote all right scalar multiples of ξ. Note that if $\lambda \cdot \xi \ = \ 0$ for one pair of homogeneous coordinate choices, then the same is true for every other pair. For the most general change of representatives is given by $k\,\lambda$ and $\xi\,m$ so that

$$(k\,\lambda) \cdot (\xi\,m) \ = \ k\,(\lambda \cdot \xi)\,m \ = \ 0$$

by associativity of multiplication.

Definition.   Given a line L in $\left(\mathbb{FP}^2\right)^* \ \cong \ \mathcal{S}_1(\mathbb{F}^{1,3})$, we say that a nonzero vector $\lambda \in \mathbb{F}^{1,3}$ is a set of homogeneous coordinates for L if the latter is the set of all points $\mathbf{x}$ whose homogeneous coordinates ξ satisfy $\lambda \cdot \xi \ = \ 0$.

By construction, three points in $\mathbb{FP}^2$ are collinear if and only if their homogeneous coordinates span a 2-dimensional right vector subspace of $\mathbb{F}^{3,1}$. The dualization of this to homogeneous line coordinates is an easy consequence of Theorem 1.

THEOREM V.2. *Let L, M and N be three distinct lines in $\mathbb{FP}^2$. Then they are concurrent if and only if their homogeneous line coordinates are linearly dependent.*■

NOTATIONAL CONVENTIONS. Throughout this chapter we shall be passing back and forth between geometric points and lines and the algebraic homogeneous coordinates which represent them. Needless to say, it is convenient to have some standard guidelines for passing back and forth between the geometric and algebraic objects. Normally we shall denote the geometric objects by Roman letters and appropriate homogeneous coordinates by corresponding Greek letters (strictly speaking, we use mathematicians' versions of Greek letters). For example, if X and Y are points, then we shall normally use ξ or η for homogeneous coordinates, and if L is a line we shall normally use λ.

Homogeneous coordinate formulas

In the remainder of this section, we shall describe some useful formulas which are valid provided the skew-field $\mathbb{F}$ is **commutative**. Of course, if $\mathbb{F}$ is commutative the distinction between left and right vector subspaces is unnecessary.

We begin by stating two obvious problems:

1.   If L is a line determined by $\mathbf{x}$ and $\mathbf{y}$, express homogeneous coordinates for L in terms of homogeneous coordinates for $\mathbf{x}$ and $\mathbf{y}$.

2.   If $\mathbf{x}$ is the point of intersection for lines L and M, express homogeneous coordinates for $\mathbf{x}$ in terms of homogeneous coordinates for L and M.

Consider the first problem. By the definition of homogeneous coordinates of lines, a set of homogeneous coordinates λ for L must satisfy $\lambda \cdot \xi = \lambda \cdot \eta = 0$. If $\mathbb{F}$ is the real numbers, this means that the transpose of λ is perpendicular to ξ and η. Since $\mathbf{x} \neq \mathbf{y}$, it follows that ξ and η are linearly independent; consequently, the subspace of vectors perpendicular to both of the latter must be 1-dimensional. A nonzero (and hence spanning) vector in the subspace of vectors perpendicular to ξ and η is given by the *cross product* $\xi \times \eta$, where the latter are viewed as ordinary 3-dimensional vectors (see Section A.5 in Appendix A). It follows that λ may be chosen to be an arbitrary nonzero multiple of $\xi \times \eta$. We shall generalize this formula to other fields.

THEOREM V.3. *Let $\mathbb{F}$ be a (commutative) field, and let $\mathbf{x}$ and $\mathbf{y}$ be distinct points in $\mathbb{F}\mathbb{P}^2$ having homogeneous coordinates ξ and η. Then the line $\mathbf{xy}$ has homogeneous coordinates given by the* transpose *of $\xi \times \eta$.*

Proof. The definition of cross product implies that

$$^{\mathbf{T}}(\xi \times \eta) \cdot \xi = {}^{\mathbf{T}}(\xi \times \eta) \cdot \eta = 0$$

so it is only necessary to show that if ξ and η are linearly independent then $\xi \times \eta \neq \mathbf{0}$.

Let the entries of ξ be given by x_i, let the entries of η be given by y_j, and consider the 3×2 matrix B whose entries are the entries of the 3×1 matrices ξ and η:

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}$$

Since the columns are linearly independent, the rank of this matrix is 2. By Theorem A.11 this means there is 2×2 submatrix of B with nonzero determinant. If $k \in \{1,2,3\}$ is such that the matrix obtained by deleting the kth row is nonzero, then by the definition of cross product the kth entry of the latter must be nonzero. Therefore the transpose of $\xi \times \eta$ is a set of homogeneous coordinates for L.■

Dually, we have the following result:

THEOREM V.4. *Let $\mathbb{F}$ be a (commutative) field, and let L and M be distinct lines in $\mathbb{F}\mathbb{P}^2$ having homogeneous coordinates λ and μ. Then the intersection point of L and M has homogeneous coordinates given by the* transpose *of $\lambda \times \mu$.*■

The "Back-Cab Rule" for triple cross products

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$$

(see Theorem A.20) implies the following useful formula:

THEOREM V.5. *Let L be a line in $\mathbb{F}\mathbb{P}^2$, and let $\mathbf{x}$ and $\mathbf{y}$ be points of $\mathbb{F}\mathbb{P}^2$ not on L. Let λ, ξ and η be homogeneous coordinates for L, $\mathbf{x}$ and $\mathbf{y}$ respectively. Then the common point of the lines L and $\mathbf{xy}$ has homogeneous coordinates equal to $(\lambda \cdot \xi)\,{}^{\mathbf{T}}\eta - (\lambda \cdot \eta)\,{}^{\mathbf{T}}\xi$.*

Proof. By Theorem 3 the line $\mathbf{xy}$ has homogeneous coordinates $^{\mathbf{T}}\xi \times {}^{\mathbf{T}}\eta$, and thus by Theorem 4 the common point of L and $\mathbf{xy}$ has homogeneous coordinates equal to

$$^{\mathbf{T}}\left(\lambda \times {}^{\mathbf{T}}\left(^{\mathbf{T}}\xi \times {}^{\mathbf{T}}\eta\right)\right) \ .$$

The latter is equal to $(\lambda \cdot \xi) \, {}^{\mathbf{T}}\eta \ - \ (\lambda \cdot \eta) \, {}^{\mathbf{T}}\xi$ by the Back-Cab Rule.■

EXERCISES

1. Consider the affine line in $\mathbb{F}^2$ defined by the equation $ax + by \ = \ c$. What are the homogeneous coordinates of its extension to $\mathbb{FP}^2$? As usual, consider the 1–1 map from $\mathbb{F}^2$ to $\mathbb{FP}^2$ which sends (x, y) to the point with homogeneous coordinates

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \ .$$

2. Suppose that three affine lines are defined by the equations $a_i x + b_i y \ = \ c_i$, where $i = 1, 2, 3$. Prove that these three lines are concurrent if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \ = \ 0 \qquad \text{and} \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \ \neq \ 0 \ .$$

What can one conclude about the lines if **both** determinants vanish?

3. Using the methods of this section, find the equation of the affine line joining $(\frac{1}{4}, \frac{1}{2})$ to the point of intersection of the lines defined by the equations $x + 2y + 1 \ = \ 0$ and $2x + y + 3 \ = \ 0$.

4. Find the homogeneous coordinates of the point at which the line with homogeneous coordinates $(2 \quad 1 \quad 4)$ meets the line through the points with homogeneous coordinates

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \ .$$

5. Let A be an invertible 3×3 matrix over $\mathbb{F}$, and let T be the geometric symmetry of $\mathbb{FP}^2 \ \cong \ \mathcal{S}_1(\mathbb{F}^{3,1})$ defined by the equation

$$T(\mathbf{x}) \ = \ A \cdot \xi \cdot \mathbb{F}$$

where ξ is a set of homogeneous coordinates for $\mathbf{x}$ and the dot indicates matrix multiplication. If L is a line in $\mathbb{FP}^2$ with homogeneous coordinates λ, show that the line $f[L]$ has homogeneous coordinates given by $\lambda \cdot A^{-1}$.

2. Cross ratio

At the beginning of Chapter III, we mentioned that one forerunner of projective geometry was the development of a mathematical theory of perspective images by artists during the 15th and early 16th century. Clearly, if one compares such perspective photographic images with the physical objects they come from, it appears that some physical properties are faithfully captured by the photograph while others are not. For example, if three points on a physical object are collinear, then their photographic images are also collinear, and under suitable conditions if we have three distinct points A, B and C such that B is between A and C, then the image point B' of B will also lie between the corresponding image points A and C'.[1] However, it is also apparent that the relative distances among the three points can be greatly distorted. For example, if B is the midpoint of A and C, then we cannot conclude that B' is the midpoint of A' and C'. Similarly, if B is between A and C and the distance from B to C is twice the distance from A to B, we cannot conclude that a similar relationship holds for the corresponding relative distances among the image points A', B' and C'. HOWEVER, if we are given *four* collinear points on the physical object, then there is a number called the *cross ratio*, which is determined by relative distances among the four points, that is the same for the original four points on the physical object(s) as well as their photographic images. The cross ratio itself was apparently first defined algebraically by P. de la Hire[2] (1640–1718), but the perspective invariance property was essentially known to Pappus of Alexandria[3] (*c.* 290 – *c.* 350??) and perhaps even earlier. Throughout the rest of these notes we shall see that the cross ratio plays a fundamentally important role in projective geometry.

It will be convenient to define the cross ratio in terms of homogeneous coordinates and to give a nonvisual motivation for the concept. In problems involving coordinate projective spaces, it is often helpful to choose homogeneous coordinates in a particular way. We shall prove some results justifying such choices below and use them to give a fairly simple definition of the cross product. The discussion up to (but not including) Theorem 8 is valid for any skew-field $\mathbb{F}$. Starting with Theorem 9, we assume $\mathbb{F}$ is commutative.

THEOREM V.6. *Let A, B, C, D be four points in $\mathbb{FP}^2$, no three of which are collinear. Then there exist homogeneous coordinates α, β, γ, δ for A, B, C, D such that $\delta = \alpha + \beta + \gamma$.*

Proof. Let α_0, β_0, γ_0 be arbitrary homogeneous coordinates for A, B, C respectively. Since A, B, C are noncollinear, the vectors α_0, β_0, γ_0 form a basis for $\mathbb{F}^{3,1}$. Thus there exist x, y, $z \in \mathbb{F}$ such that homogeneous coordinates for D are given by $\delta = \alpha_0 x + \beta_0 y + \gamma_0 z$. We claim that each of x, y, z is nonzero. If $x = 0$, then it follows that $\delta = y\beta_0 + z\gamma_0$, so that D lies on the line BC; therefore it follows that $x \neq 0$, and similarly we can conclude that y and z are nonzero. But this means that $\alpha_0 x$, $\beta_0 y$, $\gamma_0 z$ are homogeneous coordinates for A, B, C respectively, and accordingly we may take $\alpha = \alpha_0 x$, $\beta = \beta_0 y$, and $\gamma = \gamma_0 z$.∎

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[1] The suitable, and physically reasonable, conditions are given in terms of the aperture of the camera at some point X and the image plane P which does not contain X: If Q is the unique plane through X which is parallel to P, then normally the plane P the physical object(s) being photographed will lie on opposite sides of the plane Q.

[2] PHILIPPE DE LA HIRE made significant contributions to geometry and several other areas of mathematics, and he also made important contributions to astronomy and geodesy.

[3] PAPPUS OF ALEXANDRIA was one of the last major figures in classical Greek mathematics; he is known for his contributions to geometry as well as his sequence of books which described much of classical Greek geometry, including information on sources that are no longer known to exist.

The next few results are true in $\mathbb{FP}^n$ for any $n \geq 1$.

THEOREM V.7. *Let A and B Let A and B be distinct points in $\mathbb{FP}^n$ with homogeneous coordinates α and β respectively. Let C be a third point on AB. Then there exist homogeneous coordinates γ for C such that $\gamma = \alpha x + \beta$ for some unique $x \in \mathbb{F}$.*

Proof.   Since A, B, and C are noncollinear, there exist u, $v \in \mathbb{F}$ such that homogeneous coordinates for C are given by $\alpha u + \beta v$. We claim that $v \neq 0$, for otherwise $\gamma = \alpha u$ would imply $C = A$. Thus $\alpha u\, v^{-1} + \beta$ is also a set of homogeneous coordinates for C, proving the existence portion of the theorem. Conversely, if $\alpha y + \beta$ is also a set of homogeneous coordinates for C, then there is a nonzero scalar k such that

$$\alpha\, y \;+\; \beta \;=\; (\alpha\, x \;+\; \beta) \cdot k \;=\; \alpha\, x\, k \;+\; \beta\, k \;.$$

Equating coefficients, we have $k = 1$ and $y = xk = x$.∎

Notation.   If $C \neq A$, the element of $\mathbb{F}$ determined by Theorem 7 is called the *nonhomogeneous coordinate* of C with respect to α and β and written $x_{(C;\alpha,\beta)}$.

THEOREM V.8. *Let A, B, and C be distinct collinear points in $\mathbb{FP}^n$. Then there exist homogeneous coordinates α, β, γ for A, B, C such that $\gamma = \alpha + \beta$. Furthermore, if α', β', γ' are arbitrary homogeneous coordinates for A, B, and C, then there is a nonzero constant $k \in \mathbb{F}$ such that $\alpha' = \alpha k$, $\beta' = \beta k$, and $\gamma' = \gamma k$.*

Proof.   By Theorem 7 there exist homogeneous coordinates α'' and β'' for A and B such that homogeneous coordinates for C are given by $\gamma = \alpha''x + \beta''$. If x were zero then B and A would be equal, and consequently we must have $x \neq 0$. Thus if we take $\alpha = \alpha''x$ and $\beta = \beta'$, then $\gamma = \alpha + \beta$ is immediate.

Suppose that α', β', γ' satisfy the condition of the theorem. Then there exist constants a, b, c such that $\alpha' = \alpha\, a$, $\beta' = \beta\, b$, and $\gamma' = \gamma\, c$. It follows that

$$\alpha\, c \;+\; \beta\, c \;=\; \gamma\, c \;=\; \alpha' \;+\; \beta' \;=\; \alpha \cdot a \;+\; \beta \cdot b \;.$$

Equating coefficients, we obtain $a \;=\; c \;=\; b$, and thus we may take $k \;=\; c$.∎

COMMUTATIVITY ASSUMPTION.   Throughout the rest of this section $\mathbb{F}$ is assumed to be commutative. The definition of cross ratio is justified by the following result:

THEOREM V.9. *Let A, B and C be distinct collinear points in $\mathbb{FP}^n$, and let D be a point on this line such with $D \neq A$. Suppose that homogeneous coordinates α, β, γ for A, B, C are chosen such that $\gamma = \alpha + \beta$, and write homogeneous coordinates for D as $\delta = u\alpha + v\beta$ in these coordinates (since $D \neq A$ we must have $v \neq 0$). Then the quotient u/v is the same for all choices of α, β, γ satisfying the given equation.*

Definition.   The scalar $u/v \in \mathbb{F}$ is called the **cross ratio** of the ordered quadruple of collinear points (A, B, C, D), and it is denoted by $\mathsf{XR}(A, B, C, D)$.

Proof.   If α', β', γ' is another triple such that $\gamma' = \alpha' + \beta'$, then by Theorem 8 there is a nonzero scalar $k \in \mathbb{F}$ such that $\alpha' = k\alpha$, $\beta' = k\beta$, and $\gamma' = k\gamma$. If δ' is another set of homogeneous coordinates for D, then $\delta' = r\delta$ for some $r \in \mathbb{F}$. Thus if $\delta' = u'\alpha' + v'\beta'$, it follows

that $\delta' = ku'\alpha + kv'\beta$. On the other hand, $\delta' = r\delta$ implies that $\delta' = ru\alpha + rv\beta$. Equating coefficients, we find that $ku' = ru$ and $kv' = rv$. Therefore we have

$$\frac{u}{v} = \frac{ru}{rv} = \frac{ku'}{kv'} = \frac{u'}{v'}$$

and therefore the ratio of the coefficients does not depend upon the choices of homogeneous coordinates.∎

The next result answers a fundamental question concerning the cross ratio:

THEOREM V.10. *Suppose that A, B and C are distinct collinear points and $r \in \mathbb{F}$ is an arbitrary constant. Then there is a unique $D \in AB$ such that $\mathsf{XR}(A, B, C, D) = r$.*

Proof. *Existence.* Choose D so that it has homogeneous coordinates equal to $\delta = r\alpha + \beta$.

Uniqueness. Suppose that $\mathsf{XR}(A, B, C, D) = \mathsf{XR}(A, B, C, E) = r$, where neither D nor E is equal to A. Choose homogeneous coordinates so that we have $\gamma = \alpha + \beta$; then we may write $\delta = u\alpha + v\beta$ and $\varepsilon = s\alpha + t\beta$, where

$$\frac{u}{v} = r = \frac{s}{t} ;$$

note that v and t are nonzero because neither D nor E is equal to A. It follows that $u = vr$ and $s = tr$, from which we conclude that $\delta = v\,t^{-1}\varepsilon$. The latter in turn implies that $D = E$.∎

Another fundamental property of the cross ratio is given by the following result, whose proof is left as an exercise:

THEOREM V.11. *Let A, B, C and D be distinct noncollinear points, and let $E \neq A$ be another point on the same line. Then $\mathsf{XR}(A, B, C, E) = \mathsf{XR}(A, B, C, D) \cdot \mathsf{XR}(A, B, D, E)$.*∎

There are 24 possible orders in which four distinct collinear points A, B, C, D may be reordered. We summarize what happens to the cross ration under reordering below.

THEOREM V.12. *Let A, B, C and D be distinct noncollinear points in $\mathbb{FP}^n$, and assume that $\mathsf{XR}(A, B, C, D) = r$. Then the other 23 cross ratios involving these points by reordering are given as follows:*

$$r = \mathsf{XR}(A, B, C, D) = \mathsf{XR}(B, A, D, C) = \mathsf{XR}(C, D, A, B) = \mathsf{XR}(D, C, B, A)$$

$$\frac{1}{r} = \mathsf{XR}(A, B, D, C) = \mathsf{XR}(B, A, C, D) = \mathsf{XR}(D, C, A, B) = \mathsf{XR}(C, D, B, A)$$

$$1 - r = \mathsf{XR}(A, C, B, D) = \mathsf{XR}(C, A, B, D) = \mathsf{XR}(B, D, A, C) = \mathsf{XR}(D, B, C, A)$$

$$\frac{1}{1-r} = \mathsf{XR}(A, C, D, B) = \mathsf{XR}(C, A, B, D) = \mathsf{XR}(D, B, A, C) = \mathsf{XR}(B, D, C, A)$$

$$\frac{1-r}{r} = \mathsf{XR}(A, D, B, C) = \mathsf{XR}(D, A, C, B) = \mathsf{XR}(B, C, A, D) = \mathsf{XR}(C, B, D, A)$$

$$\frac{r}{1-r} = \mathsf{XR}(A, D, C, B) = \mathsf{XR}(D, A, B, C) = \mathsf{XR}(C, B, A, D) = \mathsf{XR}(B, C, D, A)$$

The proof is a sequence of elementary and eventually boring calculations, and it is left as an exercise.■

The next result gives a useful expression for the cross ratio:

THEOREM V.13. *Let A_1, A_2, A_3 be distinct collinear points, and let $A_4 \neq A_1$ lie on this line. Let B_1, B_2, B_3 be distinct collinear points on this line with $B_1 \neq A_i$ for all i, and suppose that $\mathsf{XR}(B_1, B_2, B_3, A_i) = x_i$ for $i = 1, 2, 3, 4$. Then the following holds:*

$$\mathsf{XR}(A_1, A_2, A_3, A_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}$$

Proof.   Choose homogeneous coordinates β_i for Q_i such that $\beta_3 = \beta_1 + \beta_2$. Then homogeneous coordinates α_j for the points A_j are given by $x_j \beta_1 + \beta_2$. It is not difficult to verify that

$$(x_1 - x_2)\alpha_3 = (x_3 - x_2)\alpha_1 + (x_1 - x_3)\alpha_2$$

is true and similarly

$$(x_1 - x_2)\alpha_3 = (x_3 - x_2)\alpha_1 + (x_1 - x_3)\alpha_2 \ .$$

Thus if $\alpha_1' = (x_3 - x_2)\beta_1$ and $\alpha_2' = (x_1 - x_3)\beta_2$, then we have

$$(x_1 - x_2)\alpha_4 = \frac{(x_4 - x_2)}{(x_3 - x_2)}\alpha_1' + \frac{(x_1 - x_4)}{(x_1 - x_3)}\alpha_2' \ .$$

The cross ratio formula in the theorem is an immediate consequence of these formulas.■

DUALIZATION. The preceding discussion can be dualized to yield the *cross ratio of four concurrent lines* in $\mathbb{FP}^2$. Cross rations of collinear points and concurrent lines are interrelated as follows.

THEOREM V.14. *Let L_1, L_2, L_3 be distinct concurrent lines, and let $L_4 \neq L_1$ be another line through this point. Let M be a line in $\mathbb{FP}^2$ which does not contain this common point, and let A_i be the point where L_i meets M, where $i = 1, 2, 3, 4$. Then the point $A_4 \in M$ lies on L_4 if and only if $\mathsf{XR}(A_1, A_2, A_3, A_4) = \mathsf{XR}(L_1, L_2, L_3, L_4)$.*

Proof.   *Suppose that $A_4 \in M \cap L_4$.* Let r be the cross ratio of the lines, and let s be the cross ratio of the points. Choose homogeneous coordinates for the points A_i and lines L_i such that $\alpha_3 = \alpha_1 + \alpha_2$ and $\lambda_3 = \lambda_1 + \lambda_2$, so that $\alpha_4 = s\,\alpha_1 + \alpha_2$ and $\lambda_4 = r\,\lambda_1 + \lambda_2$. Since $A_i \in L_i$ for all i, we have $\lambda_i \cdot \alpha_i = 0$ for all i. In particular, these equations also imply

$$0 = \lambda_3 \cdot \alpha_3 = (\lambda_1 + \lambda_2) \cdot (\alpha_1 + \alpha_2) =$$

$$\lambda_1 \cdot \alpha_1 + \lambda_1 \cdot \alpha_2 + \lambda_2 \cdot \alpha_1 + \lambda_2 \cdot \alpha_2 =$$

$$\lambda_1 \cdot \alpha_2 + \lambda_2 \cdot \alpha_1$$

so that $\lambda_1 \cdot \alpha_2 = -\lambda_2 \cdot \alpha_1$. Therefore we see that

$$0 = \lambda_r \cdot \alpha_r = (r\lambda_1 + \lambda_2) \cdot (s\alpha_1 + \alpha_2) =$$

$$r\lambda_1 \cdot \alpha_2 + s\lambda_2 \cdot \alpha_1 = (r - s)\lambda_1 \cdot \alpha_2 \ .$$

The product $\lambda_1 \cdot \alpha_2$ is nonzero because $A_2 \notin L_1$, and consequently $r - s$ must be equal to zero, so that $r = s$.■

Suppose that the cross ratios are equal. Let $C \in M \cap L_4$; then by the previous discussion we know that

$$\mathsf{XR}(A_1, A_2, A_3, C) \;=\; \mathsf{XR}(L_1, L_2, L_3, L_4) \;.$$

Therefore we also have $\mathsf{XR}(A_1, A_2, A_3, C) \;=\; \mathsf{XR}(A_1, A_2, A_3, A_4)$, so that $A_4 = C$ by Theorem 10 and in addition we have $A_4 \in L_4$.$\blacksquare$

The following consequence of Theorem 14 is the result on *perspective invariance of the cross ratio* mentioned at the beginning of this section.

THEOREM V.15. *Let L_1, L_2, L_3 and L_4 be distinct concurrent lines, and let M and N be two lines which do not contain the common point. Denote the intersection points of M and N with the lines L_i by A_i and B_i respectively. Then* $\mathsf{XR}(A_1, A_2, A_3, A_4) \;=\; \mathsf{XR}(B_1, B_2, B_3, B_4)$.
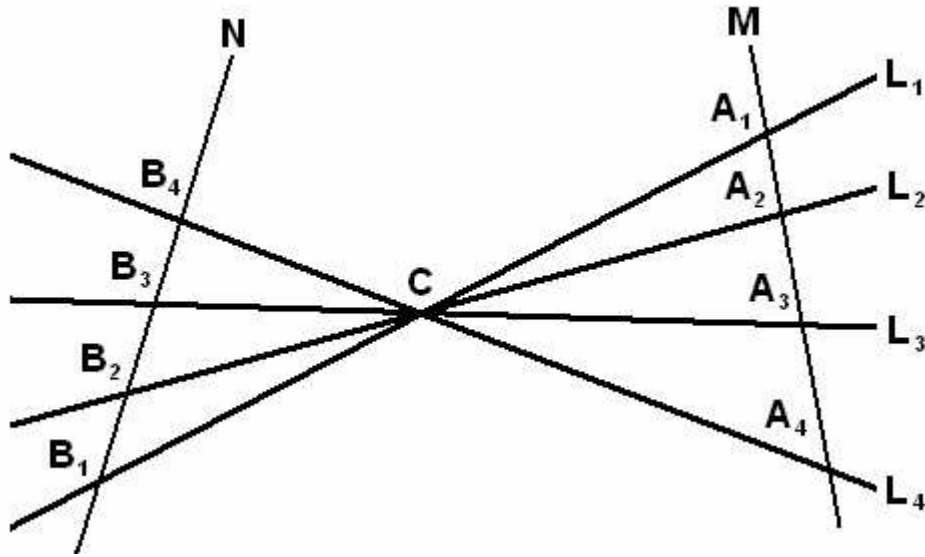
Figure V.1

Proof. Two applications of Theorem 14 imply that

$$\mathsf{XR}(A_1, A_2, A_3, A_4) \;=\; \mathsf{XR}(L_1, L_2, L_3, L_4) \;=\; \mathsf{XR}(B_1, B_2, B_3, B_4) \;.\blacksquare$$

As noted before, Theorem 15 has a visual application to the interpretation of photographs. Namely, in any photograph of a figure containing four collinear points, the cross ratio of the points is equal to the cross ratio of their photographic images (as before, think of the common point as the aperture of the camera, the line N as the film surface, and the points A_i as the points being photographed).

Finally, we explain the origin of the term *cross ratio.* If V is a vector space over $\mathbb{F}$ and $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are distinct collinear points of V, then the *ratio in which* $\mathbf{c}$ *divides* $\mathbf{a}$ *and* $\mathbf{b}$ is given by $(1 - t)/t$, where $\mathbf{c} = t\,\mathbf{a} \;+\; (1 - t)\,\mathbf{b}$. If $\mathbf{d}$ is a fourth point on the line and $\mathsf{J} : \mathbb{F}^n \to \mathbb{F}\mathbb{P}^n$ is the usual inclusion, then Theorem 16 shows that $\mathsf{XR}\big(\mathsf{J}(\mathbf{a}), \mathsf{J}(\mathbf{b}), \mathsf{J}(\mathbf{c}), \mathsf{J}(\mathbf{d})\big)$ is the ratio in which $\mathbf{c}$ divides $\mathbf{a}$ and $\mathbf{b}$ divided by the ratio in which $\mathbf{d}$ divides $\mathbf{a}$ and $\mathbf{b}$.

THEOREM V.16. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, *and* $\mathbf{d}$ *be distinct collinear points of* $\mathbb{F}^n$, *and write* $\mathbf{c} = t\,\mathbf{a} + (1-t)\,\mathbf{b}$ *and* $\mathbf{d} = s\,\mathbf{a} + (1-s)\,\mathbf{b}$. *Then*

$$\mathsf{XR}\big(\,\mathsf{J}(\mathbf{a}), \mathsf{J}(\mathbf{b}), \mathsf{J}(\mathbf{c}), \mathsf{J}(\mathbf{d})\,\big) \;=\; \frac{(1-t)\,s}{(1-s)\,t}\ .$$

Proof.   We shall identify $\mathbb{F}^{n+1,1}$ with $\mathbb{F}^{n+1}$ and $\mathbb{F}^n \times \mathbb{F}$ in the obvious manner. Recall that $\mathbf{x} \in \mathbb{F}^n$ implies that $\xi = (\mathbf{x}, 1)$ is a set of homogeneous coordinates for $\mathbf{x}$. Clearly we have

$$(\mathbf{c}, 1) \;=\; t\,(\mathbf{a}, 1) \;+\; (1-t)\,(\mathbf{b}, 1)$$

so that we also have

$$(\mathbf{d}, 1) \;=\; \frac{s}{t}\,(t\mathbf{a}, t) \;+\; \frac{1-s}{1-t}\,\big(\,(1-t)\mathbf{b}, (1-t)\,\big)$$

and the cross ratio formula is an immediate consequence of this.∎

The following formula is also useful.

THEOREM V.17. *Let* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *and be distinct collinear points of* $\mathbb{F}^n$, *and let* P_∞ *be the point at infinity on the projective extension of this line. If* $\mathbf{c} \;=\; t\mathbf{a} + (1-t)\mathbf{b}$, *then we have*

$$\mathsf{XR}\big(\,\mathsf{J}(\mathbf{a}), \mathsf{J}(\mathbf{b}), \mathsf{J}(\mathbf{c}), P_\infty\,\big) \;=\; \frac{t-1}{t}\ .$$

Proof.   Let homogeneous coordinates for $\mathsf{J}()$, $\mathsf{J}(\mathbf{b})$, and $\mathsf{J}(\mathbf{c})$ be given as usual by $(\mathbf{a}, 1)$, $(\mathbf{b}, 1)$, and $(\mathbf{c}, 1)$ respectively. Then $(\mathbf{c}, 1)$ has homogeneous coordinates

$$(\mathbf{c}, 1) \;=\; (t\mathbf{a}, t) \;+\; \big(\,(1-t)\mathbf{b}, (1-t)\,\big)$$

and P_∞ has homogeneous coordinates

$$(\mathbf{b} - \mathbf{a}, 0) \;=\; (\mathbf{b}, 1) \;-\; (\mathbf{a}, 1) \;=\; \frac{1}{1-t}\cdot\big(\,(1-t)\mathbf{b}, (1-t)\,\big) \;-\; \frac{1}{t}(t\mathbf{a}, t)\ .$$

Therefore the cross ratio is equal to

$$\frac{-1/t}{1/(1-t)} \;=\; \frac{t-1}{t}$$

which is the formula stated in the theorem.∎

<div align="center">EXERCISES</div>

1.   Prove Theorem 11.

2.   Prove Theorem 12.

3.   In the notation of Theorem 13, assume that all the hypotheses except $A_1 \neq B_1$ are valid. Prove that the cross ratio is given by $(x_2 - x_4)/(x_2 - x_3)$ if $A_1 = B_1$.

4. Find the cross ratio of the four collinear points in $\mathbb{RP}^3$ whose homogeneous coordinates are given as follows:

$$\begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 7 \\ 4 \\ 4 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ -2 \\ -2 \end{pmatrix}$$

5. In $\mathbb{RP}^2$, find the cross ratio of the lines joining the point $J(\frac{1}{4}, \frac{1}{2})$ to the points with the following homogeneous coordinates:

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

6. In $\mathbb{RP}^2$, find the cross ratio formed by the points with homogeneous coordinates $^{\mathbf{T}}(1 \quad 1 \quad 2)$ and $^{\mathbf{T}}(2 \quad 3 \quad 4)$ and the points in which their line meets the lines defined by $x_1 + x_2 + x_3 = 0$ and $2x_1 + x_3 = 0$.

7. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ be distinct collinear points of $\mathbb{R}^n$. Prove that their cross ratio is given by the following formula in which "·" denotes the usual vector dot (or inner) product:

$$\mathsf{XR}\big(\mathsf{J}(\mathbf{a}), \mathsf{J}(\mathbf{b}), \mathsf{J}(\mathbf{c}), \mathsf{J}(\mathbf{d}) \big) \;=\; \frac{[\,(\mathbf{b} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})\,]\,[\,(\mathbf{d} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})\,]}{[\,(\mathbf{c} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})\,]\,[\,(\mathbf{d} - \mathbf{d}) \cdot (\mathbf{b} - \mathbf{a})\,]}$$

Using this formula, show that the absolute value of the cross ratio is given by the following expression:

$$\frac{|\mathbf{c} - \mathbf{b}|\,|\mathbf{d} - \mathbf{a}|}{|\mathbf{c} - \mathbf{a}|\,|\mathbf{d} - \mathbf{b}|}$$

8. If A, B, C and D are four distinct collinear points of $\mathbb{FP}^n$ (where $\mathbb{F}$ is a commutative field) and $(A', B'\,C'\,D')$ is a rearrangement of (A, B, C, D), then by Theorem 12 there are at most six possible values for $\mathsf{XR}(A', B', C', D')$ as (A', B', C', D') runs through all rearrangements. Usually there are exactly six different values for all the possible rearragements, and this exercise analyzes the exceptional cases when there are fewer than six possibilities. By interchanging the roles of (A, B, C, D) if necessary, we can assume that $\mathsf{XR}(A, B, C, D) = r$ is equal to one of the other five expressions in Theorem 12.

(i) Suppose that $1 + 1 \neq 0$ in $\mathbb{F}$ and $\mathsf{XR}(A, B, C, D) = r$ is equal to one of the expressions $1/r$, $1 - r$ or $r/(r - 1)$. Prove that r belongs to the set $\{-1, 2, \frac{1}{2}\} \subset \mathbb{F}$ and that the values of the cross ratios $\mathsf{XR}(A', B', C', D')$ for the various rearrangements are precisely the elements of $\{-1, 2, \frac{1}{2}\}$. — Explain why there are three elements in this set if $1 + 1 + 1 \neq 0$ in $\mathbb{F}$ but only one if $1 + 1 + 1 = 0$ in $\mathbb{F}$.

(ii) Suppose that $\mathbb{F}$ is the complex numbers $\mathbb{C}$ and that r is equal to either $1/(1 - r)$ or $(r - 1)/r$. Prove that r belongs to the set $\{ \frac{1}{2}1 + \mathbf{i}\sqrt{3}, \frac{1}{2}(1 - \mathbf{i}\sqrt{3}, \} \subset \mathbb{C}$ and that the values of the cross ratios $\mathsf{XR}(A', B', C', D')$ for the various rearrangements are precisely the elements of $\{ \frac{1}{2}1 + \mathbf{i}\sqrt{3}, \frac{1}{2}(1 - \mathbf{i}\sqrt{3}, \}$. — Explain why $r^6 = 1$ but $r^m \neq 1$ for $1 \leq m \leq 5$. [*Hint:* Show that $r^2 - r + 1 = 0$ implies $r^3 = -1$ by multiplying both sides by $(r + 1)$, and then use this to explain why $r^6 = 1$. In particular, it follows that the possibilities in this case arise if and only if there is some element $r \in \mathbb{F}$ such that $r^6 = 1$ but no smaller positive integral power of r is equal to 1. Such elements exist in $\mathbb{Z}_p$ if p is a prime of the form $6k + 1$ — for example, if $p = 7, 13, 19, 31, 37, 43, 61, 67, 73, 79$, or 97.]

Definitions.   In Case (i) of the preceding exercise, there is a rearrangement (A', B' C' D') of (A, B, C, D) such that $\mathsf{XR}(A', B', C', D') = -1$. Ordered quadruples of collinear points satisfying this condition are called *harmonic quadruples*, and they are discussed further in Section V.4, Exercise VI.3.8 and Exercise VII.2.3. In Case (ii), the quadruple is said to form an *equianharmonic set* (the next to last word should be decomposed as equi/an/harmonic). The latter are related to topics in the theory of functions of a complex variable which go far beyond the scope of these notes, and we shall not attempt to give any contexts in which such sets arise.

9.   Let $\mathbb{F}$ be a field in which $1+1 \neq 0$ and $1+1+1 \neq 0$, and let A, B, C and D be four distinct collinear points in $\mathbb{F}\mathbb{P}^n$.

(i) Suppose that $\mathsf{XR}(A, B, C, D) = -1$. Determine all the rearrangements (A', B', C', D') of (A, B, C, D), for which $\mathsf{XR}(A', B', C', D')$ is equal to -1, 2, and $\frac{1}{2}$ respectively.

(ii) Suppose that $\mathsf{XR}(A, B, C, D) = r$, where r satisfies the quadratic equation $x^2 - x + 1 = 0$. Explain why $1 - r$ is a second root of the equation such that $r \neq 1 - r$, and using Theorem 12 determine all the rearrangements (A', B', C', D') of (A, B, C, D), for which $\mathsf{XR}(A', B', C', D')$ is equal to r and $(1 - r)$ respectively.

10.   Let $n \geq 1$, let A be an invertible $(n + 1) \times (n + 1)$ matrix over $\mathbb{F}$, and let T_A be the geometric symmetry of $\mathbb{F}\mathbb{P}^n \cong \mathcal{S}_1(\mathbb{F}^{n+1,1})$ defined by the equation

$$T_A(\mathbf{x}) \; = \; A \cdot \xi \cdot \mathbb{F}$$

where ξ is a set of homogeneous coordinates for $\mathbf{x}$ and the dot indicates matrix multiplication. Suppose that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ are collinear points such that the first three are distinct and $\mathbf{x}_4 \neq \mathbf{x}_1$. Prove that T_A preserves cross ratios; more formally, prove that

$$(\mathbf{x}_1,, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \; = \; \bigl(T_A(\mathbf{x}_1), T_A(\mathbf{x}_2), T_A(\mathbf{x}_3), T_A(\mathbf{x}_4) \bigr) \; .$$

11.   Let $\mathbb{F}$ be a field, and let $g : \mathbb{F}\mathbb{P}^1 \to \mathbb{F}\mathbb{P}^1$ be the 1–1 correspondence such that $g \circ \mathrm{J}(x) \; = \; x^{-1}$ if $0 \neq x \in \mathbb{F}$, and G interchanges the zero point $\mathrm{J}(0)$ and the point at infinity $\infty(\mathbb{F}\mathbb{P}^1)$ with homogeneous coordinates given by the transpose of $^{\mathbf{T}}(1 \quad 0)$. Prove that there is an invertible 2×2 matrix A such that $g \; = \; T_A$, where the right hand side is defined as in the preceding exercise. [*Hint:* The result extends to *linear fractional transformations* defined by

$$g \circ \mathrm{J}(x) \; = \; \frac{ax + b}{cx + d} \qquad (\text{where} \quad c \quad \text{and} \quad ad - bc \neq 0)$$

if $x \; \neq \; -d/c$, while g interchanges $-d/c$ and $\infty(\mathbb{F}\mathbb{P}^1)$.]

3. Theorems of Desargues and Pappus

We begin with a new proof of Theorem IV.5 (Desargues' Theorem) for coordinate planes within the framework of coordinate geometry. Among other things, it illustrates some of the ideas that will appear throughout the rest of this chapter. For most of this section, $\mathbb{F}$ will denote an arbitrary skew-fields, and as in the beginning of Section V.1 we shall distinguish between left and right vector spaces.

THEOREM V.18. *A coordinate projective plane $\mathbb{FP}^2$ is Desarguian.*

Proof. Let $\{A,\ B,\ C\}$ and $\{A',\ B',\ C'\}$ be two triples of noncollinear points, and let X be a point which lies on all three of the lines AA', BB' and CC'. Choose homogeneous coordinates $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ for A, B, C, A', B', C' and ξ for X such that

$$\alpha' \ = \ \xi \ + \ \alpha \cdot a, \qquad \beta' \ = \ \xi \ + \ \beta \cdot b, \qquad \gamma' \ = \ \xi \ + \ \gamma \cdot c.$$

Since $\beta' - \gamma' \ = \ \beta \cdot b \ - \ \gamma \cdot c$, it follows that the point $D \in BC \cap B'C'$ has homogeneous coordinates $\beta' - \gamma'$. Similarly, the points $E \in AC \cap A'C'$ and $F \in AB \cap A'B'$ have homogeneous coordinates $\alpha' - \gamma'$ and $\alpha' - \beta'$ respectively. The sum of these three homogeneous coordinates is equal to zero, and therefore it follows that the points they represent — which are D, E and F — must be collinear.■

Another fundamental result of projective geometry was first stated and proved by Pappus of Alexandria in a Euclidean context.

Pappus' Theorem.[4] *Let $\{A_1,\ A_2,\ A_3\}$ and $\{B_1,\ B_2,\ B_3\}$ be two coplanar triples of noncollinear points in the real projective plane or 3-space. Assume the two lines and six points are distinct. Then the cross intersection points*

$$
\begin{aligned}
X \ &\in \ A_2B_3 \ \cap \ A_3B_2 \\
Y \ &\in \ A_1B_3 \ \cap \ A_3B_1 \\
Z \ &\in \ A_1B_2 \ \cap \ A_2B_1
\end{aligned}
$$

are collinear.

---

[4]This is sometimes called the Pappus Hexagon Theorem to distinguish it from the theorems on centroids of surfaces and solids of revolution that were proven by Pappus and rediscovered independently by PAUL GULDIN (1577–1643). A more detailed discussion of Guldin's life and work is available at the following online site: `http://www.faculty.fairfield.edu/jmac/sj/scientists/guldin.htm` . For reasons discussed in the final section of Chapter VII, the Theorem of Pappus stated in this section is sometimes called *Pascal's Theorem*, and this is especially true for mathematical articles and books written in French or German.
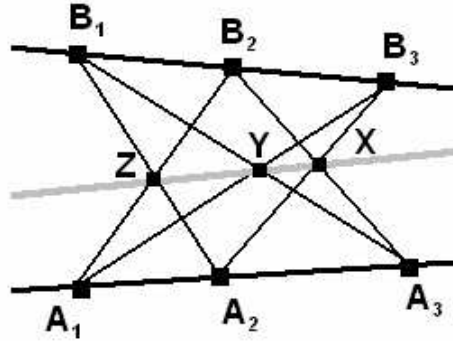
Figure V.2

THEOREM V.19. *Let $\mathbb{F}$ be a skew-field. Then Pappus' Theorem is valid in $\mathbb{F}\mathbb{P}^n$ (where $n \geq 2$) if and only if $\mathbb{F}$ is commutative.*

Proof.   At most one of the six points $\{A_1,\ A_2,\ A_3,\ B_1,\ B_2,\ B_3\}$ lies on both lines. In particular, there exist distinct numbers $j, k \in \{1, 2, 3\}$ such that $A_j,\ A_k,\ B_j,\ B_k$ do not lie on both lines. If we re-index the points using a suitable reordering of $\{1, 2, 3\}$ (explicitly, send j to 1, k to 2, and the remaining number to 3), we find that the renamed points $A_1,\ A_2,\ B_1,\ B_2$ do not lie on both lines. We shall use this revised indexing henceforth. Since the six points are coplanar, by Theorem 6 we may choose homogeneous coordinates α_i and β_j for the points so that $\beta_2\ =\ \alpha_1 + \alpha_2 + \beta_1$. Furthermore, by Theorem 7 we may write $\alpha_3\ =\ \alpha_1 + \alpha_2 \cdot a$ and $\beta_3\ =\ \beta_1 + \beta_2 \cdot b$. Since $Z \in A_1 B_2 \cap A_2 B_1$, we know there are scalars x, y, u, v such that homogeneous coordinates ζ for Z are given by

$$\zeta\ =\ \alpha_1 \cdot x\ +\ \beta_2 \cdot y\ =\ \beta_1 \cdot u\ +\ \alpha_2 \cdot v\ =\ \alpha_1\,(x + y)\ +\ \alpha_2 \cdot y\ +\ \beta_1 \cdot y\ .$$

Equating coefficients, we see that $x + y = 0$ and $y = u = v$. Hence homogeneous coordinates for Z are given by $\beta_1 + \alpha_2$. Similarly, homogeneous coordinates η for Y are given as follows:

$$\eta\ =\ \alpha_1 \cdot x\ +\ \beta_3 \cdot y\ =\ \alpha_1\,(x + by)\ +\ \beta_1\,(y + by)\ +\ \alpha_2\,(by)\ =$$
$$\alpha_3 \cdot u\ +\ \beta_1 \cdot v\ =\ \alpha_1 \cdot u\ +\ \alpha_2 \cdot au\ +\ \beta_1 \cdot v\ .$$

Equating coefficients as before, we find that homogeneous coordinates for Y are given by

$$\eta\ =\ \alpha_1\ +\ \alpha_2 \cdot a\ +\ \beta_1\,(1 + b^{-1}a)\ .$$

Still another calculation along the same lines shows that homogeneous coordinates ξ for X are given by

$$\xi\ =\ \alpha_1\ +\ \alpha_2\,(1 + b^{-1} - ab^{-1})\ +\ \beta_1\,(1 + b^{-1})\ .$$

Assume now that $\mathbb{F}$ is commutative. Then $ab^{-1}\ =\ b^{-1}a$, and hence

$$\eta\ -\ \xi\ =\ \alpha_2\,(a - 1 - b^{-1} + ab^{-1})\ +\ \beta_1\,(a + b^{-1}a - 1 - b^{-1})$$

is a scalar multiple of $\alpha_2 + \beta_1$. Since the latter vector is a set of homogeneous coordinates for Z, the conclusion $Z \in XY$ is immediate.$\square$

Conversely, assume that the Pappus' Theorem is always valid in $\mathbb{F}\mathbb{P}^n$. It suffices to prove that $ab = ba$ for all $a, b \in \mathbb{F}$ which are not equal to 0 or 1. Let A_1, A_2, B_1, B_2 be four coplanar points, no three of which are collinear, and choose homogeneous coordinates such that $\beta_2 = \alpha_1 + \alpha_2 + \beta_1$. Let $A_3 \in A_1 A_2$ and $B_3 \in B_1 B_2$ be chosen so that we have homogeneous coordinates of the form $\alpha_3 = \alpha_1 + \alpha_2 \cdot a$ and $\beta_3 = \beta_1 + \beta_2 \cdot b^{-1}$.

Since $Z \in XY$, there exist x, $y \in \mathbb{F}$ such that

$$\alpha_2 + \beta_1 = \eta \cdot x + \xi \cdot y \ .$$

By the calculations in the preceding half of the proof, the right hand side is equal to

$$\alpha_1 (x + y) + \alpha_2 \cdot z + \beta_1 \cdot w$$

where z and w are readily computable elements of $\mathbb{F}$. If we equate coefficients we find that $x + y = 0$ and hence $\alpha_2 + \beta_1 = (\eta - \xi) x$. On the other hand, previous calculations show that

$$(\eta - \xi) x = \alpha_2 (a - 1 - b - ab) x + \beta_1 (a + ba - 1 - b) x \ .$$

By construction, the coefficients of α_2 and β_1 in the above construction are equal to 1. Therefore x is nonzero and

$$a - 1 - b - ab = a + ba - 1 - b$$

from which $ab = ba$ follows. ∎

Since there exist skew-fields that are not fields (*e.g.*, the quaternions given in Example 3 at the beginning of Appendix A), Theorem 19 yields the following consequence:

> For each $n \geq 2$ there exist Desarguian projective *n*-spaces in which Pappus' Theorem is not valid.

Appendix C contains additional information on such noncommutative skew-fields and their implications for projective geometry. The logical relationship between the statements of Desargues' Theorem and Pappus' Theorem for projective spaces is discussed later in this section will be discussed following the proof of the next result, which shows that Pappus' Theorem is effectively invariant under duality.

THEOREM V.20. *If Pappus' Theorem is true in a projective plane, then the dual of Pappus' Theorem is also true in that plane.*

Proof. Suppose we are given two triples of concurrent lines $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2, \beta_3\}$ with distinct points of concurrency that we shall call A and B respectively. Let $C_{i,j}$ denote the common point of α_i and β_j. To prove the planar dual of Pappus' Theorem, we must show that the lines

$$C_{1,3} C_{3,1}, \qquad C_{2,3} C_{3,2}, \qquad C_{1,2} C_{2,1}$$

are concurrent (see the figure below):
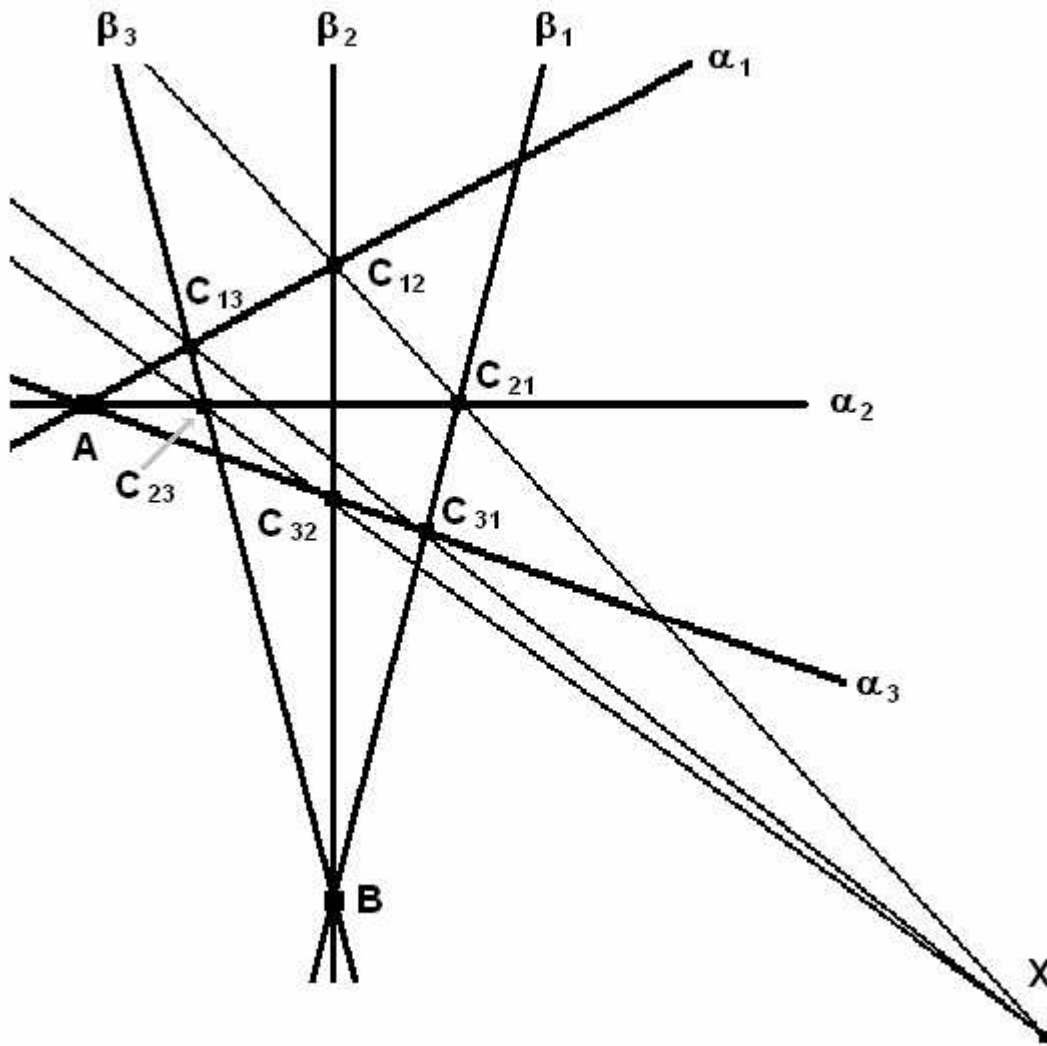
Figure V.3

Now $\{A,\ C_{3,1},\ C_{3,2}\}$ and $\{B,\ C_{2,3},\ C_{1,3}\}$ are two triples of collinear points not on the same line, and hence the three points

$$
\begin{aligned}
X &\in C_{3,1}C_{2,3}\ \cap\ C_{3,2}C_{2,3} \\
C_{2,1} &\in AC_{2,3}\ \cap\ BC_{3,1}\ =\ \alpha_2\ \cap\ \beta_1 \\
C_{1,2} &\in AC_{1,3}\ \cap\ BC_{3,2}\ =\ \alpha_1\ \cap\ \beta_2
\end{aligned}
$$

are collinear by Pappus' Theorem. Hence $X \in C_{1,2}C_{2,1}$. By the definition of X, it now follows that X lies on all three of the lines $C_{1,3}C_{3,1}$, $C_{2,3}C_{3,2}$, and $C_{1,2}C_{2,1}$.∎

A relationship between the validities of Desargues' Theorem and Pappus' Theorem was first discovered by G. Hessenberg (1874–1925). However, his proof was incomplete because it did not consider some exceptional cases, and the first totally correct argument was published by A. Cronheim (1922–2005); the paper containing Cronheim's proof is listed in the bibliography.

THEOREM V.21. *Let $(P, \mathcal{L})$ be a projective plane in which Pappus' Theorem is valid. Then $(P, \mathcal{L})$ is Desarguian.*

The original idea of the proof of Theorem 21 is fairly elementary, but the complete argument is a tedious exercise in the use of such elementary techniques. Details appear on pages 64–66 of the book by Bumcrot listed in the bibliography.

We have already noted that Pappus' Theorem is not necessarily valid in a Desarguian projective plane. However, the following result is true:

THEOREM V.22. *Let P be a* FINITE *Desarguian projective plane. Then Pappus' Theorem is valid in P.*

The main step in the proof of Theorem 22 is an algebraic result of J. H. M. Wedderburn.[5] Proofs appear on pages 375–376 of M. Hall's book on group theory and in the final chapter of the book, *Topics in Algebra*, by I. N. Herstein (more detailed information appears in the bibliography).

Theorem of Wedderburn. *Every finite skew-field is commutative.*∎

Proof of Theorem 22. (*assuming Wedderburn's Theorem*) By Theorem IV.17 we know that P is isomorphic to $\mathbb{F}\mathbb{P}^2$ for some skew-field $\mathbb{F}$. Since $\mathbb{F}\mathbb{P}^2$ is finite, so is $\mathbb{F}$, and since $\mathbb{F}$ is commutative by Wedderburn's Theorem, we know that Pappus' Theorem is valid by Theorem 19.∎

STRENGTHENED RESULT. The theory of finite fields[6] implies that the number of elements in a finite field is p^r, where p is prime and r is a positive integer and also that all finite fields with p^r elements are isomorphic. Combining this with the preceding theorem and a count of the number of elements in $\mathbb{F}\mathbb{P}^n$ if $\mathbb{F}$ has $q = p^r$, one can prove the following result:

Complement to Theorem 22. *The number of elements in a finite Desarguian projective n-space is equal to $1 + q + \cdots + q^n$ where $q = p^r$ for some r, and two finite Desarguian n-spaces with the same numbers of elements are isomorphic.*∎

A purely algebraic proof of this result is described in the exercises.

Note on the proofs of Theorem 22 *and its complement.* Since the statements of the result and its complement only involve synthetic and geometric concepts, it is natural to ask if there is a more direct proof that does not require such substantial algebraic input.[7] However, no synthetic proof of Theorem 22 is known.

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[5] JOSEPH H. M. WEDDERBURN (1882–1948) is particularly known for some fundamental results on the structure of certain important types of abstract algebraic systems. As noted in the article by K. H. Parshall in the bibliography, there is a case for attributing the cited result jointly to Wedderburn and LEONARD EUGENE DICKSON (1874–1954), who was one of the first American researchers in abstract algebra and number theory.

[6] See the books by Herstein or Hungerford.

[7] This reflects a special case of a logical principle known as *Ockham's razor*; one concsequence of the latter is that one should not introduce extraneous concepts unless it is absolutely necessary to do so. WILLIAM OF OCKHAM (*c.* 1285–1349) was an English scholar who used the principle named after him frequently and effectively; historical and other information about this maxim is given in the following online site: `http://en.wikipedia.org/wikki/Occam's_Razor`

EXERCISES

1. Fill in the omitted details of the calculations for ξ and η in Theorem 19.

2. Given three collinear points $\{A,\ B,\ C\}$ and three other collinear points on a different lines, how many different 1–1 ordered correspondences involving the two sets of points are possible?

3. In Exercise 2, assume all points belong to $\mathbb{F}\mathbb{P}^2$, where $\mathbb{F}$ is a field. Each correspondence in Exercise 2 determines a line given by Pappus' Theorem. Draw a figure illustrating the lines arising from the two given unordered triples of collinear points. Formulate and prove a statement about these lines.

4. Prove the Complement to Theorem 22. [*Hint:* If $\mathbb{F}$ has q elements, show that the number of nonzero elements of $\mathbb{F}^{n+1}$ is equal to the product of the number of 1-dimensional subspaces times the number of nonzero elements in $\mathbb{F}$. Both of the latter are easy to compute. Solve the resulting equation to obtain the number of elements in $\mathbb{F}\mathbb{P}^n$. If two finite fields have different numbers of elements, use the formula to show that their projective n-spaces also have different number of elements because $1 + q + \ \cdots\ + q^n$ is a strictly increasing function of q.]

5. Let $\mathbb{F}$ be a field, and suppose we are given the basic setting of Pappus' Theorem in $\mathbb{F}^2$: Specifically, $\{A_1,\ A_2,\ A_3\}$ and $\{B_1,\ B_2,\ B_3\}$ are triples of noncollinear points in $\mathbb{F}^2$, and the two lines and six points are distinct.

(i) Suppose that there are points X and Y such that $X\ \in\ A_2B_3\ \cap\ A_3B_2$ and $Y\ \in\ A_1B_3\ \cap\ A_3B_1$, but the lines A_1B_2 and A_2B_1 are parallel. What conclusion does Pappus' Theorem imply about these two lines and XY?

(ii) Suppose now that we are given $A_1B_3\ ||\ A_3B_1$ and $ZA_1B_2\ ||\ A_2B_1$. What conclusion does Pappus' Theorem imply about A_2B_3 and A_3B_2?

6. Let P be a projective plane with exactly 7 points. Prove that P is isomorphic to $\mathbb{Z}_2\mathbb{P}^2$ and hence is Desarguian. [*Hint:* Start by taking a 1–1 correspondence between four points in $\mathbb{Z}_2\mathbb{P}^2$, with no three collinear, to four points in P of the same type. Show that there is at most one isomorphism inducing this 1–1 correspondence, use this to define a 1–1 correspondence between the two projective planes, and verify that the constructed map is an ismorphism.]

> NOTE. By Theorem IV.15 and the Bruck-Ryser-Chowla Theorem (mentioned on the page following IV.15), if a projective plane has more than 7 points then the next few possibilities for the number of points in P are 13, 21, 31, 57, 73 and 91. A non-Desarguian projective plane with 91 points is constructed in Problems 26–29 of Hartshorne's book.[8]

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[8]In fact, every projective plane with fewer than 91 points is Desarguian. A reference for this and a chart listing all known isomorphism types of projective planes with fewer than 1000 points appears in the following online site: `http://www.uwyo.edu/moorhouse/pub/planes`

4. Complete quadrilaterals and harmonic sets

In the exercises for the preceding section, we defined the concept of a harmonic quadruple of collinear points. Since the concept plays an important role in this section, we shall repeat the definition and mention a few alternative phrases that are frequently used.

Definition. Let A, B, C, D be a set of four collinear points in $\mathbb{FP}^n$, where $\mathbb{F}$ is a skew-field such that $1 + 1 \neq 0$ in $\mathbb{F}$. We shall say that the ordered quadruple (A, B, C, D) is a *harmonic quadruple* if $\mathsf{XR}(A, B, C, D) = -1$. Frequently we shall also say that *the points A and B separate the points C and D harmonically*,[9] or that *the ordered quadruple (A, B, C, D) forms a harmonic set* (sometimes, with an abuse of language, one also says that the four points form a harmonic set, but Theorem 12 shows that one must be careful about the ordering of the points whenever this wording is used).

Here is the definition of the other basic concept in this section.

Definition. Let A, B, C, D be a set of four coplanar points in a projective incidence space such that no three are collinear. The *complete quadrilateral* determined by these four points, written $\boxtimes ABCD$, is the union of the six lines joining these four points:

$$\boxtimes ABCD = AB \cup BC \cup CD \cup DA \cup AC \cup BD$$

Each line is called a *side*, and the points

$$
\begin{aligned}
X &\in AD \cap BC \\
Y &\in AB \cap CD \\
Z &\in AC \cap BD
\end{aligned}
$$

are called the *diagonal points* of the complete quadrilateral.
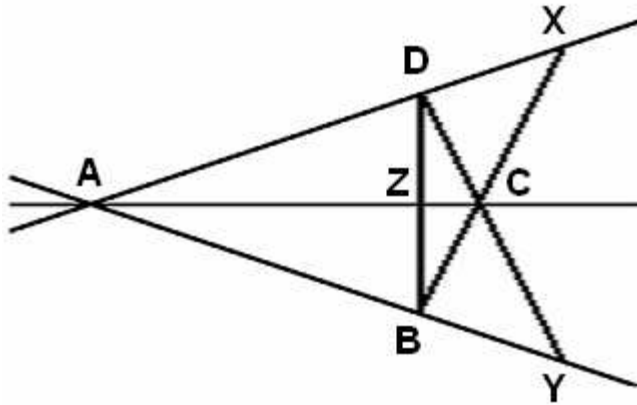
Figure V.4

Note that if the original four points form the vertices of an affine parallelogram in $\mathbb{F}^n$ where $n \geq 2$ and $1 + 1 \neq 0$ in $\mathbb{F}$, then by Theorem II.26 the first two diagonal points are ideal points but the third is not. On the other hand, if we have a field $\mathbb{F}$ such that $1 + 1 = 0$ in $\mathbb{F}$, then

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[9]This does not depend upon the order of $\{A, B\}$ or $\{C, D\}$ because $\mathsf{XR}(A, B, C, D) = -1$ and Theorem 12 imply $\mathsf{XR}(B, A, C, D)$, $\mathsf{XR}(A, B, D, C)$ and $\mathsf{XR}(B, A, D, C)$ are all equal to -1. In fact, the definition is symmetric in the two pairs of points because Theorem 12 implies $\mathsf{XR}(A, B, C, D) = \mathsf{XR}(C, D, A, B)$.

by Exercise II.4.8 then all three diagonal points are ideal points. The following result, whose significance was observed by G. Fano,[10] is a generalization of these simple observations about parallelograms.

THEOREM V.23. *Let $n \geq 2$, let $\mathbb{F}$ be a skew-field and let A, B, C, D be a set of four coplanar points in $\mathbb{FP}^n$ such that no three are collinear.*

(*i*) *If $1 + 1 \neq 0$ in $\mathbb{F}$, then the diagonal points of the complete quadrilateral $\boxtimes ABCD$ are* noncollinear.

(*ii*) *If $1 + 1 = 0$ in $\mathbb{F}$, then the diagonal points of the complete quadrilateral $\boxtimes ABCD$ are* collinear.

Proof. As usual start by choosing homogeneous coordinates α, β, γ, δ for A, B, C, D such that $\delta = \alpha + \beta + \gamma$. There exist scalars x, y, u, v such that homogeneous coordinates for X are given by

$$\beta \cdot x + \gamma \cdot y = \alpha \cdot u + \delta \cdot v = \alpha (u + v) + \beta \cdot v + \gamma \cdot v .$$

Thus we must have $x = y = v$ and $u + v = 0$, so that $\xi = \beta + \gamma$ is a set of homogeneous coordinates for X. Similarly, homogeneous coordinates for Y and Z are given by

$$\eta_0 = \alpha \cdot x + \beta \cdot y = \gamma \cdot u + \delta \cdot v = \alpha \cdot v + \beta \cdot v + \gamma (u + v)$$

and

$$\zeta_0 = \alpha \cdot x' + \gamma \cdot y' = \beta \cdot u' + \delta \cdot v' = \alpha \cdot v' + \beta (u' + v') + \gamma \cdot v'$$

respectively, where u, v, u', v' are appropriate scalars. In these equations we have $x = y = v$ and $x' = y' = v'$, so that Y and Z have homogeneous coordinates $\eta = \alpha + \beta$ and $\zeta = \alpha + \gamma$ respectively. The vectors ξ, η and ζ are linearly independent if $1 + 1 \neq 0$ in $\mathbb{F}$ and linearly dependent if $1 + 1 = 0$ in $\mathbb{F}$, and therefore the points X, Y and Z are noncollinear if are linearly independent if $1 + 1 \neq 0$ in $\mathbb{F}$ and collinear if $1 + 1 = 0$ in $\mathbb{F}$.∎

HYPOTHESIS. Throughout the rest of this section we assume $\mathbb{F}$ is a commutative field; in most but not all cases, we shall also assume that $1 + 1 \neq 0$ in $\mathbb{F}$, but for each result or discussion we shall state explicitly if we making such an assumption.

AFFINE INTERPRETATIONS OF HARMONIC SETS. Given that we have devoted so much attention to affine geometry in these notes, it is natural to ask just what harmonic quadruples look like in affine (and, of course, Euclidean) geometry. Here is one basic result which shows that harmonic quadruples often correspond to familiar concepts in "ordinary" geometry. Additional examples are given in the exercises.

THEOREM V.24. *Let $\mathbb{F}$ be a field in which $1 + 1 \neq 0$, let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three distinct points in $\mathbb{F}^n$, where $n \geq 1$, and let P_∞ be the ideal point on the projective line $J(\mathbf{a})J(\mathbf{b})$. Then $\big(J(\mathbf{a}), J(\mathbf{b}), J(\mathbf{c}), J(P_\infty)\big)$ if and only if $\mathbf{c}$ is the midpoint of $\mathbf{a}$ and $\mathbf{b}$.*

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[10]GINO FANO (1871–1952) is recognized for his contributions to the foundations of geometry and to algebraic geometry; an important class of objects in the latter subject is named after him, the projective plane over $\mathbb{Z}_2$ is frequently called the *Fano plane*, and the noncollinearity conclusion in Theorem 23 below is often called *Fano's axiom*.

Proof.   According to Theorem 17, if $\mathbf{c} = t\mathbf{a} + (1-t)\mathbf{b}$, then

$$\big(\mathrm{J}(\mathbf{a}),\, \mathrm{J}(\mathbf{b}),\, \mathrm{J}(\mathbf{c}),\, \mathrm{J}(P_\infty) \big) = -\frac{1-t}{t}\ .$$

It is a straightforward algebraic exercise to verify that the right hand side is equal to -1 if and only if $t = \frac{1}{2}$.■

In view of the preceding theorem, the next result may be interpreted as a generalization of the familiar theorem, "The diagonals of a parallelogram bisect each other" (see Theorem II.26).

THEOREM V.25. *Let $\boxtimes ABCD$ be a complete quadrilateral in $\mathbb{FP}^n$, and let X, Y ,Z be its diagonal points as in the definition. Let $W \in XY \cap BD$ and let $V \in AC \cap XY$. Then we have* $\mathrm{XR}(B, D, W, Z) = \mathrm{XR}(X, Y, W, V) = -1.$
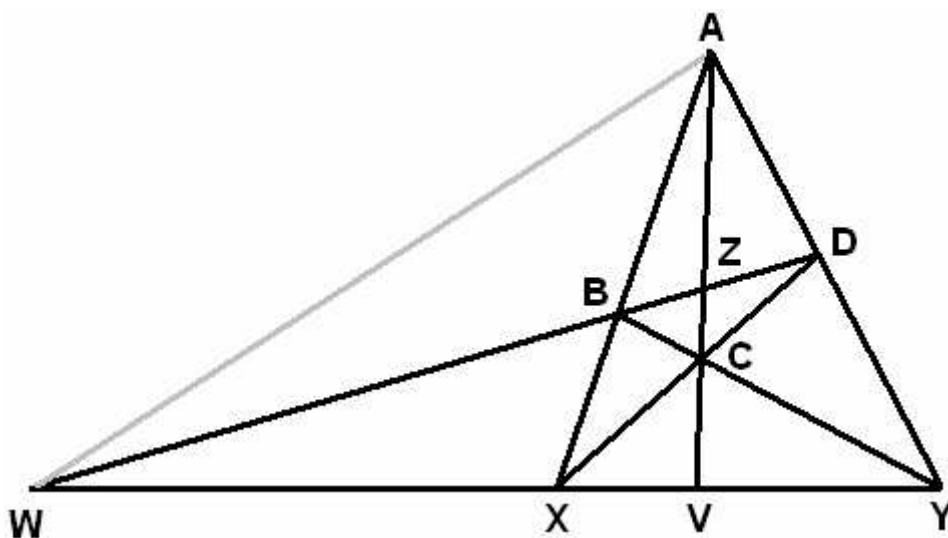
Figure V.5

Proof.   Since $X \in AB$, $Y \in AD$, $W \in AW$ and $V \in AZ$, clearly the two cross ratios agree. Choose homogeneous coordinates so that $\delta = \alpha + \beta + \gamma$ (as usual α, β, γ, δ are homogeneous coordinates for A, B, C, D respectively). We have already seen that homogeneous coordinates ξ, η, ζ for X, Y, Z are given by $\xi = \alpha + \beta$, $\eta = \beta + \gamma$, and $\zeta = \alpha + \gamma$. To find homogeneous coordinates ω for W, note that $\xi + \eta = \alpha + 2\beta + \gamma = \beta + \delta$. Thus $\omega = \alpha + 2\beta + \gamma - \xi + \eta = \beta + \delta$. However, the formulas above imply that

$$\zeta = \alpha + \gamma = -\beta + \delta$$

and therefore the desired cross ratio formula $\mathrm{XR}(B, D, W, Z) = -1$ follows.■

REMARK. One can use the conclusion of the preceding result to give a purely synthetic definition of harmonic quadruples for arbitrary projective planes. Details appear in many of the references in the bibliography.

Definition.   Let L be a line in a projective plane, and let X_i (where $1 \le i \le 6$) be six different points on L. The points X_i are said to form a *quadrangular set* if there is a complete quadrilateral in the plane whose six sides intersect L in the points X_i.

The next result was first shown by Desargues.

THEOREM V.26. *Let $\mathbb{F}$ be a field. Then any five points in a quadrangular set uniquely determine the sixth.*

We should note that the theorem is also true if $\mathbb{F}$ is a skew-field; the commutativity assumption allows us to simplify the algebra in the proof.

Proof.   Let L be the given line, and let $\boxtimes ABCD$ be a complete quadrilateral. Define X_1, X_2, X_3, X_4, X_5, X_6 to be the intersections of L with AB, AC, BC, BD, AD, CD respectively. It suffices to prove that X_6 is uniquely determined by the points X_i for $i \leq 5$. The other five cases follow by interchanging the roles of A, B, C and D.

Choose homogeneous coordinates as usual so that $\delta = \alpha + \beta + \gamma$, and let λ be a set of homogeneous coordinates for L. Define $a = \lambda \cdot \alpha$, $b = \lambda \cdot \beta$, $c = \lambda \cdot \gamma$, and $d = \lambda \cdot \delta$. By construction, we have $d = a + b + c$. Using Theorem 5, we obtain homogeneous coordinates for the points X_i as follows:

$$\xi_1 = \lambda \cdot (\alpha \times \beta) = (\lambda \cdot \beta)\alpha - (\lambda \cdot \alpha)\beta = b\alpha - a\beta$$
$$\xi_2 = c\alpha - a\gamma$$
$$\xi_3 = c\beta - b\gamma$$
$$\xi_4 = (a + c)\beta - b(\alpha + \gamma)$$
$$\xi_5 = (b + c)\alpha - a(\beta + \gamma)$$
$$\xi_6 = (a + b)\gamma - c(\alpha + \delta)$$

The preceding equations immediately imply that

$$\xi_4 = \xi_3 - \xi_1 , \qquad \xi_5 = \xi_1 + \xi_2 , \qquad \xi_6 = \xi_3 - \xi_2$$

for the above choices of ξ_i. Furthermore, we have $c\xi_1 - b\xi_2 = a\xi_3$, and hence we may write the equations above as follows:

$$\xi_3 = \frac{c}{a}\xi_1 - \frac{b}{a}\xi_2 , \qquad \xi_4 = \frac{c-a}{a}\xi_1 - \frac{b}{a}\xi_2 , \qquad \xi_6 = \frac{c}{a}\xi_1 - \frac{b+a}{a}\xi_2 .$$

By definition, the above equations imply the following cross ratio properties:

$$\mathsf{XR}(X_1, X_2, X_5, X_3) = -\frac{c}{b}$$

$$\mathsf{XR}(X_1, X_2, X_5, X_4) = -\frac{c-a}{b}$$

$$\mathsf{XR}(X_1, X_2, X_5, X_6) = -\frac{c}{a+b}$$

If the first cross ratio is denoted by r and the second by s, then the third is equal to

$$\frac{r}{s - r + 1} .$$

Thus the third cross ratio only depends upon the points X_i for $i \leq 5$; in particular, if

$$\{X_1, X_2, X_3, X_4, X_5, Y\}$$

is an arbitrary quadrangular set, then $\mathsf{XR}(X_1, X_2, X_5, Y) = \mathsf{XR}(X_1, X_2, X_5, X_6)$. By Theorem 10, this implies that $Y = X_6$.∎

Harmonic quadruples and von Staudt's Theorem

The importance of harmonic quadruples in projective geometry is reflected by the following remarkable result of K. von Staudt.[11]

THEOREM V.27. *Let $\varphi : \mathbb{RP}^1 \to \mathbb{RP}^1$ be a $1-1$ onto map which preserves harmonic quadruples; specifically, for all distinct collinear ordered quadruples $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ we have $\mathsf{XR}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = -1$ if and only if $\mathsf{XR}(\varphi(\mathbf{a}), \varphi(\mathbf{b}), \varphi(\mathbf{c}), \varphi(\mathbf{d})) = -1$. Then there is an invertible 2×2 matrix A over $\mathbb{R}$ such that φ corresponds to $\mathcal{S}_1(A)$ under the standard identification of $\mathbb{RP}^1$ with $\mathcal{S}_1(\mathbb{R}^{2,1})$, and for all distinct collinear ordered quadruples $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ we have*

$$\mathsf{XR}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \mathsf{XR}(\varphi(\mathbf{a}), \varphi(\mathbf{b}), \varphi(\mathbf{c}), \varphi(\mathbf{d})) .$$

The second part of the conclusion follows from the first (see Exercise V.2.10). One very accessible proof of the first conclusion of the theorem is the following online document:

http://www-m10.ma.tum.de/~richter/Vorlesungen/ProjectiveGeometrie/Kapitel/Chap5.pdf

Note that von Staudt's result is only stated for the case $\mathbb{F} = \mathbb{R}$. The corresponding result for more general fields in which $1 + 1 \neq 0$ is discussed in Section VI.3 following Theorem VI.11 (see the subheading *Collineations of $\mathbb{FP}^1$*).

EXERCISES

1. In $\mathbb{RP}^2$, show that the pair of points whose homogeneous coordinates satisfy
$$x_1^2 - 4x_1 x_2 - 3x_2^2 = x_3 = 0$$
separate harmonically the pair whose coordinates satisfy
$$x_1^2 + 2x_1 x_2 - x_2^2 = x_3 = 0 .$$

2. Prove that a complete quadrilateral is completely determined by one vertex and its diagonal points.

3. State the plane dual theorem to the result established in Theorem 26.

4. In the Euclidean plane $\mathbb{R}^2$, let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be noncollinear points, and let the lines bisecting $\angle \mathbf{bac}$ and its supplement meet $\mathbf{bc}$ in the points $\mathbf{e}$ and $\mathbf{d}$ of $\mathbb{RP}^2$ respectively. Prove that $\mathsf{XR}(\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) = -1$.

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[11]KARL G. C. VON STAUDT (1798–1867) is best known for his contributions to projective geometry and his work on a fundamentally important sequence in number theory called the *Bernoulli numbers*. In his work on projective geometry, von Staudt's viewpoint was highly synthetic, and his best known writings provide a purely synthetic approach to the subject.
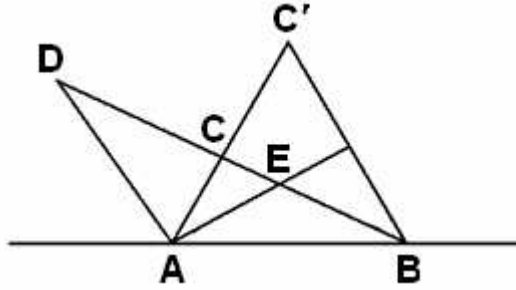
Figure V.6

[*Hint:* Explain why it suffices to consider the triple of points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}'$ where $\mathbf{c}'$ satisfies $\mathbf{c}' - \mathbf{a} = t(\mathbf{c} - \mathbf{a})$ for some $t > 0$ and $|\mathbf{c}' - \mathbf{a}| = |\mathbf{b} - \mathbf{a}|$. Why is $\mathbf{ae}$ the perpendicular bisector of $\mathbf{b}$ and $\mathbf{c}'$, and why is $\mathbf{bc}'||\mathbf{ad}$? If $\mathbf{e}'$ is the point where $\mathbf{ae}$ meets $\mathbf{bc}'$ and $\mathbf{j}$ is the ideal point on the line $\mathbf{bc}'$, what are $\mathsf{XR}(\mathbf{b}, \mathbf{c}', \mathbf{e}', \mathbf{j})$ and $\mathsf{XR}(\mathbf{b}, \mathbf{c}', \mathbf{j}, \mathbf{e}')$?][12]

5. Let $\mathbb{F}$ be a field in which $1 + 1 \neq 0$, let $A \neq B$ in $\mathbb{FP}^n$, where $n \geq 1$, and let α and β be homogeneous coordinates for A and B. For $1 = 1, 2, 3, 4$ let X_i have homogeneous coordinates $x_i\alpha + \beta$, and assume that $\mathsf{XR}(X_1, X_2, X_3, X_4) = -1$. Prove that

$$\frac{2}{x_4 - x_3} = \frac{1}{x_1 - x_2} + \frac{1}{x_1 - x_4} .$$

6. Let $\mathbb{F}$ be as in the previous exercise, and let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in \mathbb{F}^2$ be distinct points with coordinates $(0,0)$, $(b,0)$, $(c,0)$ and $(d,0)$ respectively. Assume that $\mathsf{XR}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = -1$. Prove that

$$\frac{1}{b} = \frac{1}{2} \cdot \left(\frac{1}{c} + \frac{1}{d} \right) .$$

[*Hint:* Apply the preceding exercise, taking A and B to be the points U and V whose homogeneous coordinates are $^\mathbf{T}(1 \quad 0 \quad 0)$ and $^\mathbf{T}(0 \quad 0 \quad 1)$ respectively.]

> REMARK. The right hand side of the equation above is called the **harmonic mean** of c and d. The harmonic mean was well-known to ancient Greek mathematicians; the name itself[13] was first used by Archytas of Tarentum (*c.* 428 B. C. E. – *c.* 350 B. C. E.), but the concept had been known since the time of the Pythagoreans in the 6th century B. C. E.

7. Let $\mathbb{F}$ be the real numbers $\mathbb{R}$, let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in \mathbb{R}^2$ be distinct points with coordinates $(0,0)$, $(b,0)$, $(c,0)$ and $(d,0)$ respectively, and assume that b, c and d are all positive. Prove that $\mathsf{XR}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is positive if either b is less than or greater than both c and d, but $\mathsf{XR}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is negative if either $0 < c < b < d$ or $0 < d < b < c$. — In other words, the cross ratio is negative if and only if the points 0 and b *separate* the points c and d in the sense that one of the latter lies on the bounded open interval defined by $0 < x < b$ and the other lies on the unbounded interval defined by $x > b$. More will be said about this concept of separation in the final section of Chapter VI. [*Hint:* Use the same methods as in the preceding exercise to express the cross ratio in terms of the coordinates of the four points.]

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[12]See Moïse, Moïse and Downs, Prenowitz and Jordan, or the online site mentioned in the Preface for mathematically sound treatments of the Euclidean geometry needed for this exercise.

[13]More correctly, the corresponding name in Classical Greek.

5. Interpretation of addition and multiplication

Theorem IV.15 states that if an n-dimensional projective incidence space is isomorphic to $\mathbb{FP}^n$ for some skew-field $\mathbb{F}$, then the latter is unique up to algebraic isomorphism. In particular, if $\mathbb{E}$ is a skew-field such that $\mathbb{EP}^n$ is isomorphic to $\mathbb{FP}^n$, then $\mathbb{E}$ is algebraically isomorphic to $\mathbb{F}$. The reason for this is that addition and multiplication in the underlying skew-field have synthetic interpretations in terms of certain geometric constructions which are motivated by ordinary Euclidean geometry. If the two coordinate projective spaces as above are isomorphic, this means that the algebraic operations in each are characterized by the same synthetic constructions, and therefore the algebraic operations in the two underlying skew-fields must be isomorphic.

We shall take the preceding discussion further in Chapter VI, where we shall use the synthetic interpretations of addition and multiplication to give an complete description of all geometrical (incidence space) automorphisms of $\mathbb{FP}^n$, where $\mathbb{F}$ is a field and $n \geq 2$.

In order to simplify the algebra, we again restrict attention to commutative fields; however, all the results in this section are equally valid for arbitrary skew-fields (and the result on automorphisms in Chapter VI also extend to the noncommutative case).

Euclidean addition of lengths. If L is a line in the Euclidean plane containing the points X_1, A and B such that the lengths of the segments $[X_1A]$ and $[X_1B]$ have lengths a and b respectively, then the figure below indicates one method for finding a point $C \in L$ such that the segment $[X_1C]$ has length $a + b$.
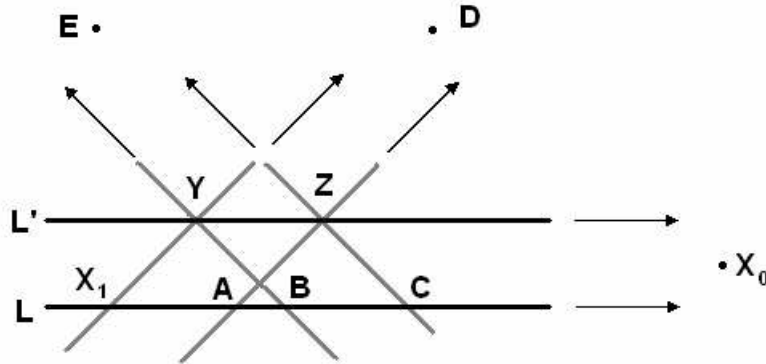
$L||L'$, $X_1Y||AZ$, $YB||ZC$ with ideal points X_0, E and D respectively.

Figure V.7

This example motivates the following abstract result:

THEOREM V.28. *Let $\mathbb{F}$ be a field, and let $n \geq 2$. Let L be a line in $\mathbb{FP}^n$ containing a point X_0, let M be another line containing X_0, let X_1 be another point on L, and let U be a third point on $L = X_0X_1$. Let A and B be points of $L - \{P_0\}$, and let Y be a point in the plane of L and M which does not line on either line. Let D be the (unique) intersection point of $X_1Y \cap M$ (note that $D \neq X_1$, for that would imply X_1 and X_0 both lie on $L \cap M$), and let*

$$Z \in AD \cap X_0Y \,, \quad E \in YB \cap M \,, \quad C \in RE \cap L \,.$$

Then $C \neq X_0$ and $\mathsf{XR}(X_0, X_1, U, C) = \mathsf{XR}(X_0, X_1, U, A) + \mathsf{XR}(X_0, X_1, U, B)$.
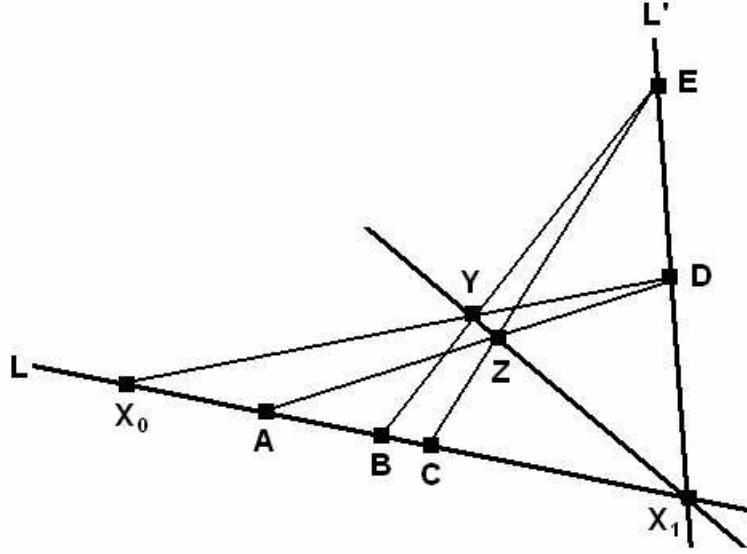
Figure V.8

Proof. Let V be the point where X_0Y meets DU. Then no three of the points $\{X_0,\ X_1,\ D,\ V\}$ are collinear. Choose homogeneous coordinates $\xi_0,\ \xi_1,\ \delta,\ \psi$ for $X_0,\ X_1,\ D,\ V$ so that $\psi\ =\ \xi_0 + \xi_2 + \delta$. Since $Y \in DX_1 \cap X_0V$ it follows that $\eta = \xi_1 + \delta$, and since $U \in X_0Y \cap DV$ it follows that $\eta = \xi_1 + \delta$.

By the definition of cross ratios, there are homogeneous coordinates α and β for A and B respectively such that $\alpha\ =\ a\xi_0 + \xi_1$ and $\beta\ =\ b\xi_0 + \xi_1$, where $a = \mathsf{XR}(X_0, X_1, U, A)$ and $b = \mathsf{XR}(X_0, X_1, U, B)$. Since $Z \in AD \cap X_0Y$, there exist $x,\ y,\ u,\ v \in \mathbb{F}$ such that homogeneous coordinates for ζ are given by

$$x\alpha\ +\ y \cdot a\ =\ u\xi_0\ +\ v\eta\ .$$

Using the preceding equations, this equation may be expanded to

$$xa\xi_0\ +\ x\xi_1\ +\ y\delta\ =\ u\xi_0\ +\ v\xi_1\ +\ v\delta\ .$$

Therefore we must have $x = v$ and $xa = u$, so that $\zeta\ =\ a\xi_0 + \xi_1 + \delta$. Similarly, $D \in YB \cap X_0D$ implies an equation of the form $x\eta + y\beta\ =\ u\xi_0 + \delta$, which is equivalent to

$$x\xi_1\ +\ x\delta\ +\ yb\xi_0\ +\ y\xi_1\ =\ u\xi_0\ +\ v\delta\ .$$

Therefore $yb = u$, $x = v$ and $x + y = 0$ imply that homogeneous coordinates ε for E are given by

$$\varepsilon\ =\ -b\xi_0\ +\ \delta\ .$$

Finally, $C \in ZE \cap X_0X_1$ implies an equation

$$x\zeta\ +\ y\varepsilon\ =\ u\xi_0\ +\ v\xi_1$$

which is equivalent to

$$xa\xi_0\ +\ x\xi_1\ +\ x\delta\ -\ y\delta\ -\ yb\xi_0\ =\ u\xi_0\ +\ v\xi_1\ .$$

Thus $x + y = 0$, $xa - yb = u$ and $x = v$ imply that homogeneous coordinates γ for C are given by

$$\gamma\ =\ (a + b)\xi_0\ +\ \xi_1\ .$$

In particular, it follows that $C \neq X_0$ and $\mathsf{XR}(X_0, X_1, U, C)\ =\ a + b.\blacksquare$

Euclidean multiplication of lengths. Similarly, if L is a line in the Euclidean plane containing the points X_1, U, A and B such that the lengths of the segments $[X_1U]$, $[X_1A]$, and $[X_1B]$ have lengths 1, a and b respectively, then the figure below indicates one method for finding a point $K \in L$ such that the segment $[X_1K]$ has length $a \cdot b$.
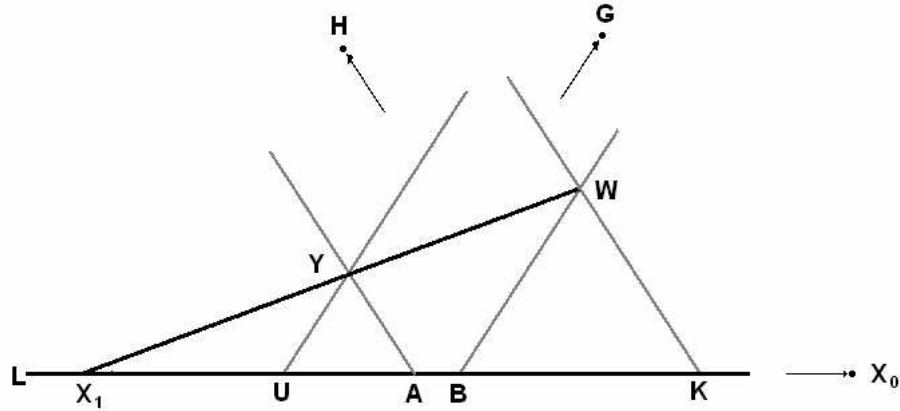
$YA||WK$ and $YU||BW$ with ideal points H and G respectively, and X_0 is the ideal point of L.

Figure V.9

Here is the corresponding abstract result:

THEOREM V.29. *Let L, X_0, X_1, U, A, B, Y, D satisfy the conditions of the previous theorem, and let*

$$G \in YU \cap L, \qquad H \in YA \cap L, \qquad W \in X_1D \cap BG, \qquad K \in HW \cap X_0X_1.$$
$$\text{Then } K \neq X_0 \text{ and } \mathsf{XR}(X_0, X_1, U, C) = \mathsf{XR}(X_0, X_1, U, A) \cdot \mathsf{XR}(X_0, X_1, U, B).$$
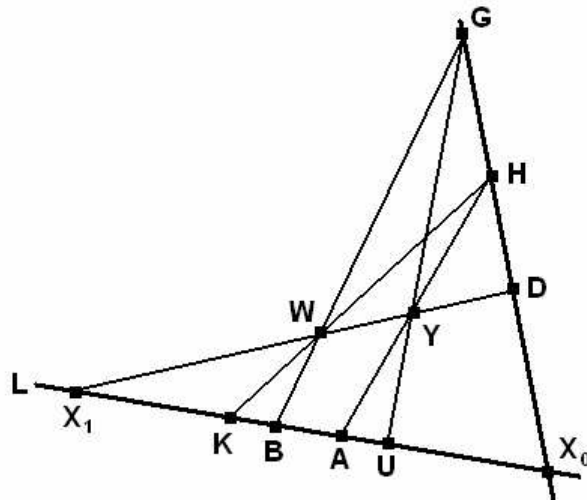
Figure V.10

Proof. The problem here is to find homogeneous coordinates for G, H, W and K. Unless otherwise specified, we shall use the same symbols for homogeneous coordinates representing

L, X_0, X_1, A, B, Y, D as in the previous argument, and choose homogeneous coordinates for U of the form $\xi_0 + \xi_1$. Since $G \in YU \cap X_0D$, there is an equation of the form

$$x\eta \; + \; y(\xi_0 + \xi_1) \;\; = \;\; u\xi_0 \; + \; v\delta$$

which is equivalent to

$$x(\xi_1 + \delta) \; + \; y(\xi_0 + \xi_1) \;\; = \;\; u\xi_0 \; + \; v\delta \; .$$

Therefore $y = u$, $x + y = 0$, and $x = v$ imply that homogeneous coordinates χ for G are given by

$$\chi \;\; = \;\; \xi_0 - \delta \; .$$

Since $H \in YA \cap X_0D$, there is an equation of the form

$$x\xi_0 \; + \; y\alpha \;\; = \;\; u\xi_0 \; + \; v\delta$$

which is equivalent to

$$x(\xi_0 + \delta) \;\; = \;\; y(a\xi_0 + \xi_1) \;\; = \;\; u\xi_0 \; + \; v\delta \; .$$

Therefore $ya = u$, $x + y = 0$, and $x = v$ imply that homogeneous coordinates θ for H are given by

$$\theta \;\; = \;\; -a\xi_0 \; + \; \delta \; .$$

Since $W \in X_1U \cap BG$, there is an equation

$$x\xi_1 \; + \; y\delta \;\; = \;\; u\beta \; + \; v\chi$$

which is equivalent to

$$x\xi_1 \; + \; y\delta \;\; = \;\; u(b\xi_0 + \xi_1) \; + \; v(\xi_0 - \delta) \;\; = \;\; (ub + v)\xi_0 \; + \; u\xi_1 \; - \; v\delta \; .$$

Thus $ub + v = 0$, $-v = y$, and $u = x$ imply that homogeneous coordinates ω for W are given by

$$\omega \;\; = \;\; \xi_1 \; - \; \delta \; .$$

Finally, $K \in HW \cap X_0X_1$ implies an equation of the form

$$x\theta \; + \; y\omega \;\; = \;\; u\xi_0 \; + \; v\xi_1$$

which is equivalent to

$$u\xi_0 \; + \; v\xi_1 \;\; = \;\; x(-a\xi_0 + \delta) \; + \; y(\xi - b\delta) \;\; = \;\; -ax\xi_0 \; + \; y\xi_1 \; - \; (ax + by)\delta \; .$$

Therefore $u = -ax$, $v = y$.and $x + by = 0$ imply

$$u \;\; = \;\; -ax \;\; = \;\; -a(-by) \;\; = \;\; aby \;\; = \;\; abv$$

and hence homogeneous coordinates κ for K are given by $\kappa \;\; = \;\; ab\xi_0 \; + \; \xi_1.\blacksquare$

CHAPTER VI

MULTIDIMENSIONAL PROJECTIVE GEOMETRY

In this chapter we shall study coordinate projective spaces of arbitrary dimension. As in the previous chapter, we shall use concepts from linear algebra extensively. Although some portions of this chapter contain results of the previous one as special cases, most of the material involves concepts not covered earlier in these notes.

One major difference between this chapter and the previous one is that we are mainly interested in somewhat different types of results. In particular, we are interested in the geometric automorphisms of a coordinate projective space $\mathbb{F}\mathbb{P}^n$, and the results of this chapter give a simple but complete description of them. In the final section of this chapter we shall assume that we have a field (or skew-field) of scalars $\mathbb{F}$ which has a notion of ordering with the same basic properties of the orderings of the real or rational numbers, and we shall analyze the geometrical implications of such algebraic orderings.

1. Linear varieties and bundles

Our first objective is to extend the results of Section V.1 on duality and homogeneous coordinates from $\mathbb{F}\mathbb{P}^2$ to $\mathbb{F}\mathbb{P}^n$, where $n \geq 3$ is arbitrary. As indicated in Theorem IV.16, if (S, Π, d) is an n-dimensional projective (incidence) space, then the "points" of the dual projective n-space (S^*, Π^*, d^*) are the hyperplanes of S. Suppose now that $S = \mathbb{F}\mathbb{P}^n$ for some skew-field $\mathbb{F}$; by the results of Section V.1, if $n = 2$ then we can introduce homogeneous coordinates into the dual projective plane $\left(\mathbb{F}\mathbb{P}^2 \right)^*$. We shall extend this to all $n \geq 2$, showing that one can define well-behaved homogeneous coordinates for the hyperplanes of $\mathbb{F}\mathbb{P}^n$ for all $n \geq 2$ such that most of the fundamental results from Section V.1 also extend to this more general setting.

According to Theorem III.12, a hyperplane in $\mathbb{F}\mathbb{P}^n$ is definable by a right homogeneous linear equation

$$\sum_{i=1}^{n+1} u_i \, x_i \;=\; 0$$

where the coefficients u_i are not all zero. Furthermore, two n-tuples $(u_1, \; \cdots \; , u_{n+1})$ and $(v_1, \; \cdots \; , v_{n+1})$ define the same hyperplane if and only if there is a nonzero $k \in \mathbb{F}$ such that $u_i \;=\; k \, v_i$ for all i (compare Section V.1). This immediately yields the following analog of Theorem V.1:

THEOREM VI.1. *Let $\mathbb{F}$ be a skew-field, and let $n \geq 2$. Then the set of hyperplanes in $\mathbb{F}\mathbb{P}^n$ is in $1 - 1$ correspondence with $\mathcal{S}_1(\mathbb{F}^{1,n+1})$. Furthermore, if the hyperplane H corresponds to the left 1-dimensional vector subspace $\mathbb{F} \cdot \theta$ and $X \in \mathbb{F}\mathbb{P}^n$ is given by $\xi \cdot \mathbb{F}$, then $X \in H$ if and only if $\theta \cdot \xi \;=\; 0$.*∎

As before, if the hyperplane H corresponds to the left 1-dimensional vector subspace Ω of $\mathbb{F}^{1,n+1}$, then a set of *homogeneous coordinates* for H is any nonzero vector in Ω.

Motivated by the preceding description of hyperplanes, we define a *linear variety* in $\mathbb{F}\mathbb{P}^n$ to be the set of all points whose homogeneous coordinates satisfy a system of linear homogeneous equations

$$\sum_{j=1}^{n+1} u_{i.j}x_j = 0 \qquad 1 \le i \le m .$$

The following result shows that linear varieties are the same as geometrical subspaces.

THEOREM VI.2. *Let V be a linear variety defined by a system of linear homogeneous equations as above, and suppose that the (left) row rank of the matrix $B = (u_{i,j})$ is equal to r. Then V is an r-plane in $\mathbb{F}\mathbb{P}^n$.*

Proof.   If V_0 is the solution space of the system of equations, then clearly $V = \mathcal{S}_1(V_0)$. Since the rank of B is r, then dimension of V_0 is equal to $n+1-r$ by Theorem A.10, and hence V is an $(n-r)$-plane in $\mathbb{F}\mathbb{P}^n$.

On the other hand, assume that W is a $(k+1)$-dimensional vector subspace of $\mathbb{F}^{n+1,1}$, so that $\mathcal{S}_1(W)$ is a k-plane. Let $\mathbf{w}_1, \cdots \mathbf{w}_{k+1}$ be a basis for W, and write $\mathbf{w}_i = {}^{\mathbf{T}}(w_{1,i} \quad \cdots \quad w_{k+1,i})$. Consider the left-homogeneous system of linear equations

$$\sum_i y_i\, w_{j,i} = 0 \qquad (1 \le i \le k+1) .$$

Since the right column rank of the matrix $C = (w_{j,i})$ is equal to k+1, the left subspace of solutions has dimension equal to $n-k$ (again by Theorem A.10). Let $\mathbf{v}_1, \cdots \mathbf{v}_{n-k}$ be a basis for the space of solutions, and write $\mathbf{v}_i = (v_{i,1} \quad \cdots \quad v_{i,n+1})$. Then, by construction, the vector subspace W is contained in the space of solutions of the system

$$\sum_j v_{j,i}\, x_j = 0 \qquad (1 \le j \le n-k) .$$

On the other hand, the space of solutions W' has dimension equal to

$$(n+1) - (n-k) = k + 1 .$$

Since this is the dimension of W, we must have $W' = W$, and this proves the second half of the theorem.∎

Similarly, we may define a *linear variety of hyperplanes* to be the set of all hyperplanes whose homogeneous coordinates satisfy a system of left-homogeneous linear equations

$$\sum_i u_i\, x_{i,j} = 0 \qquad (1 \le j \le m) .$$

If the right rank of $X = (x_{i,j})$ is r, the variety of hyperplanes is said to be $(n-r)$-dimensional. The following result shows that linear varieties of hyperplanes are also equivalent to geometrical subspaces of $\mathbb{F}\mathbb{P}^n$.

THEOREM VI.3. *An r-dimensional linear variety of hyperplanes in $\mathbb{F}\mathbb{P}^n$ consists of all hyperplanes containing a fixed $(n - r - 1)$-plane in the terminology of Chapter IV, a linear bundle with the given $(n - r - 1)$-plane as center). Conversely, every $(n - r - 1)$-plane in $\mathbb{F}\mathbb{P}^n$ is the center of some linear variety of hyperplanes.*

Proof. The ideas are similar to those employed in Theorem 2. Let C_0 be the span of the rows of the matrix $(x_{i,j})$. By hypothesis, $\dim C_0 = n-r$. Thus $C = \mathcal{S}_1(C_0)$ is an $(n-r-1)$-plane in $\mathbb{FP}^n$, and every hyperplane containing it automatically belongs to the linear variety. Conversely, if $\mathbf{y}_0 \in C_0$, then we may write $\mathbf{y} = \sum_i \mathbf{x}_i r_i$ where $\mathbf{x}_i = (x_{1,i}, \cdots, x_{n+1,i})$ and $r_i \in \mathbb{F}$, so that if H lies in the variety and θ is a set of homogeneous coordinates for H then we have

$$\theta \cdot \mathbf{y} = \sum_i (\theta \cdot \mathbf{x}_i)\, r_i = 0 \ .$$

Thus every hyperplane in the variety contains every point of C. This proves the first half of the theorem.

Now suppose that we are given an $(n-r-1)$-plane $Z = \mathcal{S}_1(Z_0)$. Let $\mathbf{z}_1, \cdots, \mathbf{z}_{n-r}$ be a basis for Z_0, and write $\mathbf{z}_j = (z_{1,j}, \cdots, z_{n+1,j})$. Consider the variety of hyperplanes defined by the system of homogeneous equations

$$\sum_j u_i z_{i,j} = 0 \qquad (1 \leq j \leq n-r) \ .$$

Since the right rank of the matrix $(z_{i,j})$ is equal to $(n-r)$, this bundle is r-dimensional. Furthermore, its center Z' is an $(n-r-1)$-plane which contains every point of Z by the reasoning of the previous paragraph. Therefore we have $Z = Z'$.■

The preceding result has some useful consequences.

THEOREM VI.4. *Let $(\mathbb{FP}^n)^*$ be the set of hyperplanes in $\mathbb{FP}^n$, and let Π^* and d^* be defined as in Section IV.3. Then a subset $S \subset (\mathbb{FP}^n)^*$ is in Π^* if and only if it is a linear variety of hyperplanes, in which case $d^*(S)$ is the dimension as defined above.*■

THEOREM VI.5. (compare Theorem V.1) *The triple*

$$\left((\mathbb{FP}^n)^*, \ \Pi^*, \ d^*\right)$$

is a projective n-space which is isomorphic to $\mathcal{S}_1(\mathbb{F}^{1,n+1})$.

Since Theorem 4 is basically a restatement of Theorem 3, we shall not give a proof. However, a few remarks on Theorem 5 are in order.

Proof of Theorem 5. By Theorem 1 we have a 1–1 correspondence between $(\mathbb{FP}^n)^*$ and $\mathcal{S}_1(\mathbb{F}^{1,n+1})$. Furthermore, the argument used to prove Theorem 2 shows that r-dimensional varieties of hyperplanes correspond to set of the form $\mathcal{S}_1(V)$, where V is an $(r+1)$-dimensional left subspace of $\mathbb{F}^{1,n+1}$ (merely interchange the roles of left and right in the proof, switch the orders of the factors in products, and switch the orders of double subscripts). But r-dimensional linear bundles correspond to r-dimensional linear varieties of hyperplanes by Theorems 3 and 4. Combining these, we see that r-dimensional linear bundles of hyperplanes correspond to r-planes in $\mathcal{S}_1(\mathbb{F}^{1,n+1})$ under the 1–1 correspondence between $(\mathbb{FP}^n)^*$ and $\mathcal{S}_1(\mathbb{F}^{1,n+1})$.■

By the Coordinatization Theorem (Theorem IV.18), this result implies the first half of Theorem IV.16. On the other hand, if we interchange the roles of left and right, column vectors and row vectors, and the orders of multiplication and indices in the reasoning of this section, we find

that the dual of $\mathcal{S}_1(\mathbb{F}^{1,n+1})$ is isomorphic to $\mathbb{FP}^{n+1}$. In fact, this isomorphism $h : \mathcal{S}_1(\mathbb{F}^{1,n+1})^* \to \mathcal{S}_1(\mathbb{F}^{n+1,1})$ is readily seen to have the property that the composite $h \circ f^* \circ e$ given by

$$\mathbb{FP}^n \xrightarrow[\cong]{\ e\ } (\mathbb{FP}^n)^{**} \xrightarrow{\ f^*\ } \mathcal{S}_1(F^{1,n+1})^* \xrightarrow{\ h\ } \mathbb{FP}^n$$

is the identity (here f^* is an isomorphism of dual spaces induced by f as in Exercise IV.3.4). This establishes the second half of Theorem IV.16 and allows us to state the **Principle of Duality in Higher Dimensions:**

METATHEOREM VI.6. *A theorem in projective geometry in dimension $n \geq 2$ remains true if we interchange the expressions* point *and* hyperplane, *the phrases* r-planes in an n-space *and* $(n-r-1)$-planes in an n-space, *and the words* contains *and* is contained in.∎

We shall now assume that $\mathbb{F}$ is commutative. Since $\mathbb{F} = \mathbb{F}^{\mathsf{OP}}$ in this case, the dual of $\mathbb{FP}^n$ is isomorphic to $\mathbb{FP}^n$. Hence the metatheorem may be modified in an obvious way to treat statements about projective n-spaces **over fields**.

The cross ratio of four hyperplanes four hyperplanes in $\mathbb{FP}^n$ containing a common $(n-2)$-plane may be defined in complete analogy with the case $n = 2$, which was treated in Section V.2. In particular, Theorem V.14 generalizes as follows.

THEOREM VI.7. *Let H_1, H_2, H_3 be distinct hyperplanes through an $(n-2)$-plane K in $\mathbb{FP}^n$, and let $H_4 \neq H_1$ be another hyperplane through K. Let L be a line disjoint from K, and let X_i be the unique point where L meets H_i for $1 = 1, 2, 3$. Then the point $X_4 \in L$ lies on H_4 if and only if we have*

$$\mathsf{XR}(X_1, X_2, X_3, X_4) \ = \ \mathsf{XR}(H_1, H_2, H_3, H_4) \ .$$

The proof of this result is formally identical to the proof of Theorem V.1.∎

EXERCISES

1. Let $\mathbb{F}$ be a field, and let X, Y, $Z \in \mathbb{FP}^3$ be noncollinear points. Suppose that homogeneous coordinates for these points are respectively given as follows:

$$\xi \ = \ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \eta \ = \ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \qquad \zeta \ = \ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

Prove that the plane they determine has the following homogeneous coordinates:

$$\left(\begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_{,4} \end{vmatrix}, \ - \begin{vmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ z_1 & z_3 & z_{,4} \end{vmatrix}, \ \begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{vmatrix}, \ - \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \right)$$

By Theorem A.11, not all of the four determinants vanish because ξ, η and ζ are linearly independent. To see that X ,Y, Z lie on the above hyperplane, consider the determinants of the three 4×4 matrices whose rows are given by $^{\mathbf{T}}\omega$, $^{\mathbf{T}}\xi$, $^{\mathbf{T}}\eta$ and $^{\mathbf{T}}\zeta$, where ω runs through the three vectors in the set ξ, η, ζ.

2. Explain why four points $\mathbf{p}_i = (x_i, y_i, z_i) \in \mathbb{F}^3$ (where $1 \leq i \leq 4$) are coplanar if and only if the 4×4 determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

is zero, where $\mathbb{F}$ is a field. Formulate an analogous statement for n dimensions. [*Hint:* For both parts, use the properties of determinants as described in Appendix A and the characterization of hyperplanes in terms of n-dimensional vector subspaces of $\mathbb{F}^{1,n+1}$.]

3. Write out the 3-dimensional projective duals of the following concepts:

(a) A set of collinear points.

(b) A set of concurrent lines.

(c) The set of all planes through a given point.

(d) Four coplanar points, no three of which are collinear.

(e) A set of noncoplanar lines.

4. What is the 3-dimensional dual of Pappus' Theorem?

5. What is the 4-dimensional dual of two planes whose intersection consists of a line? What kind of set is the intersection of the dualized objects?

6. Let $\{A, B, C, D\}$ and $\{A', B', C'', D'\}$ be two triples of noncoplanar points in a projective 3-space, and assume that the lines AA', BB', CC' and DD' are concurrent. Prove that the lines of intersection

$$\begin{aligned} G &= \mathsf{plane}(ABC) \cap \mathsf{plane}(A'B'C') \\ H &= \mathsf{plane}(ABD) \cap \mathsf{plane}(A'B'D') \\ K &= \mathsf{plane}(ACD) \cap \mathsf{plane}(A'C'D') \\ L &= \mathsf{plane}(BCD) \cap \mathsf{plane}(B'C'D') \end{aligned}$$

are coplanar, and state and prove the converse.

7. Find the equations of the hyperplanes through the following quadruples of points in $\mathbb{RP}^4$.

(a)

$$\begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 4 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 3 \\ 4 \\ 0 \\ 0 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 0 \\ 5 \\ 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

2. Projective coordinate systems

Theorems V.6 and V.7, which yielded particularly convenient choices for homogeneous coordinates in one or two dimensions, proved to be extremely useful in Chapter V. We shall prove a corresponding result for higher dimensions in this section; as one might expect, this result has corresponding importance in higher dimensional projective geometry. All the results of this section (except for Exercise 1) are valid if $\mathbb{F}$ is an arbitrary skew-field.

THEOREM VI.8. *Let $\{B_0, \cdots, B_r\}$ be a set of independent points in $\mathbb{FP}^n$, and let U be a point in the r-plane $B_0 \cdots B_r$ such that every proper subset of $\{B_0, \cdots, B_r, U\}$ is independent. Then homogeneous coordinates β_i and ψ can be chosen for these points may be chosen such that*

$$(\ddagger) \qquad \psi = \beta_0 + \cdots + \beta_r .$$

Furthermore, if β_i' and ψ' is another collection of homogeneous for these points such that ($\ddagger$) holds, then there is a nonzero $a \in \mathbb{F}$ such that $\psi = \psi' a$ and $\beta_i = \beta_i' a$ for $i = 0, \cdots, r$.

Proof. . Since the points B_i are independent, if we take arbitrary homogeneous coordinates $\widetilde{\beta_i}$ and ψ then there exist unique scalars c_i such that

$$\widetilde{\psi} = \widetilde{\beta_0}\, c_0 + \cdots + \widetilde{\beta_r}\, c_r .$$

None of the coefficients c_i can be equal to zero, for otherwise a proper subset of $\{B_0, \cdots, B_r, U\}$ would be independent, contradicting our assumption about such proper subsets. If we now take $\beta_i = \widetilde{\beta_i}\, c_i$ for each i, we then have the desired relation ($\ddagger$) which is stated above.$\square$

Conversely, suppose that ($\ddagger$) is satisfied. If we are given arbitrary homogeneous coordinates β_i' and ψ' for the points B_i and U, then there exist nonzero scalars a and q_i such that $\psi = \psi' a$ and $\beta_i = \beta_i' q_i$. The new homogeneous coordinate vectors satisfy a relation of the form

$$\psi' = \beta_0'\, q_0\, a^{-1} + \cdots + \beta_r'\, q_r\, a^{-1}$$

and if ($\ddagger$) is valid then all the coefficients on the right hand side must be equal to 1. In other words, we must have $b_i\, a^{-1} = 1$ for all i or equivalently $b_i = a$ for all i, which is exactly what we wanted to prove.$\blacksquare$

Assume now that ($\ddagger$) is valid, and let X be any point of the r-plane $L = B_0 \cdots B_r$. A set of homogeneous coordinates ξ for X is then a linear combination of the form $\xi = \sum_i \beta_i\, x_i$. Since ξ is defined up to multiplication by a scalar factor and the vectors β_j are defined up to multiplication by a common scalar factor, it follows that the coefficients x_i are also determined up to multiplication by a common scalar factor, and such an ordered $(r+1)$-tuple $(x_0, \cdots, x_r)$ of coefficients is called a set of *homogeneous coordinates for $X \in L$ relative to the projective coordinate system* $(B_0 \cdots B_r \,|\, U)$. It is frequently denoted by notation such as $\vec{X}(B_0 \cdots B_r \,|\, U)$. The set $\{B_0, \cdots, B_r\}$ is often called the *coordinate simplex* or *fundamental simplex*, the points B_i are said to be the *vertices* of this coordinate simplex, and the point U is often called the *unit point* because homogeneous coordinates for this point in the projective coordinate system are given by $(1, \cdots, 1)$.

The homogeneous coordinates given in the definition of projective space may be viewed as a special case of the preceding construction; specifically, if the unit vectors in $\mathbb{F}^{n+1,1}$ are given by $\mathbf{e}_i$, then the appropriate coordinate simplex has vertices $\mathbf{e}_i \cdot \mathbb{F}$ and the corresponding unit point is $\mathbf{d} \cdot \mathbb{F}$, where $\mathbf{d} = \sum_i \mathbf{e}_i$. This is often called the *standard coordinate system*.

The next result describes the change in homogeneous coordinates which occurs if we switch from one projective coordinate system to another.

THEOREM VI.9. *Let* $(B_0 \; \cdots \; B_r \,|\, U)$ *and* $(B_0^* \; \cdots \; B_r^* \,|\, U^*)$ *be two projective coordinate systems for an* r-*plane in* $\mathbb{FP}^n$, *and let* X *be a point in this* r-*plane. Then the homogeneous coordinates* x_i *and* x_i^* *of* X *relative to these respective coordinate systems are related by the coefficients of an invertible matrix* $A \;=\; (\,a_{i,j}\,)$ *as follows:*

$$x_i^* \cdot \rho \;=\; \sum_{k=0}^{r} a_{i,k} \, x_k$$

Here ρ *is a nonzero scalar in* $\mathbb{F}$.

Proof. Suppose that the coordinate vectors are chosen as before so that $\psi \;=\; \sum_i \beta_i$ and $\psi' \;=\; \sum_i \beta_i'$. If ξ is a set of homogeneous coordinates for X, then homogeneous coordinates for ξ are defined by the two following two equations:

$$\xi \;=\; \sum_i \beta_i \, x_i \qquad\qquad \xi^* \;=\; \sum_i \beta_i^* \, x_i^*$$

Since the points lie in the same r-plane, we have

$$\beta_i^* \;=\; \sum_k \beta_k \, a_{i,k}$$

for sutiable scalars $a_{i,k}$, and the matrix A with these entries must be invertible because the set $\{B_0 \; \cdots \; B_r\}$ is independent. Straightforward calculation shows that

$$\xi \;=\; \sum_k \beta_k \, x_k \;+\; \sum_{i,k} \beta_i^* \, a_{i,k} \, x_k \;=\; \sum_i \beta_i^* \, x_i^*$$

which implies that $x_i^* \;=\; \sum_k a_{i,k} \, x_k$. These are the desired equations; we have added a factor ρ because the homogeneous coordinates are defined only up to a common factor.■

EXERCISES

1. Take the projective coordinate system on $\mathbb{RP}^3$ whose fundamental simplex points B_i have homogeneous coordinates

$$\beta_0 \;=\; \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \beta_1 \;=\; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \beta_2 \;=\; \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \beta_3 \;=\; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and whose unit point U has homogeneous coordinates

$$\psi \;=\; \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Find the homogeneous coordinates of the point A with respect to the system $(B_0 \, B_1 \, B_2 \, B_3 \mid U)$ where ordinary homogeneous coordinates α for A are given below; there are two parts to this exercise corresponding to the two possibilities for A.

$$\alpha = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}.$$

2. Let T be an invertible linear transformation on $\mathbb{F}^{n+1,1}$ with associated invertible matrix A, let T_* be the associated geometric symmetry of $\mathbb{FP}^n$, let $(B_0 \; \cdots \; B_n \mid U)$ define the standard homogeneous coordinate system, and let $X \in \mathbb{FP}^n$ have homogeneous coordinates given by $x_0, \cdots, x_n$. What are the homogeneous coordinates of X with respect to the coordinate system

$$\big(T_*(B_0) \; \cdots \; T_*(B_n) \mid T_*(U) \big) \; ?$$

3. Let $(B_0 \; \cdots \; B_n \mid U)$ and $(B_0^* \; \cdots \; B_n^* \mid U^*)$ be two projective coordinate systems for $\mathbb{FP}^n$. Suppose that for all points $X \in \mathbb{FP}^n$ we know that $\vec{X}(B_0 \; \cdots \; B_n \mid U)$ and $\vec{X}(B_0^* \; \cdots \; B_n^* \mid U^*)$ determine the same 1-dimensional subspace of $\mathbb{F}^{n+1}$. Prove that $B_i = B_i^*$ for all i and also that $U = U^*$.

4. Let $(B_0, B_1, B_2 \mid U)$ be a projective coordinate system for $\mathbb{FP}^2$, and let $X \in \mathbb{FP}^2$. Give a formula relating the homogeneous coordinates $\eta = (y_1, y_2, y_3, y_4)$ and $\zeta = (z_1, z_2, z_3, z_4)$ of X with respect to the systems $(B_0, B_1, B_2 \mid U)$ and $(B_0, B_1, U \mid B_2)$, and prove that this formula is correct.

5. Let $(B_0 \; \cdots \; B_n \mid U)$ be a projective coordinate system for $\mathbb{FP}^n$, and suppose that V is another point such that no $n + 1$ points in the set $\{B_0, \cdots, B_n, U, V\}$ are dependent. Explain why all the coordinates of $\vec{V}(B_0 \; \cdots \; B_n \mid U)$ are nonzero and no two coordinates are equal.

3. Collineations

At the beginning of Section II.6, we noted that an appropriate notion of isomorphism for figures in Euclidean space is given by certain 1–1 correspondences with special properties. If one analyzes the situation further, it turns out that the relevant class of 1–1 correspondences is given by maps which extend to *isometries* of the Euclidean n-space $\mathbb{R}^n$. Specifically, these are 1–1 mappings T from $\mathbb{R}^n$ to itself with the following two properties:

(i) If $\mathbf{x}$ and $\mathbf{y}$ are distinct points in $\mathbb{R}^n$, then T satisfies the identity
$$d(\mathbf{x}, \mathbf{y}) \;=\; d\bigl(T(\mathbf{x}), T(\mathbf{y})\bigr) \;;$$
in other words, T preserves distances between points.

(ii) If $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are distinct points in $\mathbb{R}^n$, then T satisfies the identity
$$\mathsf{measure}(\angle\,\mathbf{xyz}) \;=\; \mathsf{measure}\bigl(\angle\,T(\mathbf{x})T(\mathbf{y})T(\mathbf{z})\bigr) \;;$$
in other words, T preserves angle measurements.

Further information on such mappings and closely related issues can be found in the Addendum to Appendix A and the references cited there. For our purposes here, two important points are (i) all such isometries of $\mathbb{R}^n$ are describable in a very simple and explicit manner, (ii) isometries are geometric symmetries. Specifically, every such isometry of $\mathbb{R}^n \cong \mathbb{R}^{n,1}$ is given by a mapping of the form $T(\mathbf{x}) \;=\; A\mathbf{x} + \mathbf{b}$, where $\mathbf{b} \in \mathbb{R}^n \cong \mathbb{R}^{n,1}$ and A is an $n \times n$ matrix which is *orthogonal*; the latter means that ${}^{\mathsf{T}}A$ is equal to A^{-1} or equivalently that the rows and columns of A define orthonormal sets of vectors. In this section we shall prove similar results for symmetries of projective spaces, showing that all geometric symmetries of $\mathbb{FP}^n$ are also given by some fairly basic constructions using linear algebra.

Frequently in this section we shall use the term **collineation** to denote an isomorphism from one n-dimensional incidence space to another (assuming $n \geq 2$). This name dates back to the 19$^{\text{th}}$ century, and at the time collineations were the first types of incidence space isomorphisms to be considered abstractly.

Algebraic automorphisms and geometric symmetries

We have seen that every invertible $(n+1) \times (n+1)$ matrix A determines a collineation f_A of $\mathbb{FP}^n$ which is defined by the formula
$$f_A(\mathbf{x} \cdot \mathbb{F}) \;=\; A\mathbf{x} \cdot \mathbb{F} \;.$$
However, for many choices of $\mathbb{F}$ there are examples which do not have this form. In particular, if $\mathbb{F}$ is the complex numbers $\mathbb{C}$ and χ denotes the map on $\mathbb{F}^{n+1,1}$ which takes a column vector with entries z_j to the column vector whose entries are the complex conjugates[1] $\overline{z_j}$, then there is a well-defined collineation g_χ on $\mathbb{CP}^n$ such that
$$g_\chi(\mathbf{x} \cdot \mathbb{C}) \;=\; \chi(\mathbf{x}) \cdot \mathbb{C}$$
that can also be defined, but it turns out that such a map is not equal to any of the maps f_A described previously. The proof that g_χ is a collineation depends upon the fact that complex conjugation is an automorphism; *i.e.*, we have $\overline{z_1 + z_2} \;=\; \overline{z_1} + \overline{z_2}$, $\overline{z_1 \cdot z_2} \;=\; \overline{z_1} \cdot \overline{z_2}$, and conjugation is a 1–1 correspondence to $\mathbb{C}$ to itself because this map is equal to its own inverse.

---

[1]Recall that if a complex number is given by $u + v\mathbf{i}$, where $\mathbf{i}^2 = -1$, then its conjugate is equal to $a - b\mathbf{i}$.

More generally, if $\mathbb{F}$ is an arbitrary skew-field and χ is an automorphism of $\mathbb{F}$, then one can construct a similar collineation g_χ on $\mathbb{FP}^n$ that is not expressible as f_A for some A. One major objective of this section is to prove that mappings of the form f_A and g_χ for the various choices of A and χ determine **all** collineations of $\mathbb{FP}^n$. In order to simplify the arguments, for the rest of this section we shall assume that the skew-field $\mathbb{F}$ is commutative; at the end of the section we shall discuss some aspects of the noncommutative case.

In fact, one of the most important prepreties of a collineation f from one coordinate projective space to another (perhaps over a different field) is that the collineation determines an isomorphism Φ_f of the underlying fields; if the two projective spaces are identical, this isomorphism becomes an automorphism. The first result of this section establishes the relationship between collineations and field isomorphisms.

THEOREM VI.10. *Let f be a collineation from the projective space $\mathbb{FP}^n$ to the projective space $\mathbb{EP}^n$, where $\mathbb{F}$ and $\mathbb{E}$ are fields and $n \geq 2$. Then there is an isomorphism*

$$\Phi_f : \mathbb{F} \longrightarrow \mathbb{E}$$

characterized by the equation

$$\Phi_f\big(\mathsf{XR}(Y_1,\, Y_2,\, Y_3,\, Y_4) \big) \;=\; \mathsf{XR}\big(\Phi_f(Y_1),\, \Phi_f(Y_2),\, \Phi_f(Y_3),\, \Phi_f(Y_4) \big)$$

where Y_1, Y_2, Y_3, Y_4 is an arbitrary sequence of collinear points such that the first three are distinct and $Y_4 \neq Y_2$.

There are three basic steps in the proof; namely, defining a map from $\mathbb{F}$ to $\mathbb{E}$ which dependes upon some choices, showing that such a map is an isomorphism of fields, and finally showing that the map is independent of the choices that were made at the first step. The second part of the proof uses the results from Section V.5, and the third part — which is by far the longest — relies heavily on results from Chapter V on cross ratios.

Proof of Theorem 10.   *Construction of a mapping from $\mathbb{F}$ to $\mathbb{E}$.*   Let X_0, X_1 and U be three distinct collinear points, and let $q \in \mathbb{F}$. Then there is a unique point $Q \in X_0X_1$ such that $Q \notin P_0$ and $\mathsf{XR}(X_0, X_1, U, Q) \;=\; q$. Define $\Phi_f(q) \;=\; \mathsf{XR}\big(f(X_0), f(X_1), f(U), f(Q) \big)$. Strictly speaking, one should write this as $\Phi_{f,X_0,X_1,U}$ to indicate that it depends upon the choices of X_0, X_1 and U.

We claim that *the map $\Phi_{f,X_0,X_1,U}$ defines an isomorphism from $\mathbb{F}$ to $\mathbb{E}$.*   — Since the elements of $\mathbb{E}$ are in 1–1 correspondence with the elements of $f(X_0)f(X_1) - \{f(X_0)\}$ and f maps $X_0X_1 - \{X_0\}$ bijectively to $f(X_0)f(X_1) - \{f(X_0)\}$, it follows that $\Phi_{f,X_0,X_1,U}$ is 1–1 and onto. Furthermore, to see that the latter map is an isomorphism, take another line L through X_0, coplanar points Z_0 and D, and points A, $B \in X_0X_1$ as in Section V.3. Let $f(X_i) = X_i'$, $f[L] = L'$, $f(Z_0) - Z_0'$, $f(A) = A'$, $f(D) = D'$, and $f(B) = B'$. If X is any point constructed from the unprimed point as in Section V.5, let X' be the corresponding point constructed from the primed points. Since f is a collineation, it is easy to verify that $f(X) = X'$ for all point X constructed in Section V.5. In particular, $f(C) = C'$ and $f(K) = K'$. But the latter equalities combined with Theorem V.28 and V.29 imply that

$$\Phi_{f,X_0,X_1,U}(a + b) \;=\; \Phi_{f,X_0,X_1,U}(a) \;+\; \Phi_{f,X_0,X_1,U}(b) \;.$$

Proof that the map $\Phi_{f,X_0,X_1,U}$ *depends only on the line* M *containing* X_0, X_1 *and* U.   — It suffices to show that

$$\Phi_{f,X_0,X_1,U}\big(\mathsf{XR}(A,B,C,D)\big) \;=\; \mathsf{XR}\big(f(A),f(B),f(C),f(D)\big)$$

for every quadruple of distinct points on $X_0 X_1$. There are several cases to be considered.

CASE 1.   None of the points A, B, C, D is X_0. If we choose homogeneous coordinates ξ_i for X_i and ψ for U such that $\xi_0 + \xi_1 = \psi$, then homogeneous coordinates α, β, γ and δ for A, B, C and D are given as follows:

$$\begin{aligned}
\alpha &= \mathsf{XR}(X_0,X_1,U,A)\xi_0 \;+\; \xi_i \\
\beta &= \mathsf{XR}(X_0,X_1,U,B)\xi_0 \;+\; \xi_i \\
\gamma &= \mathsf{XR}(X_0,X_1,U,C)\xi_0 \;+\; \xi_i \\
\delta &= \mathsf{XR}(X_0,X_1,U,D)\xi_0 \;+\; \xi_i
\end{aligned}$$

and therefore we have

$$\Phi_{f,X_0,X_1,U}\big(\mathsf{XR}(A,B,C,D)\big) \;=\; \mathsf{XR}\big(f(A),f(B),f(C),f(D)\big)$$

by the formula established in Theorem V.13 and the fact that $\Phi_{f,X_0,X_1,U}$ is an isomorphism of fields.

CASE 2.   One of the points is X_0. We claim it suffices to consider the case $A = X_0$. For by Theorem V.12 there is a reordering $\big(\sigma(A),\sigma(B),\sigma(C),\sigma(D)\big)$ of (A,B,C,D) such that $\sigma(A) = X_0$ and

$$\mathsf{XR}\big(\sigma(A),\sigma(B),\sigma(C),\sigma(D)\big) \;=\; \mathsf{XR}(A,B,C,D)\ .$$

If the assertion is correct for quadruples whose first term is X_0, then

$$\Phi_{f,X_0,X_1,U}\Big(\big(\sigma(A),\sigma(B),\sigma(C),\sigma(D)\big)\Big) = \Big(\sigma\big(f(A)\big),\sigma\big(f(B)\big),\sigma\big(f(C)\big),\sigma\big(f(D)\big)\Big)\ .$$

Since the right hand side is equal to $\Phi_{f,X_0,X_1,U}\big(\mathsf{XR}(A,B,C,D)\big)$ and the right hand side is equal to $\mathsf{XR}\big(f(A),f(B),f(C),f(D)\big)$, the cases where X_0 is one of B, C or D follow.

By the preceding discussion, we might as well assume that $X_0 = A$ in Case 2. The remainder of the argument for Case 2 splits into subcases depending upon whether X_1 is equal to one of the remaining points.

SUBCASE 2.1.   Suppose that $A = X_0$ and $B = X_1$. Then by Theorem V.11 we have

$$\mathsf{XR}(A,B,C,D) \;=\; \frac{\mathsf{XR}(A,B,U,D)}{\mathsf{XR}(A,B,U,C)} \;=\; \frac{\mathsf{XR}(X_0,X_1,U,D)}{\mathsf{XR}(X_0,X_1,U,C)}\ .$$

Note that the cross ratio $\mathsf{XR}(A,B,U,C)$ is nonzero because $B \neq C$. The assertion in this case follows from the formula above and the fact that Φ is an automorphism.

SUBCASE 2.2.   Suppose that $A = X_0$ and $C = X_1$. Then $\mathsf{XR}(A,B,C,D) = 1 - \mathsf{XR}(A,C,B,D)$, and hence the assertion in this subcase follows from Subcase 2.1 and the fact that Φ is an automorphism.

SUBCASE 2.3.   Suppose that $A = X_0$ but neither B nor C is equal to X_1. Let

$$\begin{aligned}
b &= \mathsf{XR}(X_0,X_1,U,B) \\
c &= \mathsf{XR}(X_0,X_1,U,C) \\
d &= \mathsf{XR}(X_0,X_1,U,D)
\end{aligned}$$

so that homogeneous coordinates β, ξ_0 and ξ_1 for the points B, X_0 and X_1 can be chosen such that $\beta = b\xi_0 + \xi_1$, and hence the corresponding homogeneous coordinates $\gamma = c\xi_0 + \xi_1$ for C satisfy

$$\gamma \;=\; c\xi_0 \;+\; \xi_1 \;=\; (c - b)\xi_0 \;+\; (b\xi_0 + \xi_1)\;.$$

Since $B \neq C$, it follows that $c - b = 0$. Therefore homogeneous coordinates δ for D are given by

$$\delta \;=\; d\xi_0 \;+\; \xi_1 \;=\; \frac{d - b}{c - b}(c - b)\xi_0 \;+\; (b\xi_0 + \xi_1)\;.$$

Therefore we have the identity

$$\mathsf{XR}(A, B, C, D) \;=\; \frac{d - b}{c - b}\;.$$

The assertion in this subcase follows from the above formula and the fact that Φ is an isomorphism. This concludes the proof that Φ only depends upon the line $L = X_0 X_1$.

Proof that the isomorphism $\Phi_f = \Phi_{f,M}$ does not depend upon the choice of the line M. — Once again, there are two cases.

CASE 1.   Suppose we are given two lines M and M' which have a point in common; we claim that $\Phi_{f,M} = \Phi_{f,M'}$. Let V be a point in the plane of M and M' which is not on either line. If $X \in M$, let $X' \in M' \cap VX$; then

$$f(X') \;\in\; f[M'] \;\cap\; f(V)f(X)$$

because f is a collineation. Thus two applications of Theorem 15 imply

$$\mathsf{XR}(A, B, C, D) \;=\; \mathsf{XR}(A', B', C', D')$$
$$\mathsf{XR}\big(f(A), f(B), f(C), f(D)\big) \;=\; \mathsf{XR}\big(f(A'), f(B'), f(C'), f(D')\big)\;.$$

On the other hand, we also have

$$\Phi_{f,M}\big(\mathsf{XR}(A, B, C, D)\big) \;=\; \mathsf{XR}\big(f(A), f(B), f(C), f(D)\big) \text{ and}$$
$$\Phi_{f,M'}\big(\mathsf{XR}(A', B', C', D')\big) \;=\; \mathsf{XR}\big(f(A'), f(B'), f(C'), f(D')\big)\;.$$

Since every element of $\mathbb{F}$ has the form $\mathsf{XR}(A, B, C, D)$ for suitable points, the equations above imply that $\Phi_{f,M} = \Phi_{f,M'}$.

CASE 2.   Suppose we are given two lines M and M' which have no points in common; we claim that $\Phi_{f,M} = \Phi_{f,M'}$. Let M'' be a line joining one point in M to one point in M'. Then two applications of the first case imply that $\Phi_{f,M} = \Phi_{f,M''} = \Phi_{f,M'}$.■

The characterization of Φ_f in terms of the cross ratio implies some useful properties of the construction Φ which sends a collineation $\mathbb{F}\mathbb{P}^n \to \mathbb{E}\mathbb{P}^n$ to the field isomorphism $\Phi_f : \mathbb{F} \to \mathbb{E}$.

THEOREM VI.11.   *If $f : P \to P'$ and $f' : P' \to P''$ are collineations of coordinate projective n-spaces (where $n \geq 2$), then $\Phi_{gf} = \Phi_g \circ \Phi_f$. If $f : P \to P$ is the identity, then Φ_f is the identity on the underlying field. Finally, if $g : P' \to P$ is equal to f^{-1}, then $\Phi_g = (\Phi_f)^{-1}$.*

Proof.   If f is the identity, then we have

$$\Phi_f\big(\mathsf{XR}(A, B, C, D)\big) \;=\; \mathsf{XR}\big(f(A), f(B), f(C), f(D)\big) \;=\; \mathsf{XR}(A, B, C, D)$$

because $f(X) = X$ for all X. If g and f are collineations then

$$\Phi_g \circ \Phi_f\big(\mathsf{XR}(A, B, C, D)\big) \;=\; \Phi_g\Big(\mathsf{XR}\big(f(A), f(B), f(C), f(D)\big)\Big) \;=\;$$

$$\mathsf{XR}\big(g \circ f(A), g \circ f(B), g \circ f(C), g \circ f(D)\big) \;=\; \Phi_{gf}\big(\mathsf{XR}(A, B, C, D)\big) \; .$$

To prove that $\Phi_{f^{-1}} \;=\; (\Phi_f)^{-1}$, note that $f \circ f^{-1} \;=\;$ identity and $f^{-1} \circ f \;=\;$ identity combine with the first two identities to show that the composites $\Phi_{f^{-1}} \circ \Phi_f$ and $\Phi_f \circ \Phi_{f^{-1}}$ are both identity maps, and these identities imply that $\Phi_{f^{-1}} \;=\; (\Phi_f)^{-1}$.∎

Collineations of $\mathbb{F}\mathbb{P}^1$

Of course, an incidence-theoretic definition of collineations for coordinate projective lines is meaningless. However, if $1 + 1 \neq 0$ in $\mathbb{F}$, then as in Section V.4 it is possible to define collineations of $\mathbb{F}\mathbb{P}^1$ as 1–1 correspondences which preserve harmonic quadruples. With this definition, an analog of Theorem 10 is valid. Details appear on pages 85–87 of the book by Bumcrot listed in the bibliography (this is related to the discussion of von Staudt's Theorem at the end of Section V.4).

Examples

We have already noted that every invertible $(n+1) \times (n+1)$ matrix A over $\mathbb{F}$ defines a geometric symmetry f_A of $\mathbb{F}\mathbb{P}^n$, and by a straightforward extension of Exercise V.2.10 the mapping f_A preserves cross ratios; therefore, the automorphism Φ_{f_A} is the identity. On the other hand, if χ is an automorphism of $\mathbb{F}$ as above and g_χ is defined as at the beginning of this section, then for all distinct collinear points A, B, C, D in $\mathbb{F}\mathbb{P}^n$ we have

$$\chi\big(\mathsf{XR}(A, B, C, D)\big) \;=\; \mathsf{XR}\big(g_\chi(A), g_\chi(B), g_\chi(C), g_\chi(D)\big)$$

and therefore $\Phi_{g_\chi} \;=\; \chi$. In particular, the latter implies the following:

> For every field $\mathbb{F}$, every automorphism χ of $\mathbb{F}$, and every $n > 0$, there is a collineation g from $\mathbb{F}\mathbb{P}^n$ to itself such that $\Phi_g \;=\; \chi$.∎

Later in the section we shall prove a much stronger result of this type.

The Fundamental Theorem of Projective Geometry

Before stating and proving this result, we need to state and prove some variants of standard results from linear algebra. Let V and W be vector spaces over a field $\mathbb{F}$, and let α be an automorphism of $\mathbb{F}$. A mapping $T : V \to V$ is said to be an *α-semilinear transformation* if it satisfies the following conditions:

(1) $T(\mathbf{x} + \mathbf{y}) \;=\; T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}$, $\mathbf{y} \in V$.

(2) $T(c\mathbf{x} + \mathbf{y}) \;=\; \alpha(c)T(\mathbf{x})$ for all $\mathbf{x} \in V$ and $c \in \mathbb{F}$.

If α is the identity mapping, this reduces to the usual definition of a linear transformation.

THEOREM VI.12. *Let V, W, $\mathbb{F}, \alpha$ be as above. If $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is a basis for V and $\mathbf{w}_1, \cdots, \mathbf{w}_n \in W$, then there is a unique α-semilinear transformation $T : V \to W$ such that $T(\mathbf{v}_i) \;=\; \mathbf{w}_i$ for all i.*

Proof. Suppose that $v \in V$ and that T and S are α-semilinear transformations from V to W satisfying the conditions of the theorem. Write $\mathbf{v}$ as a linear combination $\sum_i c_i \mathbf{v}_i$. Then we have

$$T(\mathbf{v}) \; = \; T\left(\sum_i c_i \mathbf{v}_i\right) \; = \; \sum_i \alpha(c_i) T(\mathbf{v}_i) \; = \; \sum_i \alpha(c_i)\mathbf{w}_i \; =$$

$$\sum_i \alpha(c_i)S(\mathbf{v}_i) \; = \; S\left(\sum_i c_i \mathbf{v}_i\right) \; = \; S(\mathbf{v})$$

and hence $S = T$. Conversely, if $\mathbf{v}$ is given as above, then $T(\mathbf{v}) \; = \; \sum_{\mathbf{i}} \alpha(\mathbf{c_i})\mathbf{w_i}$ is a well-defined α-semilinear transformation.■

This result has the following basic consequence:

THEOREM VI.13. *In the setting above, the mapping T is $1-1$ and onto if and only if the vectors $\mathbf{w}_1, \; \cdots, \mathbf{w}_n$ form a basis for W. In this case the inverse map T^{-1} is an α^{-1}-semilinear transformation.*

Proof. Since the image of T is contained in the subspace spanned by the vectors $\mathbf{w}_i$, it follows that T cannot be onto if these vectors do not span W. Conversely, suppose that these vectors do form a basis. Then by the previous result there is an α^{-1}-semilinear transformation $S : W \to V$ such that $S(\mathbf{w}_i) \; = \; \mathbf{v}_i$ for all i. It follows that $S \circ T$ is an $\alpha^{-1} \circ \alpha$-semilinear (hence linear) transformation from V to itself which sends $\mathbf{v}_i$ to $\mathbf{v}_i$ for all i, and hence $S \circ T$. Reversing the roles of V and W and also the roles of S and T in this argument, we conclude similarly that $T \circ S$ is the identity. Therefore the α-semilinear map S is an inverse to T and the latter is 1–1 and onto.■

If $\mathbb{F}$ and n are as in Theorem 12 and T is an invertible α-semilinear transformation from $\mathbb{F}^{n+1,1}$ to itself, then as in Section 4.3 there is a collineation f_T from $\mathbb{FP}^n$ to itself defined by

$$f_T(X) \; = \; T(\xi) \cdot \mathbb{F}$$

where ξ is an arbitrary set of homogeneous coordinates for X; this does not depend upon the choice of homogeneous coordinates, for if $\xi' = c\xi$ is another set of homogeneous coordinates for X we have

$$T(c\xi) \cdot \mathbb{F} \; = \; \alpha(c) \cdot T(\xi) \cdot \mathbb{F} \; = \; T(\xi) \cdot \mathbb{F} \; .$$

The proof that this map defines a collineation proceeds exactly as in the case of linear transformations, the only change being the need to substitute $T(c\mathbf{v}) \; = \; \alpha(c) \cdot T(\mathbf{v})$ in place of $T(c\mathbf{v}) \; = \; c \cdot T(\mathbf{v})$ when the latter appears.

The Fundamental Theorem of Projective Geometry is a converse to the preceding construction, and it shows that every collineation of $\mathbb{FP}^n$ to itself has the form f_T for a suitably chosen invertible α-linear mapping T from $\mathbb{F}^{n+1,1}$ to itself.

THEOREM VI.14. (Fundamental Theorem of Projective Geometry)   *Let $\{X_0, \; \cdots, X_n, A\}$ and $\{Y_0, \; \cdots, Y_n, B\}$ be two sets of $(n+2)$ points in $\mathbb{FP}^n$ (where $n \geq 2$) such that no proper subset of either is dependent, and let χ be an automorphism of $\mathbb{F}$. Then there is a unique collineation f of $\mathbb{FP}^n$ to itself satisfying the following conditions:*

(i) $f(X_i) \; = \; Y_i$ for $0 \leq i \leq n$.

(ii) $f(A) \; = \; B$.

(iii) $\Phi_f \; = \; \chi$.

The theorem (with the proof given here) is also valid if $n = 1$ and $1 + 1 \neq 0$ in $\mathbb{F}$, provided collineations of $\mathbb{FP}^1$ are defined in the previously described manner (*i.e.*, preserving harmonic quadruples).

Proof.   EXISTENCE. According to Theorem 8 we can choose homogeneous coordinates ξ for X_i, η_i for Y_i, α for A, and β for B so that $\alpha = \sum_i \xi_i$ and $\beta = \sum_i \eta_i$. The hypotheses imply that the vectors ξ_i and η_i form bases for $\mathbb{F}^{n+1,1}$, and therefore there is an invertible α-semilinear transformation of the latter such that $T(\xi_i) = \eta_i$ for all i. Then f_T is a collineation of $\mathbb{FP}^n$ sending X_i to Y_i and A to B. In order to compute the automorphism induced by f, let Q_1, Q_2, Q_3, Q_4 be distinct collinear points with homogeneous coordinates θ_i for Q_i chosen such that $\theta_3 = \theta_1 + \theta_2$ and $\theta_4 = q\theta_1 + \theta_2$, where $q = \mathsf{XR}(Q_1, Q_2, Q_3, Q_4)$. We then have $T(\theta_3) = {}_T(\theta_1) +_T (\theta_2)$ and $T(\theta_4) = \chi(q) \cdot T(\theta_1) + T(\theta_2)$, so that

$$\chi\Big(\mathsf{XR}\big(Q_1, Q_2, Q_3, Q_4 \big) \Big) = \mathsf{XR}\big(f_T(Q_1), f_T(Q_2), f_T(Q_3), f_T(Q_4) \big) .$$

It follows that $\Phi_{f_T} = \chi.\square$

UNIQUENESS. Suppose that f and g are collineations of $\mathbb{FP}^n$ which satisfy $f(X_i) = g(X_i) = Y_i$ for $0 \leq i \leq n$, $f(A) = g(A) = B$, and $\Phi_f = \Phi_g = \chi$. Then $h = g^{-1} \circ f$ is a collineation which satisfies $h(X_i) = X_i$ for $0 \leq i \leq n$, $h(A) = A$, and Φ_h is the identity. If suffices to show that a collineation h satisfying these conditions must be the identity.

Let h be a collineation such that Φ_h is the identity, and suppose that h leaves three distinct points on a line fixed; we claim that h leaves every point on the line fixed. To see this, suppose that X_1, X_2 and X_3 are distinct collinear points such that $h(X_i) = X_i$ for all i, and let $Y \in X_1 X_2$. Then we have

$$\mathsf{XR}\big(X_1, X_2, X_3, h(Y) \big) = \Phi_h\big(\mathsf{XR}(X_1, X_2, X_3, Y) \big) = \mathsf{XR}(X_1, X_2, X_3, Y)$$

and hence $h(Y) = Y$ by Theorem V.10.

Now assume that h satisfies the conditions in the first paragraph of this argument. We shall prove, by induction on r, that h fixes every point in the r-plane $X_0 \cdots X_r$. The statement is trivially true for $r = 0$, so assume that it is true for $r - 1$, where $r \geq 1$. By the dimension formula, the intersection of the subspaces $X_0 \cdots X_r$ and $A X_{r+1} \cdots X_n$ is a point which we shall call B_r. In fact, homogeneous coordinates β_r for B_r are given by

$$\beta_r = \xi_0 + \cdots + \xi_r$$

because the right hand side is set of homogeneous coordinates for a point in the intersection. Since $h(X_i) = X_i$ and $h(A) = A$, it follows that h maps the subspaces $X_0 \cdots X_r$ and $A X_{r+1} \cdots X_n$ into themselves. Thus the intersection of these subspaces (namely, the one point set $\{B_r\}$) must be mapped into itself and hence $h(B_r) = B_r$.

We claim that h fixes every point on the line $X_r B_r$ fixed. By hypothesis $h(X_r) = X_r$, and by the preceding paragraph $h(B_r) = B_r$. Hence h maps $X_r B_r$ into itself. Since $X_r B_r$ and $X_0 \cdots X_{r-1}$ are both contained in $X_0 \cdots X_r$, the dimension formula implies that they intersect in a point W. Since $W \in X_0 \cdots X_{r-1}$, the induction hypothesis implies that $h(W) = W$. Homogeneous coordinates ω for W are given by

$$\omega = \beta_r - \xi_r = \xi_0 + \cdots + \xi_{r-1}$$

and hence the points X_r, V_r and W are distinct collinear points. Since each is left fixed by h, it follows that every other point in $X_r B_r$ is also left fixed by h.
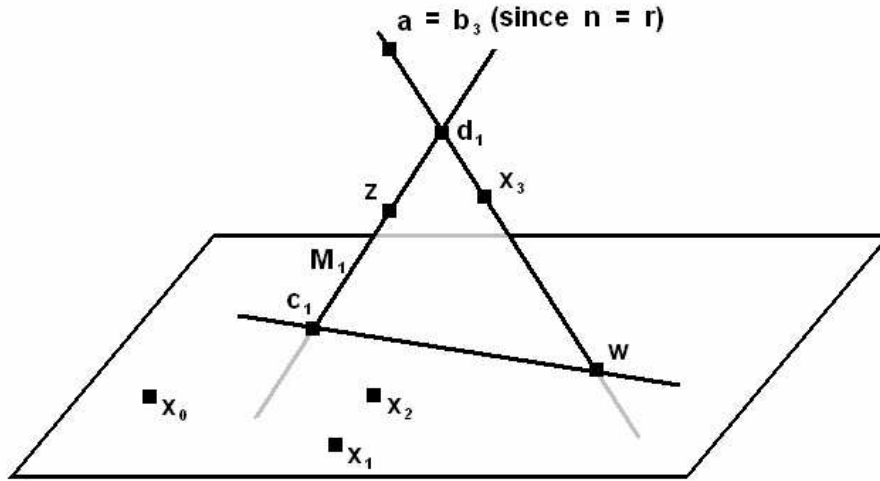
Figure VI.1

$(r = n = 3)$

Now let Z be any point of $X_0 \cdots X_r$ not on either $X_0 \cdots X_{r-1}$ or $X_r B_r$. We claim that $h(Z) = Z$. Since $X_r B_r$ and $X_0 \cdots X_{r-1}$ are both contained in $X_0 \cdots X_r$, the dimension formula implies that $X_0 \cdots X_{r-1}$ and the plane $Z X_r B_r$ intersect in a line we shall call L. The assumption that $Z \notin X_0 \cdots X_{r-1}$ implies that $Z \notin L$.

Let M_1 and M_2 be two distinct lines in $Z X_r B_r$ containing Z; since there are at least three lines in the plane $Z X_r B_r$ containing Z, we may choose M_1 and M_2 such that neither line contains the point B_r in which L meets $X_r B_r$; in particular, this means that the intersections of M_i with L and $X_r B_r$ are distinct points.

Let $S_i \in M_i \cap L$, and let $D_i \in M_2 \cap X_r B_r$ (here $i = 1$ or 2). Then $h(C_i) = C_i$ and $h(D_i) = D_i$. Since the intersections of M_i with L and $X_r B_r$ are distinct points, it follows that h leaves two distinct points of M_i fixed and hence h maps each line M_i into itself. Therefore it also follows that h maps $M_1 \cap M_2 = \{Z\}$ into itself, so that $h(Z) = Z$.

The preceding argument shows that h leaves every point of $X_0 \cdots X_r$ fixed, completing the inductive step of the argument. Therefore, by induction we conclude that h is the identity on $\mathbb{FP}^n = X_0 \cdots X_n$.∎

One immediate consequence of the Fundamental Theorem is particularly worth stating at this point:

THEOREM VI.15. *Let f be a collineation of $\mathbb{FP}^n$, and let $\Phi_f = \alpha$. Then there is an invertible α-semilinear transformation T of $\mathbb{F}^{n+1,1}$ such that if $X \in \mathbb{FP}^n$ and ξ is a set of homogeneous coordinates for X then $f(X) = T(\xi) \cdot \mathbb{F}$.*

Proof. Let $\{X_0, \cdots, X_n, A\}$ be a set of of $(n+2)$ points in $\mathbb{FP}^n$ such that no proper subset is dependent. By the proof of existence in the Fundamental Theorem there is an invertible α-semilinear transformation T such that the associated collineation f_T satisfies the following conditions:

(i) $f(X_i) = f_T(X_i)$ for $0 \le i \le n$.

(ii) $f(A) = f_T(A)$.

Also, by construction the maps f and f_T determine the same automorphism of $\mathbb{F}$. We may now apply the uniqueness portion of the Fundamental Theorem to conclude that $f = f_T$.■

Definition. A collineation f of $\mathbb{FP}^n$ is *projective* if the associated automorphism Φ_f is the identity. Theorem 11 implies that the set of projective collineations is a subgroup — in fact, a normal subgroup — of the collineation group, and by the previous construction of examples we know that the quotient of the collineation group by the subgroup of projective collineations is equal to the automorphism group of $\mathbb{F}$. Further information along these lines is discussed in Exercise 9 below.■

<div align="center">Special cases</div>

We conclude this section with some remarks on collineations if $\mathbb{F}$ is the real or complex numbers.

THEOREM VI.16. *For each $n \ge 2$, every collineation of real projective n-space $\mathbb{RP}^n$ is projective.*

By the previous results of this section, the proof of Theorem 15 reduces to showing the following:

THEOREM VI.17. *The only automorphism of the real numbers is the identity map.*

Proof(s). If χ is an isomorphism of $\mathbb{R}$, then $\chi(0) = 0$ and $\chi(0) = 1$. Suppose $\chi(r) = r$ for $r \ge 1$. Then $\chi(r+1) = \chi(r) + \chi(1) = r + 1$, and hence χ agrees with the identity on all nonnegative integers. If k is a negative integer and $k = -m$, then

$$\chi(k) = \chi(-m) = -\chi(m) = -m = k$$

and hence χ is the identity on integers. If r is a rational number, write $r = m/n$ where m is an integer and n is a positive integer. Then $n = rm$ implies that

$$m = \chi(m) = \chi(nr) = \chi(n) \cdot \chi(r) = n \cdot \chi(r)$$

which implies that $\chi(r) = m/n = r$, and hence we see that χ fixes every rational number.

Suppose now that x is an arbitrary nonnegative real number. We claim that $\chi(x) \ge 0$. Recall that $x \ge 0$ if and only if $x = y^2$ for some y. Therefore $x \ge 0$ and $x = y^2$ imply that $\chi(x) = \chi(y)^2 \ge 0$. Similarly, if a and b are real numbers such that $a \ge b$, then

$$\chi(a) - \chi(b) = \chi(a - b) \ge 0$$

implies that $\chi(a) \ge \chi(b)$. Since χ is 1–1 it also follows that $a > b$ implies $\chi(a) > \chi(b)$.

Finally, suppose that we have an element $r \in \mathbb{R}$ such that $\chi(r) \ne r$. If $\chi(r) < r$, then there is a rational number q such that $\chi(r) < q < r$. But this implies $\chi(r) < \chi(q) = q$, and this contradicts the conclusion $\chi(r) > \chi(q)$ which follows from the previous paragraph. Therefore $\chi(r) < r$ is impossible, so that $\chi(r) \ge r$ for all real numbers r.

Now χ^{-1} is also an automorphism of $\mathbb{R}$, and if we apply the previous reasoning to this automorphism we conclude that $\chi^{-1}(r) \geq r$ for all r. Since we had previously shown that automorphisms are strictly increasing functions, if we apply χ to the previous inequality we obtain

$$r \;=\; \chi \circ \chi^{-1}(r) \;\geq\; \chi(r)$$

and if we combine this with the final inequality of the preceding paragraph we conclude that $\chi(r) = r$ for all real numbers r.∎

The analog of Theorem 16 does not hold for the complex numbers. In particular, at the beginning of this section we showed that the map g_χ of $\mathbb{CP}^n$ given by conjugating homogeneous coordinates is a collineation that is not projective. Further information on automorphisms of the complex numbers and their applications to projective geometry appears in Appendix D.

EXERCISES

In the problems below, assume that $\mathbb{F}$ is a field and χ is an automorphism of $\mathbb{F}$.

1. Let A be an invertible $n \times n$ matrix over $\mathbb{F}$, and let f_A be the projective collineation of $\mathbb{FP}^n$ defined by A (in other words, if ξ are homogeneous coordinates for X, then $A\xi \cdot \mathbb{F}$ are homogeneous coordinates for $f_A(X)$). If H is a hyperplane in $\mathbb{FP}^n$ with homogeneous coordinates θ, prove that $T[H]$ has homogeneous coordinates $\theta \cdot A^{-1}$ (compare Exercise V.1.5).

2. In the notation of Exercise 1, suppose that a collineation T is defined such that if ξ are homogeneous coordinates for X, then $A\chi(\xi)$ are homogeneous coordinates for $T(X)$. Express homogeneous coordinates for $T[H]$ in terms of θ, A^{-1} and χ. You may use the product formula $\chi(A \cdot B) \;=\; \chi(A) \cdot \chi(B)$ for matrix multiplication. Also, recall that $\chi(0) \;=\; 0$.

3. Suppose that f is a collineation of $\mathbb{FP}^n$ with induced automorphism Φ_f, and suppose that H_1, H_2, H_3, H_4 are distinct hyperplanes containing a common $(n-2)$-plane. Prove that the cross ratio formula

$$\Phi_f\big(\mathsf{XR}(H_1, H_2, H_3, H_4)\big) \;=\; \mathsf{XR}\big(f[H_1], f[H_2], f[H_3], f[H_4]\big)$$

holds without using Exercise 2.

4. Suppose that T is an invertible χ-semilinear transformation of $\mathbb{F}^{n+1,1}$ where $n \geq 1$ such that the associated collineation f_T of $\mathbb{FP}^n$ is the identity. Prove that T is a scalar multiple of the identity. [*Hint:* By assumption, for each nonzero vector $\mathbf{x}$ there is a nonzero scalar $c_\mathbf{x}$ such that $T(\mathbf{x}) \;=\; c_\mathbf{x} \cdot \mathbf{x}$. If $c_\mathbf{x} \neq c_\mathbf{y}$, explain why $\mathbf{x}$ and $\mathbf{y}$ must be linearly independent. Consider $T(\mathbf{x} + \mathbf{y})$ in this case.]

5. (*a*) Let T be an invertible χ-semilinear transformation of $\mathbb{F}^n$ where $n \geq 1$, and let $\mathbf{z} \in \mathbb{F}^n$. Show that

$$G(\mathbf{x}) \;=\; T(\mathbf{x}) \;+\; \mathbf{z}$$

is a geometric symmetry of the affine incidence n-space $\mathbb{F}^n$. [*Hint:* Compare this statement to the examples following Theorem II.39.]

(*b*) Prove that G extends to a collineation g of $\mathbb{FP}^n$ for which $\Phi_g \;=\; \chi$; in other words, we have $g \circ \mathsf{J} \;=\; \mathsf{J} \circ G$ on $\mathbb{F}^n$. [*Hint:* Compare Exercise IV.4.14.]

(c) If $n \geq 2$, prove that every geometric symmetry f of $\mathbb{F}^n$ is given by a transformation of the type described in (a). [*Hint:* By Exercise 2 at the end of Chapter IV, the map f extends to a collineation g of $\mathbb{FP}^n$. Since the collineation leaves the hyperplane at infinity fixed, certain entries of an $(n+1) \times (n+1)$ matrix inducing g must vanish. But this implies the matrix has the form of one constructible by (b).]

(d) Determine whether $\mathrm{AFF}(\mathbb{F}^n)$ is the entire group of geometric symmetries of $\mathbb{F}^n$ when $\mathbb{F}$ is the real and complex numbers respectively.

6. Suppose that A is an invertible $m \times m$ matrix over a field $\mathbb{F}$ such that $1+1 \neq 0$ in $\mathbb{F}$. Prove that $\mathbb{F}^{m,1}$ contains two vector subspaces W_+ and W_- with the following properties:

(i) $A\mathbf{x} = \mathbf{x}$ if $\mathbf{x} \in W_+$ and $A\mathbf{x} = -\mathbf{x}$ if $\mathbf{x} \in W_-$.

(ii) $W_+ + W_- = \mathbb{F}^{m,1}$ and $W_+ \cap W_- = \{\mathbf{0}\}$.

[*Hint:* Let $W_\pm$ be the image of $A \pm I$. This yields the first part. To prove the rest, use the identity
$$I = \tfrac{1}{2}(A+I) - \tfrac{1}{2}(A-I) .]$$

> **Definition.** An *involution* of $\mathbb{FP}^n$ is a collineation f such that $f \circ f$ is the identity but f itself is not the identity. If $f(X) = X$, then X is called a **fixed point** of the involution.

7. (a) Let T be an invertible χ-semilinear transformation of $\mathbb{F}^{n+1,1}$ such that the induced collineation f_T of $\mathbb{FP}^n$ is an involution. Prove that T^2 is a scalar multiple of the identity. [*Hint:* Use Exercise 4.]

(b) Suppose that T is an involution of $\mathbb{RP}^n$. Prove that T is induced by an invertible $(n+1) \times (n+1)$ matrix A such that $A^2 = \pm 1$.

(c) In the previous part, prove that T has no fixed points if $A^2 = -I$. Using Exercise 6, prove that T has fixed points if $A^2 = 1$. [*Hint:* For the first part, suppose that X is a fixed point with homogeneous coordinates ξ such that $A \cdot \xi = c \cdot \xi$ for some real number c. However, $A^2 = -I$ implies that $c^2 = -1$.] — NOTATION. An involution is called *elliptic* if no fixed points exist and *hyperbolic* if fixed points exist.

(d) Using Exercise 6, prove that the fixed point set of a hyperbolic involution of $\mathbb{RP}^n$ has the form $Q_1 \cup Q_2$, where Q_1 and Q_2 are disjoint n_1- and n_2-planes and $n_1 + n_2 + 1 = n$.

8. Suppose that $A \neq B$, and that A and B are the only two points of the line AB left fixed by an involution f of $\mathbb{RP}^n$. Prove that $\mathsf{XR}\big(A, B, C, f(C)\big) = -1$ for all points C on $AB - \{A, B\}$. [*Hint:* Find an equation relating $\mathsf{XR}\big(A, B, C, f(C)\big)$ and $\mathsf{XR}\big(A, B, f(C), C\big)$.]

9. Let $\mathrm{COLL}(\mathbb{FP}^n)$ denote the group of all collineations of $\mathbb{FP}^n$, let $\mathrm{AUT}(\mathbb{F})$ denote the group of (field) automorphisms of $\mathbb{F}$, and let $\Phi : \mathrm{COLL}(\mathbb{FP}^n) \to \mathrm{AUT}(\mathbb{F})$ denote the homomorphism given by Theorem VI.10.

(a) Why is the kernel of Φ the group $\mathrm{PROJ}(\mathbb{FP}^n)$ of all projective collineations, and why does this imply that the latter is a normal subgroup of $\mathrm{COLL}(\mathbb{FP}^n)$?

(b) Show that $\mathrm{COLL}(\mathbb{FP}^n)$ contains a subgroup Γ isomorphic to $\mathrm{AUT}(\mathbb{F})$ such that the restricted homomorphism $\Phi|\Gamma$ is an isomorphism, and using this prove that every element of $\mathrm{COLL}(\mathbb{FP}^n)$ is expressible as a product of an element in $\mathrm{PROJ}(\mathbb{FP}^n)$ with an element in Γ. [*Hint:* Look at the set of all collineations of the form g_χ constructed at the top of the second page of this

section, where $\chi \in \mathrm{AUT}(\mathbb{F})$, and show that the set of all such collineations forms a subgroup of $\mathrm{COLL}\,(\mathbb{F}\mathbb{P}^n)$ which is isomorphic to $\mathrm{AUT}(\mathbb{F})$.]

(c) Suppose that A is an invertible $(n+1) \times (n+1)$ matrix over $\mathbb{F}$ and χ is an automorphism of $\mathbb{F}$, and let f_A and g_χ be the collineations of $\mathbb{F}\mathbb{P}^n$ defined at the beginning of this section. By the previous parts of this exercise and Theorem 15, we know that $g_\chi \circ f_A \circ (g_\chi)^{-1}$ has the form f_B for some invertible $(n+1) \times (n+1)$ matrix B over $\mathbb{F}$. Prove that we can take B to be the matrix $\chi(A)$ obtained by applying χ to each entry of A. [*Note:*   As usual, if two invertible matrices are nonzero scalar multiples of each other then they define the same projective collineation, and in particular we know that $f_B = f_{cB}$ for all nonzero scalars c; this is why we say that we take B to be equal to $\chi(A)$ and not that B is equal to $\chi(A)$.]

10.   Let $\mathbb{F}$ be a field, let $0 < r \le n$ where $n \ge 2$, let Q be an r-plane in $\mathbb{F}\mathbb{P}^n$. Let $\{X_0, \cdots, X_r, A\}$ and $\{Y_0, \cdots, Y_r, B\}$ be two sets of $(r+2)$ points in Q such that no proper subset of either is dependent. Then there is a projective collineation f of $\mathbb{F}\mathbb{P}^n$ to itself such that $f(X_i) = Y_i$ for $0 \le i \le r$ and $f(A) = B$. [*Hint:*   Let W be the vector subspace of $\mathbb{F}^{n+1,1}$ such that $Q = \mathcal{S}_1(W)$, define an invertible linear transformation G on W which passes to a projective collineation of Q with the required properties as in the proof of the Fundamental Theorem, extend G to an invertible linear transformation $\overline{G}$ of $\mathbb{F}^{n+1,1}$, and consider the projective collineation associated to $\overline{G}$.]

11.   (i) If T is a projective collineation of the complex projective space $\mathbb{C}\mathbb{P}^n$, prove that there is a point $\mathbf{x} \in \mathbb{C}\mathbb{P}^n$ such that $T(\mathbf{x}) = \mathbf{x}$ (*i.e.*, a **fixed point**). [*Hint:*   Every square matrix A over the complex numbers has a nonzero *eigenvector* $\mathbf{y}$ such that $A(\mathbf{y}) = c\mathbf{y}$ for some scalar $c \in \mathbb{C}$.]

(ii) Explain why a similar result holds for the real projective space $\mathbb{R}\mathbb{P}^n$ if n is even. [*Hint:* Every square matrix A over the reals with an odd number of rows and columns has a nonzero real eigenvector $\mathbf{y}$ such that $A(\mathbf{y}) = c\mathbf{y}$ for some scalar $c \in \mathbb{R}$.]

> REMARK. The corresponding result for $\mathbb{R}\mathbb{P}^{2n+1}$ is false. An explicit example is
> defined by the linear transformation of $\mathbb{R}^{2n+2}$ sending $(x_1, y_1, \cdots, x_{n+1}, y_{n+1})$
> to $(y_1, -x_1, \cdots, y_{n+1}, -x_{n+1})$.

4. Order and separation

All of the analytic projective geometry done up to this point is valid for an arbitrary $\mathbb{F}$ for which $1 + 1 \neq 0$. Certainly one would expect that real projective spaces have many properties not shared by other coordinate projective spaces just as the field of real numbers has many properties not shared by other fields. The distinguishing features of the real numbers are that it is an ordered field and is complete with respect to this ordering. In this section we shall discuss some properties of projective spaces over arbitrary ordered fields and mention properties that uniquely characterize real projective spaces.

Given points $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^n$, let $d(\mathbf{u}, \mathbf{v})$ be the distance from $\mathbf{u}$ to $\mathbf{v}$. One characterization of the statement

$$\mathbf{y} \text{ is between } \mathbf{x} \text{ and } \mathbf{z}$$

is that it holds if and only if $d(\mathbf{x}, \mathbf{z}) = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$. Another more algebraic characterization follows immediately from this.

THEOREM VI.18. *If* $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in \mathbb{R}^n$ *are distinct points, then* $d(\mathbf{x}, \mathbf{z}) = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ *holds if and only if* $\mathbf{y} = t\mathbf{x} + (1 - t)\mathbf{z}$ *for some* t *satisfying* $0 < t < 1$.

Proof.   Recall that $d(\mathbf{u}, \mathbf{v})$ is the square root of

$$(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = |\mathbf{u} - \mathbf{v}|^2 .$$

The proof of the Triangle Inequality for inner (or dot) products is a consequence of the Cauchy-Schwarz inequality

$$(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{y} - \mathbf{z}) \leq \left| (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{y} - \mathbf{z}) \right| \leq |\mathbf{x} - \mathbf{y}| \cdot |\mathbf{y} - \mathbf{z}|$$

and equality holds in the Triangle Inequality if and only if the end terms of the Cauchy-Schwarz inequality are equal.[2] However, the Cauchy-Schwarz inequality states that the middle term and right hand term are equal if and only if $\mathbf{x} - \mathbf{y}$ and $\mathbf{y} - \mathbf{z}$ are linearly dependent. Since both are nonzero, this means that $(\mathbf{y} - \mathbf{z}) = k(\mathbf{x} - \mathbf{y})$ for some $k \neq 0$. On the other hand, the left and right hand terms are equal if and only if both are nonnegative. Consequently, if the end terms are equal, then $(\mathbf{y} - \mathbf{z}) = k(\mathbf{x} - \mathbf{y})$ and also $k|\mathbf{x} - \mathbf{y}| \geq 0$. This implies that k must be positive. Conversely, if $k > 0$ then the end terms of the Cauchy-Schwarz inequality are equal.

Thus $d(\mathbf{x}, \mathbf{z}) = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ if and only if $\mathbf{y} - \mathbf{z}$ is a positive multiple of $\mathbf{x} - \mathbf{y}$. But if $\mathbf{y} - \mathbf{z} = k(\mathbf{x} - \mathbf{y})$, then

$$\mathbf{y} = \frac{k}{k+1} \mathbf{x} + \frac{1}{k+1} \mathbf{z} .$$

Since $k > 0$ implies

$$0 < \frac{k}{k+1} < 1$$

it follows that if $d(\mathbf{x}, \mathbf{z}) = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ then $\mathbf{y} = t\mathbf{x} + (1 - t)\mathbf{z}$ for some t satisfying $0 < t < 1$.

Conversely, if $\mathbf{y} = t\mathbf{x} + (1 - t)\mathbf{z}$ for some t satisfying $0 < t < 1$, then

$$\mathbf{y} - \mathbf{z} = \frac{t}{1 - t} (\mathbf{x} - \mathbf{y}) .$$

---

[2]See pp. 177–178 of Birkhoff and MacLane or pp. 277–278 of Hoffman and Kunze for further details.

Since $t/(1-t)$ is positive if $0 < t < 1$, it follows that $d(\mathbf{x}, \mathbf{z}) = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$.■

With this motivation, we define betweenness for arbitrary vector spaces over arbitrary ordered fields.

Definition.   Let $\mathbb{F}$ be an ordered field, let V be a vector space over $\mathbb{F}$, and let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be distinct points of V. We shall say that $\mathbf{y}$ is **between** $\mathbf{x}$ and $\mathbf{z}$, written $\boldsymbol{\mathcal{B}}(\mathbf{x}, \mathbf{y}, \mathbf{z})$, if there is some $t \in \mathbb{F}$ such that $0 < t < 1$ and $\mathbf{y} = t\mathbf{x} + (1-t)\mathbf{z}$. Frequently we shall also say that the points $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ satisfy the *ordering relation* $\boldsymbol{\mathcal{B}}(\mathbf{x}, \mathbf{y}, \mathbf{z})$. The *closed segment* $[\mathbf{x}; \mathbf{z}]$ consists of $\mathbf{x}$, $\mathbf{z}$, and all $\mathbf{y}$ such that $\mathbf{y}$ is between $\mathbf{x}$ and $\mathbf{z}$. In Exercise 1 below this is compared to the usual definition of closed interval in $\mathbb{R}$.
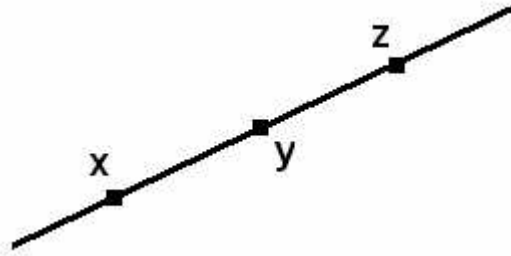
Figure VI.2

The *open segment* $(\mathbf{x}; \mathbf{z})$ consists of all $\mathbf{y}$ such that $\mathbf{y}$ is between $\mathbf{x}$ and $\mathbf{z}$.

The next results show that our definition of betweenness satisfies some properties that are probably very apparent. However, since we are dealing with a fairly abstract setting, it is necessary to give rigorous proofs.

THEOREM VI.19. *Let $\mathbb{F}$ be an ordered field, let V be a vector space over $\mathbb{F}$, and let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be distinct vectors in V. If $\boldsymbol{\mathcal{B}}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is true, then so is $\boldsymbol{\mathcal{B}}(\mathbf{c}, \mathbf{b}, \mathbf{a})$. However, each of the four statements $\boldsymbol{\mathcal{B}}(\mathbf{b}, \mathbf{a}, \mathbf{c})$, $\boldsymbol{\mathcal{B}}(\mathbf{c}, \mathbf{a}, \mathbf{b})$, $\boldsymbol{\mathcal{B}}(\mathbf{a}, \mathbf{c}, \mathbf{b})$, $\boldsymbol{\mathcal{B}}(\mathbf{b}, \mathbf{c}, \mathbf{a})$ is false.*

Proof. By assumption $\mathbf{b} = t\mathbf{a} + (1-t)\mathbf{c}$ for some t satisfying $0 < t < 1$. The latter inequalities imply $0 < (1-t) < 1$, and since $t = 1 - (1-t)$ it follows that $\boldsymbol{\mathcal{B}}(\mathbf{c}, \mathbf{b}, \mathbf{a})$ is true.

The equation $\mathbf{b} = t\mathbf{a} + (1-t)\mathbf{c}$ (where $0 < t < 1$) implies that $-t\mathbf{a} = -\mathbf{b} + (1-t)\mathbf{c}$, which in turn means that
$$t^{-1}\mathbf{b} - t^{-1}(1-t)\mathbf{c} = t^{-1}\mathbf{b} + (1 - t^{-1})\mathbf{c} \ .$$
Therefore $\mathbf{a} = s\mathbf{b} + (1-s)\mathbf{c}$ then implies $s = t^{-1}$. Since $0 < t < 1$ implies $t^{-1} > 1$, it follows that $\boldsymbol{\mathcal{B}}(\mathbf{b}, \mathbf{c}, \mathbf{a})$ is false. Furthermore, it also follows that $\boldsymbol{\mathcal{B}}(\mathbf{a}, \mathbf{c}, \mathbf{b})$ is false, for if the latter were true then by the preceding paragraph the order relation $\boldsymbol{\mathcal{B}}(\mathbf{b}, \mathbf{c}, \mathbf{a})$ would also be true.

Finally, $\mathbf{b} = t\mathbf{a} + (1-t)\mathbf{c}$ (where $0 < t < 1$) implies that $(t-1)\mathbf{c} = t\mathbf{a} - \mathbf{b}$, which in turn implies that
$$\mathbf{c} = \frac{t}{t-1}\mathbf{a} + \frac{-1}{t-1}\mathbf{b} \ .$$
Now $0 < t < 1$ implies $t - 1 < 0$, so that
$$\frac{t}{t-1} < 0 \ .$$

The latter means that $\mathcal{B}(\mathbf{c}, \mathbf{a}, \mathbf{b})$ is false, and as in the previous paragraph it follows that $\mathcal{B}(\mathbf{a}, \mathbf{c}, \mathbf{b})$ is also false.■

THEOREM VI.20. *Let $\mathbb{F}$ and V be as above, and let $\mathbf{a}$ and $\mathbf{b}$ be distinct vectors in V. Then $\mathbf{c} \in V$ lies on the line $\mathbf{ab}$ if and only if one of $\mathbf{c} = \mathbf{a}$, $\mathbf{c} = \mathbf{b}$, $\mathcal{B}(\mathbf{a}, \mathbf{b}, \mathbf{c})$, $\mathcal{B}(\mathbf{c}, \mathbf{a}, \mathbf{b})$ or $\mathcal{B}(\mathbf{b}, \mathbf{c}, \mathbf{a})$ is true. Furthermore, these conditions are mutually exclusive.*

Proof.   We know that $\mathbf{c} \in \mathbf{ab}$ if and only if $\mathbf{c} = t\mathbf{a} + (1 - t)\mathbf{b}$ for some t. We claim the five conditions are equivalent to $t = 1$, $t = 0$, $t < 0$, $t > 1$ and $0 < t < 1$ respectively. Thus it will suffice to verify the following:

  (1) $\mathcal{B}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is true if and only if $\mathbf{c} = t\mathbf{a} + (1 - t)\mathbf{b}$ for some $t < 0$.

  (2) $\mathcal{B}(\mathbf{c}, \mathbf{a}, \mathbf{b})$ is true if and only if $\mathbf{c} = t\mathbf{a} + (1 - t)\mathbf{b}$ for some $t > 1$.

PROOF OF (1).   The condition $\mathbf{c} = t\mathbf{a} + (1 - t)\mathbf{b}$ with $t < 0$ is equivalent to

$$\mathbf{b} = \frac{t}{t - 1} \mathbf{a} + \frac{-1}{t - 1} \mathbf{c} .$$

The conclusion in this case follows because the map sending t to $t/(t - 1)$ is a 1–1 correspondence from the unbounded set $\{ u \in \mathbb{F} \mid u < 0 \}$ to the bounded open interval $\{ v \in \mathbb{F} \mid 0 < v < 1 \}$.□

PROOF OF (2).   The condition $\mathbf{c} = t\mathbf{a} + (1 - t)\mathbf{b}$ with $t > 1$ is equivalent to

$$\mathbf{a} = \frac{1}{t} \mathbf{c} + \left(1 - \frac{1}{t} \right) \mathbf{b} .$$

The conclusion in this case follows because the map sending t to $1/t$ is a 1–1 correspondence from the unbounded set $\{ u \in \mathbb{F} \mid u > 1 \}$ to the bounded open interval $\{ v \in \mathbb{F} \mid 0 < v < 1 \}$.■

Betweenness and cross ratios

Not surprisingly, there are important relationships between the concept of betweenness and the notion of cross ratio. Here is the most basic result.

THEOREM VI.21. *Let $\mathbb{F}$ be an ordered field, and let $\mathrm{J} : \mathbb{F}^n \to \mathbb{FP}^n$ be the usual projective extension mapping. Then three collinear points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ of V satisfy the order relation $\mathcal{B}(\mathbf{a}, \mathbf{c}, \mathbf{b})$ if and only if*

$$\mathsf{XR}\big(\mathrm{J}(\mathbf{a}), \mathrm{J}(\mathbf{b}), \mathrm{J}(\mathbf{c}), L_\infty \big) < 0$$

where L_∞ is the ideal point of the line L containing $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Proof.   By Theorem V.17, if $\mathbf{c} = t\mathbf{a} + (1 - t)\mathbf{b}$ then

$$\mathsf{XR}\big(\mathrm{J}(\mathbf{a}), \mathrm{J}(\mathbf{b}), \mathrm{J}(\mathbf{c}), L_\infty \big) = \frac{t - 1}{t} .$$

This is negative if $0 < t < 1$ because $t - 1 < 0 < t$. We claim that the cross ratio is positive if either $t < 0$ or $t > 1$. If $t > 1$, then $t - 1 > 0$ and therefore the cross ratio is positive. Similarly, if $t < 0$, then $t - 1 < t < 0$ implies that the cross ratio is positive.■

Affine transformations obviously preserve betweenness (see Exercise 10 below). However, if $\mathcal{B}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ in $\mathbb{F}^n$ and T is a projective collineation of $\mathbb{FP}^n$ such that the images $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ of

a, **b**, **c** under T lie in (the image of) $\mathbb{F}^n$, then $\mathcal{B}(\mathbf{a}', \mathbf{b}', \mathbf{c}')$ is not necessarily true. Specific examples are given by projective collineations which interchange **a** and **b**, and send **c** to itself.

If one wants some aspect of order and betweenness which IS preserved by projective collineations, it is natural to try something involving the cross ratio, and the preceding result may be viewed as motivation for the following definition and theorem:

Definition. Let $\mathbb{F}$ be an ordered field, let V be a vector space over $\mathbb{F}$, and let **a**, **b**, **c**, **d** be collinear vectors in V. Then **a** and **b** *separate* **c** and **d** if one of $\{\mathbf{c}, \mathbf{d}\}$ is between **a** and **b** but the other is not. We shall write this as $\mathsf{sep}(\mathbf{a}, \mathbf{b} : \mathbf{c}, \mathbf{d})$. It is trivial to see that $\mathsf{sep}(\mathbf{a}, \mathbf{b} : \mathbf{c}, \mathbf{d})$ is equivalent to $\mathsf{sep}(\mathbf{a}, \mathbf{b} : \mathbf{d}, \mathbf{c})$ and $\mathsf{sep}(\mathbf{c}, \mathbf{d} : \mathbf{a}, \mathbf{b})$ (and one can also derive several other equivalent cross ratio statements from these).

There is a very simple and important characterization of separation in terms of cross ratios.

THEOREM VI.22. *Let $\mathbb{F}$ be an ordered field, let V be a vector space over $\mathbb{F}$, and let **a**, **b**, **c**, **d** be collinear vectors in V. Then $\mathsf{sep}(\mathbf{a}, \mathbf{b} : \mathbf{c}, \mathbf{d})$ is true if and only if*

$$\big(\, \mathrm{J}(\mathbf{a}), \mathrm{J}(\mathbf{b}), \mathrm{J}(\mathbf{c}), \mathrm{J}(\mathbf{d}) \,\big) \;<\; 0 \; .$$

Proof. *Suppose that $\mathsf{sep}(\mathbf{a}, \mathbf{b} : \mathbf{c}, \mathbf{d})$ is true.* Without loss of generality, we may assume that $\mathcal{B}(\mathbf{a}, \mathbf{c}, \mathbf{b})$ is true but $\mathcal{B}(\mathbf{a}, \mathbf{d}, \mathbf{b})$ is false (either this holds or else the corresponding statement with **c** and **d** interchanged is true – in the latter case, reverse the roles of the two points). Under these conditions we have $\mathbf{c} \;=\; t\mathbf{a} + (1-t)\mathbf{b}$ where $0 < t < 1$ and $\mathbf{d} \;=\; s\mathbf{a} + (1-s)\mathbf{b}$ where $s < 0$ or $s > 1$. By Theorem V.17 we have

$$\mathsf{XR}\big(\, \mathrm{J}(\mathbf{a}), \mathrm{J}(\mathbf{b}), \mathrm{J}(\mathbf{c}), \mathrm{J}(\mathbf{d}) \,\big) \;=\; \frac{s(1-t)}{t(1-s)} \; .$$

The sign of this cross ratio equals the sign of $s/(1-s)$, and the latter is negative if either $s < 0$ or $s > 1$.□

Suppose that the cross ratio is negative. We need to show that one of s and t satisfies $0 < u < 1$ and the other does not. To do this, we eliminate all the other possibilities.

CASE 1. Suppose we have $0 < s,\, t < 1$. Then all the factors of the numerator and denominator are positive.

CASE 2. Suppose neither satisfies $0 < u < 1$. Then the previous argument shows that one of s and $1 - s$ is positive and likewise for t and $1 - t$. Therefore the formula above implies that the cross ratio must be positive.■

We thus make the general definition in $\mathbb{F}\mathbb{P}^n$ that two points A and B **separate** two points C and D on AB if and only if $\mathsf{XR}(A, B, C, D) \;<\; 0$. If all four of these points are ordinary, then Theorem 22 provides a geometrical description of separation. The cases where one point is ideal can be described using the following two special cases:

(1) $\mathsf{sep}\big(\, \mathrm{J}(\mathbf{a}), \mathrm{J}(\mathbf{b}) : \infty, \mathrm{J}(\mathbf{c}) \,\big)$ and $\mathsf{sep}(\mathrm{J}(\mathbf{a}), \mathrm{J}(\mathbf{b}) : \mathrm{J}(\mathbf{c}), \infty)$ hold if and only if $\mathcal{B}(\mathbf{a}, \mathbf{c}, \mathbf{b})$ is true (see Theorems V.12 and V.17).

(2) $\mathsf{sep}\big(\, \mathrm{J}(\mathbf{a}), \infty : \mathrm{J}(\mathbf{c}), \mathrm{J}(\mathbf{d}) \,\big)$ and $\mathsf{sep}\big(\, \infty, \mathrm{J}(\mathbf{a}) : \mathrm{J}(\mathbf{c}), \mathrm{J}(\mathbf{d}) \,\big)$ hold if and only if $\mathcal{B}(\mathbf{c}, \mathbf{a}, \mathbf{d})$ is true because

$$\mathsf{XR}\big(\, \mathrm{J}(\mathbf{a}), \infty : \mathrm{J}(\mathbf{c}), \mathrm{J}(\mathbf{d}) \,\big) \;=\; \mathsf{XR}\big(\, \mathrm{J}(\mathbf{d}), \mathrm{J}(\mathbf{c}) : \infty, \mathrm{J}(\mathbf{a}) \,\big) \; .$$

The following observation is an immediate consequence of the definitions:

> Let $\mathbb{F}$ be an ordered field, let A, B, C, D be distinct collinear points in $\mathbb{FP}^n$, and let T be a projective collineation of $\mathbb{FP}^n$. Then $\mathsf{sep}(A, B : C, D)$ is true if and only if $\mathsf{sep}\big(T(A), T(B) : T(C), T(D)\big)$ is true.∎

A comprehensive visualization of separation for points on a real projective line may be given as follows:

As indicated in the picture below, there is a standard 1–1 correspondence (*stereographic projection*) between the points of $\mathbb{RP}^1$ and the points on the circle Γ in $\mathbb{R}^2$ which is tangent to the x-axis at the origin and whose center is $(0, \frac{1}{2})$. An ordinary point with standard affine coordinate u is sent to the intersection of Γ with the line joining $(u, 0)$ to $(0, 1)$, and the point at infinity is sent to $(0, 1)$. It is straightforward to check that this map σ defines a 1–1 correspondence from $\mathbb{RP}^1$ to Γ.[3]
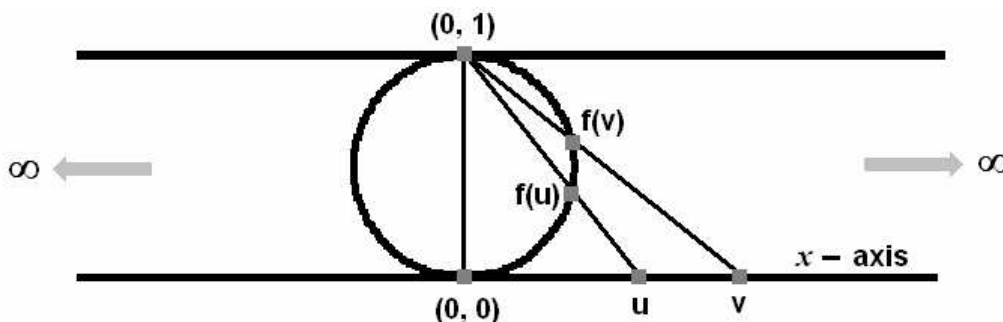
Figure VI.3

With respect to this correspondence, separation has the following interpretation. If a, $b \in \mathbb{RP}^1$, then $\Gamma - \{\sigma(a), \sigma(b)\}$ consists of two open arcs, and separation means that each arc contains exactly one of the points $\{c, d\}$.

We now summarize some basic properties of separation by means of the following theorem:

THEOREM VI.23. *If $\mathbb{F}$ is an ordered field and A, B, C, D are distinct collinear points of $\mathbb{FP}^n$, then the following hold:*

(a) $\mathsf{sep}(A, B : C, D)$ *implies* $\mathsf{sep}(A, B : D, C)$ *and* $\mathsf{sep}(C, D : A, B)$.

(b) *One and only one of the relations* $\mathsf{sep}(A, B : C, D)$, $\mathsf{sep}(B, C : D, A)$, *or and* $\mathsf{sep}(C, A : B, D)$ *is true.*

(c) *If* $\mathsf{sep}(A, B : C, D)$ *and* $\mathsf{sep}(B, C : D, E)$ *are true, then so is* $\mathsf{sep}(C, D : E, A)$.

(d) *If L is a line meeting AB, Y is a coplanar point on neither line, and X' is the intersection point of PX and L for $X = A$, B, C, D, then* $\mathsf{sep}(A, B : C, D)$ *implies* $\mathsf{sep}(A', B' : C', D')$.

---

[3]If we rotate the above picture about the y-axis in $\mathbb{R}^3$ we obtain a similar 1–1 correspondence between the complex projective line $\mathbb{CP}^1$ and the sphere of diameter 1 tangent to the xz-plane at the origin.

The proof is straightforward and is left as an exercise.■

One reason for listing the preceding four properties is that they come close to providing a complete characterization of separation.

THEOREM VI.24. *Let P be a Desarguian projective n-space, where $n \geq 2$, and suppose that P has an abstract notion of separation $\Sigma(\cdots, \cdots \| \cdots, \cdots)$ which satisfies the four properties in the previous theorem. Assume that some (hence every) line contains at least four points. Then P is isomorphic to $\mathbb{F}\mathbb{P}^n$, where $\mathbb{F}$ is an ordered skew-field, and the ordering of $\mathbb{F}$ has the property that $\mathsf{sep}(A, B : C, D)$ is true if and only if $\Sigma(\cdots, \cdots \| \cdots, \cdots)$ is.*

In principle, this result is proved on pages 239–244 of Artzy, *Linear Geometry*. We say "in principle" because the result is only stated for projective planes in which Pappus' Theorem holds. However, the latter is not used explicitly in the argument on these pages,[4] and the restriction to planes is easily removed.

We would need only one more axiom to give a completely synthetic characterization of the real projective plane (and similarly for higher dimensional real projective spaces). Fairly readable formulations of the required *continuity condition* (as it is called) may be found in Coxeter, *The Real Projective Plane*, pages 161–162, and Artzy (*op. cit.*), page 244.

EXERCISES

Throughout these exercises $\mathbb{F}$ denotes an ordered field, and the ordering is given by the usual symbolism.

1.   In the real numbers $\mathbb{R}$, prove that the closed interval $[a, b]$, consisting of all x such that $a \leq x \leq b$, is equal to the closed segment $[a; b]$ joining a to b as defined here, and likewise for their open analogs (a, b) and $(a; b)$. [*Hint:* If $a \leq c \leq b$ and $t = (b-a)/(c-a)$, consider $ta + (1-t)b$. If $c = ta + (1-t)b$ for $0 \leq t \leq 1$, why does this and $a \leq b$ imply that $a \leq c \leq b$?]

Definition.   A subset $K \subset \mathbb{F}^n$ is **convex** if $\mathbf{x}$ and $\mathbf{y}$ in K imply that the closed segment $[\mathbf{x}; \mathbf{y}]$ is contained in K. — In physical terms for, say, $\mathbb{R}^2$ or $\mathbb{R}^3$, this means that K has "no dents or holes."

2.   Prove that the following subsets of $\mathbb{F}$ are convex for an arbitrary $b \in \mathbb{F}$:

(*i*) The set $\{ x \in \mathbb{F} \mid x > b\}$.

(*ii*) The set $\{ x \in \mathbb{F} \mid x < b\}$.

(*iii*) The set $\{ x \in \mathbb{F} \mid x \geq b\}$.

(*iv*) The set $\{ x \in \mathbb{F} \mid x \leq b\}$.

3.   Prove that the intersection of an arbitrary family of convex subsets of $\mathbb{F}^n$ is also convex.

4.   Let $f : \mathbb{F}^n \to \mathbb{F}$ be a linear function of the form $f(\mathbf{x}) = \sum_i a_i x_i - b$.

---

[4]An explicit recognition that Pappus' Theorem is unnecessary appears in Forder, *Foundations of Euclidean Geometry*, pp. 196–197 and 203–206.

(a) Prove that for all $t \in \mathbb{F}$ we have $f\big(t\mathbf{x} + (1-t)\mathbf{y}\big) = t \cdot f(\mathbf{x}) + (1-t)f(\mathbf{y})$.

(b) Prove that if $K \subset \mathbb{F}^n$ is convex, then so is its image $f[K]$.

(c) Prove that if $C \subset \mathbb{F}$ is convex, then so is its inverse image $f^{-1}[C]$.

5. Let f be as in Exercise 4. Then the subsets of $\mathbb{F}^n$ on which f is positive and negative are called the *(two)* sides *of the hyperplane* H_f *defined by* f or the *(two)* half-spaces *determined by the hyperplane* H_f. Prove that each half-space is (nonempty and) convex, and if we have points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{F}^n$ such that $\mathbf{x}$ lies on one of the half-spaces and $\mathbf{y}$ lies on the other, then the closed segment $[\mathbf{x}; \mathbf{y}]$ contains a point of the hyperplane H_f defined by the equation $f(\mathbf{z}) = 0$. — This statement is called the *hyperplane separation property* for $\mathbb{F}^n$.
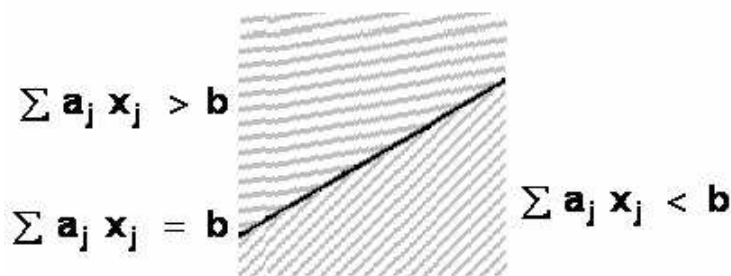
Figure VI.4

Also, explain why the hyperplane and its two sides are three pairwise disjoint subsets whose union is all of $\mathbb{F}^n$.

6. Formulate and prove a similar result to Exercise 5 for the set of all points in a k-plane $M \subset \mathbb{F}^n$ which are not in a $(k-1)$-plane $Q \subset M$.

7. Suppose that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are noncollinear points in $\mathbb{F}^2$. Define the **classical triangle** $\Delta^c\mathbf{xyz}$ to be the union of the closed segments $[\mathbf{x}; \mathbf{y}]$, $[\mathbf{x}; \mathbf{z}]$, and $[\mathbf{y}; \mathbf{z}]$. Prove the **Theorem of Pasch:**[5] A line L containing a point $\mathbf{w}$ in an open side $(\mathbf{x}; \mathbf{y})$ of $\Delta^c\mathbf{xyz}$ either passes through $\mathbf{z}$ or else meets one of the other open sides $(\mathbf{x}; \mathbf{z})$ or $(\mathbf{x}; \mathbf{z})$. [*Hint:* Explain why $\mathbf{x}$ and $\mathbf{y}$ are on opposite sides of the L through $\mathbf{w}$. What can be said about $\mathbf{z}$ if it does not lie on this line?]

Figure VI.5

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[5]MORITZ PASCH (1843–1930) is mainly known for his work on the foundations of geometry, and especially for recognizing the logical deficiencies in Euclid's *Elements* and developing logically rigorous methods for addressing such issues. The theorem in the exercise is one example of a geometrical result that is tacitly assumed – but not proved — in the *Elements*.

7. If $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$ are distinct points in $\mathbb{F}^2$ such that no three are noncollinear, then the *classical quadrilateral* $\square^c\,\mathbf{xyzw}$ is the set

$$[\mathbf{x};\mathbf{y}] \ \cup \ [\mathbf{y};\mathbf{z}] \ \cup \ [\mathbf{z};\mathbf{w}] \ \cup \ [\mathbf{w};\mathbf{x}]\ .$$

Note that the definition depends upon the ordering of the four points. The figure is called a *convex quadrilateral* if the following conditions hold:

$\mathbf{x}$ and $\mathbf{y}$ lie on the same side of $\mathbf{zw}$.
$\mathbf{y}$ and $\mathbf{z}$ lie on the same side of $\mathbf{wx}$.
$\mathbf{z}$ and $\mathbf{w}$ lie on the same side of $\mathbf{xy}$.
$\mathbf{w}$ and $\mathbf{x}$ lie on the same side of $\mathbf{yz}$.

The *diagonals* of a classical quadrilateral $\square^c\,\mathbf{xyzw}$ are the segments $[\mathbf{x};\mathbf{z}]$ and $[\mathbf{y};\mathbf{w}]$. Prove that the diagonals of a convex quadrilateral have a point in common. Why must this point lie on $(\mathbf{x};\mathbf{z}) \ \cap \ (\mathbf{y};\mathbf{w})$?
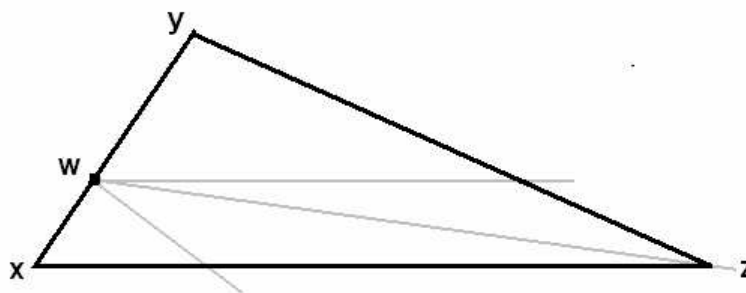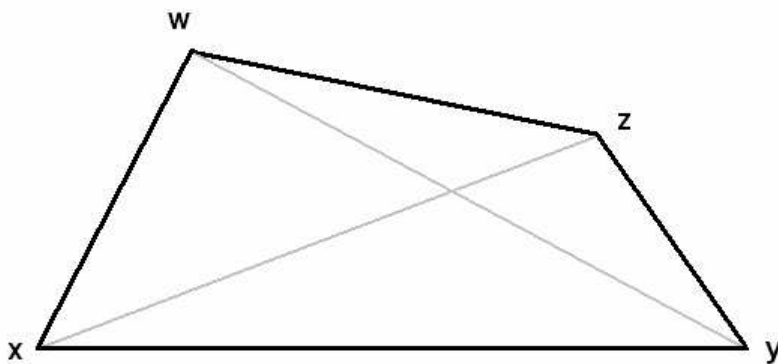

Figure VI.6

8. Give an explicit formula for the map defined by Figure VI.3 and the accompanying discussion.

9. Prove Theorem 24.

10. Suppose that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are points in $\mathbb{F}^n$ such that $\mathcal{B}(\mathbf{x},\mathbf{y},\mathbf{z})$ is true and $T \in \mathrm{AFF}(\mathbb{F}^n)$. Prove that $\mathcal{B}\big(T(\mathbf{x}), T(\mathbf{y}), T(\mathbf{z})\big)$ is also true.

11. In the notation of Exercise 5, let $\mathbf{y}_1, \cdots, \mathbf{y}_n$ be an affine basis for the hyperplane H under consideration, and let $\mathbf{y}_0 \notin H$. Prove that $\mathbf{x} \in \mathbb{F}^n$ lies on the same side of H as $\mathbf{y}_0$ if the 0th barycentric coordinate of $\mathbf{x}$ with respect to $\mathbf{y}_0$, $\mathbf{y}_1, \cdots, \mathbf{y}_n$ (an affine basis for $\mathbb{F}^n$) is positive. What is the condition for $\mathbf{x}$ and $\mathbf{y}_0$ to lie on opposite sides?

12. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be four points in $\mathbb{F}^2$ such that no three are collinear. Prove that $\mathbf{a}$, $\mathbf{b}$, bfc and $\mathbf{d}$ (in that order) determine the vertices of a convex quadrilateral if and only if the barycentric coordinate expansion $\mathbf{c} \ = \ x\mathbf{a} + y\mathbf{b} + z\mathbf{d}$ (with $x + y + z = 1$) satisfies $x > 0$, $z > 0$ and $y < 0$.

CHAPTER VII

HYPERQUADRICS

Conic sections have played an important role in projective geometry almost since the beginning of the subject. In this chapter we shall begin by defining suitable projective versions of conics in the plane, quadrics in 3-space, and more generally hyperquadrics in n-space. We shall also discuss tangents to such figures from several different viewpoints, prove a geometric classification for conics similar to familiar classifications for ordinary conics and quadrics in $\mathbb{R}^2$ and $\mathbb{R}^3$, and we shall derive an enhanced duality principle for projective spaces and hyperquadrics. Finally, we shall use a mixture of synthetic and analytic methods to prove a famous classical theorem due to B. Pascal (1623–1662)[1] on hexagons inscribed in plane conics, a dual theorem due to C. Brianchon (1783–1864),[2] and several other closely related results.

1. Definitions

> The three familiar curves which we call the "conic sections" have a long history ... It seems that they will always hold a place in the curriculum. The beginner in analytic geometry will take up these curves after he has studied the circle. Whoever looks at a circle will continue to see an ellipse, unless his eye is on the axis of the curve. The earth will continue to follow a nearly elliptical orbit around the sun, projectiles will approximate parabolic orbits, [and] a shaded light will illuminate a hyperbolic arch. —
> **J. L. Coolidge (1873–1954)**

In classical Greek geometry, conic sections were first described synthetically as intersections of a plane and a cone. On the other hand, today such curves are usually viewed as sets of points (x, y) in the Cartesian plane which satisfy a nontrivial quadratic equation of the form

$$Ax^2 \ + \ 2Bxy \ + \ Cy^2 \ + \ 2D \ + \ 2E \ + \ F \ = \ 0$$

where at least one of A, B, C is nonzero. In these notes we shall generally think of conics and quadrics in such terms. Here are some online references which relate the classical and modern approaches to these objects. The first contains some historical remarks, the second is a fairly detailed treatment which shows the equivalence of the classical and modern definitions only using material from elementary geometry, and the third contains a different proof that the definitions are equivalent using standard results from trigonometry.

http://xahlee.org/SpecialPlaneCurves_dir/ConicSections_dir/conicSections.html

http://mathdl.maa.org/convergence/1/?pa=content&sa=viewDocument&nodeId=196&bodyId=60

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[1]Incidentally, he proved this result when he was 16 years old.

[2]This result was originally discovered without using duality.

The corresponding notion of *quadric surface* in $\mathbb{R}^3$ is generally defined to be the set of zeros of a nontrivial quadratic polynomial $p(x, y, z)$ in three variables (nontriviality means that at least one term of degree two has a nonzero coefficient). One can similarly define a *hyperquadric* in $\mathbb{R}^n$ to be the set of zeros of a nonzero quadratic polynomial $p(x_1, \cdots, x_n)$. Such an equation has the form

$$\sum_{i,j} a_{i,j} x_i x_j \ + \ 2 \cdot \sum_k b_k x_k \ + \ c \ = \ 0$$

where at least one of the coefficients $a_{i,j} \ = \ 0$.

One obvious question about our definitions is to give a concise but useful description of all the different types of conics, quadrics or hyperquadrics that exist in $\mathbb{R}^n$. Using linear algebra, in each dimension it is possible to separate or classify such objects into **finitely many** types such that

> if Σ_1 and Σ_2 are hyperquadrics that are affinely equivalent *(so that there is an affine transformation T of $\mathbb{R}^n$ such that $T[\Sigma_1] \ = \ \Sigma_2$, then Σ_1 and Σ_2 have the same type. — In fact, one can choose the affine transformation to have the form $T_1 \circ T_0$, where T_0 is a linear transformation and T_1 is given by a diagonalizable invertible linear transformation; in other words, there are nonzero scalars d_i such that for each i we have $T_1(\mathbf{e}_i) \ = \ d_i \mathbf{e}_i$, where $\mathbf{e}_i$ is the ith standard unit vector in $\mathbb{R}^n$.*

For $n = 2$ and 3, the details of this classification are described explicitly in Section V.2 of the following online document:

The case of conics in $\mathbb{R}^2$ is summarized in the table on page 82 of this document, and the case of quadrics in $\mathbb{R}^3$ is summarized in the table on page 83 of the same document. In particular, there are fewer than 10 different types of possible nonempty figures in $\mathbb{R}^2$ (including degenerate cases of sets with one point or no points) and fewer than 20 different types of possible nonempty figures in $\mathbb{R}^3$ (also including an assortment degenerate cases). Later in this chapter we shall describe the analogous classification for $\mathbb{R}^n$ (with $n \geq 3$ arbitrary) in one of the exercises.

Projective extensions of hyperquadrics

We are now faced with an obvious question:

> How does one define a hyperquadric in projective space?

Let us consider the analogous situation in degree one. The sets of solutions to nontrivial linear equations $p(x_1, \cdots, x_n) = 0$ are merely hyperplanes. If $p(x_1, \cdots, x_n)s \ = \ \sum_i a_i x_i \ + \ b$, then this hyperplane is just the set of ordinary points in $\mathbb{RP}^n$ whose homogeneous coordinates satisfy the *homogeneous* linear equation

$$\sum_{i=1}^n a_i x_i \ + \ b x_{n+1} \ = \ 0 \ .$$

This suggests the following: Consider the quadratic polynomial

$$p(x_1, \cdots, x_n) \ = \ \sum_{i,j} a_{i,j} x_i x_j \ + \ 2 \cdot \sum_k b_k x_k \ + \ c$$

and turn it into a homogeneous quadratic polynomial by multiplying each degree 1 monomial in the summation by x_{n+1} and multiplying the constant term by x_{n+1}^2. We then obtain the modified quadratic polynomial

$$\overline{p}(x_1, \cdots, x_n) = \sum_{i,j} a_{i,j} x_i x_j + 2 \cdot \sum_k b_k x_k x_{n+1} + c x_{n+1}^2$$

which is homogeneous and has the following compatibility properties:

THEOREM VII.1. (i) *If X is a point in $\mathbb{RP}^n$ and ξ and ξ' are homogeneous coordinates for X, then $\overline{p}(\xi) = 0$ if and only if $\overline{p}(\xi') = 0$.*

(ii) *The set of zeros for p is equal to the set of ordinary points in $\mathbb{RP}^n$ whose homogeneous coordinates are zeros of $\overline{p}$.*

Proof.   We shall proof the two parts separately.

PROOF OF (i).   Observe that $\overline{p}(k\xi) = k^2 \cdot \overline{p}(\xi)$ by direct computation. Therefore $\xi' = k\xi$ for some $k \neq 0$ implies that $\overline{p}(\xi') = 0$ if and only if $\overline{p}(\xi) = 0.\square$

PROOF OF (ii).   If $\mathbf{x} \in \mathbb{R}^{n,1}$, then the transpose of $(x_1, \cdots, x_n, 1)$ is a set of homogeneous coordinates for $\mathbf{J}(\mathbf{x}) \in \mathbb{RP}^n$, and it is elementary to check that the solutions to the equation $p = 0$ contained in the intersection of the set of ordinary points and the points in $\mathbb{RP}^n$ whose homogeneous coordinates are solutions to the equation $\overline{p} = 0$ (in particular, we have $p(x_1, \cdots, x_n) = \overline{p}(x_1, \cdots, x_n, 1)$). Conversely, if $\overline{p}(x_1, \cdots, x_n, x_{n+1}) = 0$ where $x_{n+1} \neq 0$, then we also have

$$0 = \frac{1}{x_{n+1}^2} \cdot \overline{p}(x_1, \cdots, x_n, x_{n+1}) = \overline{p}\left(\frac{x_1}{x_{n+1}}, \cdots, \frac{x_n}{x_{n+1}}, 1 \right) = p(x_1, \cdots, x_n)$$

and hence the solutions to $\overline{p} = 0$ in the image of J are all ordinary points which are solutions to $p = 0.\blacksquare$

All of the preceding discussion makes at least formal sense over an arbitrary field $\mathbb{F}$; of course, the mathematical value of the quadrics considered depends strongly upon the solvability of quadratic equations within the given field.[3] Define a *hyperquadric* Σ in $\mathbb{FP}^n$ to be the set of zeros of a homogeneous quadratic equation:

$$\sum_{i,j=1}^{n+1} a_{i,j} x_i x_j = 0$$

In the study of hyperquadrics we generally assume that $1 + 1 \neq 0$ in $\mathbb{F}$. This condition allows us to choose the n^2 coefficients $a_{i,j}$ so that $a_{i,j} = a_{j,i}$; for if we are given an arbitrary homogeneous quadratic equation as above and set $b_{i,j} = \frac{1}{2}(a_{j,i} + a_{i,j})$, then it is easy to see that

$$\sum_{i,j=1}^{n+1} a_{i,j} x_i x_j = 0 \text{ if and only if } \sum_{i,j=1}^{n+1} b_{i,j} x_i x_j = 0$$

because we have

$$\sum_{i,j=1}^{n+1} b_{i,j} x_i x_j = \tfrac{1}{2} \left(\sum_{i,j=1}^{n+1} a_{i,j} x_i x_j + \sum_{i,j=1}^{n+1} a_{j,i} x_i x_j \right).$$

---

[3]All fields in this chapter are assumed to have commutative multiplications.

For these reasons, *we shall henceforth assume* $1 + 1 \neq 0$ *in* $\mathbb{F}$ *and* $a_{i,j} = a_{j,i}$ *for all* i *and* j.

It is natural to view the coefficients $a_{i,j}$ as the entries of a *symmetric* $(n+1) \times (n+1)$ matrix A. If we do so and Σ is the hyperquadric in $\mathbb{FP}^n$ defined by the equation $\sum_{i,j} a_{i,j} x_i x_j = 0$, then we may rewrite the defining equation for Σ as follows: A point X lies on Σ if and only if for some (equivalently, for all) homogeneous coordinates ξ representing X we have

$$^{\mathbf{T}}\xi A \xi = 0 \ .$$

If we have an affine quadric in $\mathbb{F}^n$ defined by a polynomial p as above, then an $(n+1) \times (n+1)$ matrix defining its projective extension is given in block form by

$$\begin{pmatrix} A & ^{\mathbf{T}}\mathbf{b} \\ \mathbf{b} & c \end{pmatrix}$$

where the symmetric matrix $A = (a_{i,j})$ gives the second degree terms of p, the row vector $2 \cdot \mathbf{b}$ gives the first degree terms b_i (note the coefficient!), and c gives the constant term.

Hypersurfaces of higher degree

The reader should be able to define projective hypercubics, hyperquartics, *etc.*, as well as the projective hyper—ic associated to an affine hyper—ic. Subsets of these types are generally called **projctive algebraic varieties**; they have been studied extensively over the past 300 years and have many interesting and important properties. The mathematical study of such objects has remained an important topic in mathematics ever since the development of projective geometry during the 19$^{\text{th}}$ century, but it very quickly gets into issues far beyond the scope of these notes. In particular, the theory involves a very substantial amount of input from multivariable calculus and the usual approaches also require considerably more sophisticated algebraic machinery than we introduce in these notes. The rudiments of the theory appear in Sections V.4–V.6 of the book by Bumcrot, and a more complete treatment at an advanced undergraduate level is given in Seidenberg, *Elements of the Theory of Algebraic Curves*, as well as numerous other introductory books on **algebraic geometry.**

Projective algebraic varieties also turn out to have important applications in various directions, including issues in theoretical physics, the theory of encryption, and even the proof of Fermat's Last Theorem during the 1990s which was mainly due to Andrew Wiles (the word "mainly" is included because the first complete proof required some joint work of Wiles with R. Taylor, and Wiles' work starts with some important earlier results by others). A reader who wishes to learn more about some of these matters may do so by going to the final part of Section IV.5 in the online document

<center><code>http//:math.ucr.edu~res/math133/coursenotes4b.pdf</code></center>

and checking the traditional and electronic references cited there.

EXERCISES

1.   Consider the conics in $\mathbb{R}^2$ defined by the following equations:

(*i*) The circle defined by $x^2 + y^2 - 1 = 0$.

(*ii*) The hyperbola defined by $xy - 1 = 2$.

(*iii*) The parabola defined by $y - x^2 = 0$.

Show that the associated projective conics have 0, 1 and 2 points at infinity respectively, and give homogeneous coordinates for these points.

2.   Find which points (if any) at infinity belong to the projective conics associated to the conics in $\mathbb{R}^2$ defined by the following equations.

(i) $x^2 - 2y^2 - 2xy = 0$

(ii) $3x^2 + 4y^2 - 4x + 2 = 0$

(iii) $x^2 + y^2 - 4y = 4$

(iv) $x^2 - 4xy - 4y^2 - 2y = 4$

3.   Find the points at infinity on the projective quadrics associated to the quadrics in $\mathbb{R}^3$ defined by the following equations.

(i) $x^2 + y^2 - z^2 = 1$

(ii) $x^2 + y^2 - z^2 - 6x - 8y = 0$

(iii) $x^2 + y^2 = 2z$

(iv) $x^2 - y^2 - z^2 = 1$

(v) $x^2 + y^2 = z$

(vi) $x^2 + y^2 = z^2$

4.   For each of the following affine quadrics σ in $\mathbb{R}^3$, find a symmetric 4×4 matrix such that the projective extension $\mathbb{P}(\Sigma)$ of Σ is defined by the equation ${}^{\mathrm{T}}\xi A \xi = 0$.

(i) Σ is defined by the affine equation $4x^2 + 3y^2 - z^2 + 2x + y + 2z - 1 = 0$.

(ii) Σ is defined by the affine equation $3x^2 + y^2 + 2z^2 + 3x + 3y + 4z = 0$.

(iii) Σ is defined by the affine equation $2x^2 + 4z^2 - 4x - y - 24z + 36 = 0$.

(iv) Σ is defined by the affine equation $4x^2 + 9y^2 + 5z^2 - 4xy + 8yz + 12xz + 9z - 3 = 0$.

2. Tangents

Tangent lines to circles play an important role in classical Euclidean geometry, and their generalizations to other conics were also known to classical Greek mathematicians such as Archimedes[4](287 B. C. E. – 212 B. C. E.) and Apollonius of Perga[5] (*c*. 262 B. C. E. – *c*. 190 B. C. E.). In modern mathematics they are generally defined using concepts and results from single variable or multivariable differential calculus. Of course, the latter is designed to work primarily in situations where the coordinates are real or complex numbers, and since we want to consider more general coordinates we need to develop an approach that is at least somewhat closer to the classical viewpoint.

In these notes we shall concentrate on the following two ways of viewing tangents to conics in $\mathbb{R}^2$ or quadrics in $\mathbb{R}^3$.

1. SYNTHETIC APPROACH. A line is tangent to a hyperquadric if and only if it lies wholly in the hyperquadric or has precisely one point of intersection with the hyperquadric.

2. ANALYTIC APPROACH. Let $X \in \Sigma \cap L$, where Σ is a hyperquadric and L is a line. Then L is tangent to Σ if and only if there is a differentiable curve $\gamma : (a; b) \to \mathbb{R}^n$ lying totally in Σ such that $\gamma(t_0) = \mathbf{x}$ for some $t_0 \in (a; b)$ and L is the line $\mathbf{x} + \mathbb{R} \cdot \gamma'(t_0)$.

For our purposes the first viewpoint will be more convenient; in Appendix E we shall show that the analytic approach is consistent with the synthetic viewpoint, at least in all the most important cases. Actually, the viewpoint of calculus is the better one for generalizing tangents to cubics, quartics, *etc.*, but a correct formulation is too complicated to be given in these notes.

We begin with a result on solutions to homogeneous quadratic equations in two variables:

THEOREM VII.2. *Suppose that* $\mathbb{F}$ *is a field in which* $1 + 1 \neq 0$, *and* (x_1, y_1), (x_2, y_2), (x_3, y_3) *are solutions to the homogeneous quadratic equation*

$$ax^2 + bxy + cy^2 = 0 .$$

Then either $a = b = c = 0$ *or else one of* (x_1, y_1), (x_2, y_2), (x_3, y_3) *is a nonzero multiple of another.*

Proof. If the hypothesis holds, then in matrix terminology we have

$$\begin{pmatrix} x_1^2 & x_1 y_1 & y_1^2 \\ x_2^2 & x_2 y_2 & y_2^2 \\ x_3^2 & x_3 y_3 & y_3^2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

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[4]ARCHIMEDES OF SYRACUSE is well known to be one of the most important figures in Greek mathematics; his contributions to physics and engineering innovations are also well known.

[5]APOLLONIUS OF PERGA is particularly known for his extensive study of conic sections, which goes far beyond anything previously written. He is also known for other contributions to mathematics and astronomy.

Suppose not all of a, b, c are nonzero. Then the given 3×3 matrix is not invertible and hence has a zero determinant. But the determinant of such a matrix may be computed directly, and up to a sign factor it is equal to

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \cdot \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} \cdot \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$$

The vanishing of this determinant implies that one of the 2×2 determinants in the factorization must be zero, and the latter implies that the rows of the associated 2×2 matrix are proportional to each other.∎

The preceding result has the following important geometric application:

THEOREM VII.3. *Let Σ be a hyperquadric in $\mathbb{RP}^n$, let $X \in \Sigma$, and let L be a line containing X. Then $\Sigma \cap L$ is either $\{X\}$, two points, or all of L.*

Proof. Let $Y \neq X$ where $Y \in L$, let ξ and η denote homogeneous coordinates for X and Y respectively, and suppose that Σ is defined by the equation

$$\mathbf{T}\omega A \omega \; = \; 0$$

where A is a symmetric $(n+1) \times (n+1)$ matrix and ω represents $W \in \mathbb{RP}^n$.

If $Z \in L$ and is represented by the homogeneous coordinates ζ, then $\zeta \; = \; u\xi + v\eta$ for some u, $v \in \mathbb{F}$ that are not both zero. By construction, $Z \in \Sigma$ if and only if

$$0 \; = \; \mathbf{T}\zeta A \zeta \; = \; \mathbf{T}(u\xi + v\eta)A(u\xi + v\eta) \; =$$

$$u^2 \mathbf{T}\xi A \xi \; + \; 2uv\, \mathbf{T}\eta A \xi \; + \; v^2 \mathbf{T}\eta A \eta \; = \; u^2 p \; + \; 2uvq \; + \; v^2 q$$

for suitable constants p, q, r. We claim that $\Sigma \cap L$ has at least three points if and only if $L \subset \Sigma$. The "only if" implication is trivial, so we shall focus on the "if" direction. — Suppose that Z_1, Z_2, Z_3 are points on $\Sigma \cap L$, and take homogeneous coordinates $\zeta_i \; = \; u_i \xi + v_i \eta$ for Z_i. By Theorem 2, either $p = q = r = 0$ (in which case $L \subset \Sigma$) or else one of the pairs (u_i, v_i) is proportional to the other, say $(u_j, v_j) = m(u_k, v_k)$ for some $m \neq 0$. In this case we have that $Z_j = Z_k$ and hence Z_1, Z_2, Z_3 are not distinct.∎

Definition. Let Σ be a hyperquadric, let $X \in \Sigma$, and let L be a line containing X. We shall say that L is a *tangent line* to Σ at X if either $\Sigma \cap L - \{X\}$ or $L \subset \Sigma$. In the remaining case where $\Sigma \cap L$ consists of two points, we shall say that L is a *secant line through* X. The *tangent space* to Σ at X is equal to the union of all tangent lines to Σ at X.

Singular and nonsingular points

If we consider the conic in $\mathbb{R}^2$ defined by the eqution $x^2 - y^2 = 0$ we see that the structure of the conic at the origin is different than at other points, for the conic is given by a pair of lines which intersect at the origin. Some words which may be used to describe this difference are exceptional, special or *singular*. A concise but informative overview of singular points for plane curves appears in the following online reference:

http://mathworld.wolfram.com/SingularPoint.html

There are corresponding theories of singularities for surfaces in $\mathbb{R}^3$, and more generally for hypersurfaces in $\mathbb{R}^n$. Not surprisingly, if one is only interested in hyperquadrics as in these notes, then everything simplifies considerably. We shall explain the relationship between the

theory of singular and nonsingular points for hyperquadrics and the general case in Appendix E.

We have given a purely synthetic definition of the tangent space to a hyperquadric $\Sigma \subset \mathbb{FP}^n$ at a point $X \in \Sigma$. The first step is to give an algebraic description of the tangent space in terms of homogeneous coordinates.

THEOREM VII.4. *Let* $\mathbb{F}$ *and* $\Sigma \subset \mathbb{FP}^n$ *be as above, and let* $X \in \Sigma$. *Then the tangent space to* Σ *at* X *is either a hyperplane in* $\mathbb{FP}^n$ *or all of* $\mathbb{FP}^n$. *In the former case,* X *is said to be a* nonsingular *point, and in the latter case* X *is said to be a* singular *point. Furthermore, if* Σ *is defined by the symmetric matrix* A *and* ξ *is a set of homogeneous coordinates for* X, *then in the nonsingular case* $^{\mathbf{T}}\xi A$ *is a (nonzero) set of homogeneous coordinates for the tangent hyperplane, but in the singular case we have* $^{\mathbf{T}}\xi A = \mathbf{0}$.

EXAMPLES. Suppose we consider the projectivizations of the circle $x^2 + y^2 = 1$, the hyperbola $x^2 - y^2 = 1$, the parabola $y = x^2$, and the pair of intersecting lines $x^2 = y^2$. Then the corresponding projective conics are defined by the following homogeneous quadratic equations:

$$x_1^2 + x_2^2 - x_3^2 = 0, \qquad\qquad x_1^2 - x_2^2 - x_3^2 = 0$$

$$x_1^2 - x_2 x_3 = 0, \qquad\qquad x_1^2 - x_2^2 = 0$$

In the first three cases the associated 3×3 symmetric matrix A is invertible, and hence $^{\mathbf{T}}\xi A \neq \mathbf{0}$ for all nonzero ξ, so that every point of these projective conics will be a nonsingular point. — On the other hand, in the fourth example, the symmetric matrix A is not invertible, and in fact its kernel (either on the left or right side!) consists of all vectors whose first and second coordinates are equal to zero. This implies that all points on the conic except $\mathbf{J}(\mathbf{0})$ are nonsingular but $\mathbf{J}(\mathbf{0})$ is singular. These examples are all consistent with our intuition that the first three curves behave regularly (or are nonsingular) at all points and the fourth curve behaves regularly at all points except the origin.

Proof. In the proof of the preceding theorem, we noted that if $Y \in \mathbb{FP}^n$ with homogeneous coordinates η and $Z \in XY$ has homogeneous coordinates $\zeta = u\xi + v\eta$, then $Z \in \Sigma$ if and only if

$$u^2 \left(^{\mathbf{T}}\xi A \xi \right) + 2uv \left(^{\mathbf{T}}\xi A \eta \right) + v^2 \left(^{\mathbf{T}}\eta A \eta \right) = 0$$

and the number of points on $XY \cap \Sigma$ depends upon the equivalence classes of solutions to this equation, which we shall call the INTERSECTION EQUATION.

CLAIM: The line XY is tangent to Σ if and only if $^{\mathbf{T}}\xi A \eta = {}^{\mathbf{T}}\eta A \xi = 0$.

Suppose first that XY is tangent to Σ. If XY is contained in Σ, then we have

$$^{\mathbf{T}}\xi A \xi = {}^{\mathbf{T}}\eta A \eta = {}^{\mathbf{T}}(\xi + \eta) A (\xi + \eta) = 0$$

and elementary manipulations of these equations show that $2 \cdot {}^{\mathbf{T}}\eta A \xi = 0$. On the other hand, if $XY \cap \Sigma = \{X\}$, then $^{\mathbf{T}}\eta A \eta = 0$ and the only solutions to the Intersection Equation in the first paragraph of the proof are pairs (u, v) which are nonzero scalar multiples of $(1, 0)$. Therefore, the Intersection Equation evaluated at $(1, t)$ is equal to zero if and only if $t = 0$. However, it is easy to check that the ordered pair

$$\left(1, \; -\frac{^{\mathbf{T}}\xi A \eta}{^{\mathbf{T}}\eta A \eta} \right)$$

solves the Intersection Equation because

$$\mathbf{^T}\xi A\xi \;=\; 0$$

and therefore we must have $\mathbf{^T}\xi A\eta \;=\; \mathbf{^T}\eta A\xi \;=\; 0.\square$

Conversely, suppose that $\mathbf{^T}\xi A\eta \;=\; \mathbf{^T}\eta A\xi \;=\; 0$. Since $\mathbf{^T}\xi A\xi \;=\; 0$, the Intersection Equation reduces to

$$v^2 \left(\mathbf{^T}\eta A\eta\right) \;=\; 0 \; .$$

This equation means that either $\mathbf{^T}\eta A\eta \;=\; 0$, in which case we have $L \subset \Sigma$, or else $v = 0$, in which case every solution (u, v) of the Intersection equation is proportional to the known solution $(1, 0)$, so that $\Sigma \cap XY \;=\; \{X\}$.

To conclude the proof, we have shown that the tangent space at X is the set of all points Y such that $\mathbf{^T}\xi A\eta \;=\; 0$. If $\mathbf{^T}\xi A \;=\; \mathbf{0}$, this is all of $\mathbb{FP}^n$, and if $\mathbf{^T}\xi A \;\neq\; \mathbf{0}$, this is the hyperplane with homogeneous coordinates $\mathbf{^T}\xi A.\blacksquare$

We shall say that a hyperquadric Σ is **nonsingular** if for each $X \in \Sigma$ the tangent space at X is a hyperplane (algebraically, this means that if ξ represents X then $\mathbf{^T}\xi A \neq \mathbf{0}$.

THEOREM VII.5. *If Σ is a hyperquadric defined by the symmetric matrix A, then Σ is nonsingular if and only if A is invertible.*

Proof.   Suppose first that A is invertible. Then $\xi \neq \mathbf{0}$ implies that $\mathbf{^T}\xi A$ is nonzero, and by the preceding result it follows that the tangent space at every point must be a hyperplane.$\square$

Conversely, suppose that A is not invertible. Then there is some $\xi \neq \mathbf{0}$ such that $\mathbf{^T}\xi A \;=\; \mathbf{0}$, and if ξ represents X it follows that $X \in \Sigma$ and X is a singular point of $\Sigma.\blacksquare$

By definition, each symmetric matrix A determines a hyperquadric Σ_A. This is not a 1–1 correspondence, for if c is a nonzero scalar then clearly $\Sigma_A \;=\; \Sigma_{cA}$. We shall now use the notion of tangent hyperplane to show that, in many cases, this is the only condition under which two matrices can define the same hyperquadric. Further discussion of this question is given in Section 2 of Appendix E.

THEOREM VII.6. *Let A and B be symmetric $(n+1) \times (n+1)$ matrices over the field $\mathbb{F}$ in which $1 + 1 \neq 0$, and suppose they define the same nonempty hyperquadric in $\mathbb{FP}^n$. Assume that Σ has at least one nonsingular point. Then B is a scalar multiple of A.*

Proof.   We are given that Σ has a nonsingular point X; let ξ be a set of homogeneous coordinates for X. Then both $\mathbf{^T}\xi A$ and $\mathbf{^T}\xi B$ define the same hyperplane and hence $\mathbf{^T}\xi A \;=\; k \cdot \mathbf{^T}\xi B$ for some nonzero scalar k.

Suppose now that Y does not lie on this tangent hyperplane, and let η be a set of homogeneous coordinates for Y. Then the line XY meets Σ in a second point which has homogeneous coordinates of the form $u\xi + \eta$ for some $u \in \mathbb{F}$. This scalar satisfies the following equations:

$$2u\,\mathbf{^T}\xi A\eta \;+\; \mathbf{^T}\eta A\eta \;=\; 0, \qquad\qquad 2u\,\mathbf{^T}\xi B\eta \;+\; \mathbf{^T}\eta B\eta \;=\; 0$$

Since $\mathbf{^T}\xi A \;=\; k \cdot \mathbf{^T}\xi B$ the equations above imply that

$$\mathbf{^T}\eta A\eta \;=\; k \cdot \mathbf{^T}\eta B\eta$$

for all Y whose homogeneous coordinates satisfy ${}^{\mathbf{T}}\xi A\eta \neq 0$ (*i.e.*, all vectors in $\mathbb{F}^{n+1,1}$ except those in the n-dimensional subspace defined by the tangent hyperplane to Σ and X).

To prove that ${}^{\mathbf{T}}\eta A\eta \;=\; k \cdot {}^{\mathbf{T}}\eta B\eta$ if Y lies in the tangent hyperplane at X, let Z be a point which is not on the tangent hyperplane. Then

$$ {}^{\mathbf{T}}\omega A\omega \;=\; k \cdot {}^{\mathbf{T}}\omega B\omega $$

for $\omega \;=\; \zeta,\ \eta+\zeta,\ \eta-\zeta$. Let $C = A$ or B, and write $\Psi_C(\gamma,\delta) \;=\; {}^{\mathbf{T}}\gamma C\delta$. We then have the following:

$$ \psi_C(\eta,\zeta) \;=\; \tfrac{1}{4}\Psi_C(\eta+\zeta,\eta+\zeta) \;-\; \tfrac{1}{4}\Psi_C(\eta-\zeta,\eta-\zeta) $$

$$ \Psi_C(\eta,\eta) \;=\; \Psi_C\big(\,(\eta+\zeta)-\zeta,\,(\eta+\zeta)-\zeta\,\big) $$

By the first of these and the preceding paragraph, we have $\Psi_A(\eta,\zeta) \;=\; k \cdot \Psi_B(\eta,\zeta)$. Using this, the second equation above and the preceding paragraph, we see that $\Psi_A(\eta,\eta) \;=\; k \cdot \Psi_B(\eta,\eta)$ if η represents a point Y in the tangent hyperplane to Σ at X. Applying this and the first displayed equation to arbitrary nonzero vectors $\eta,\ \zeta \in \mathbb{F}^{n+1,1}$, we see that $\Psi_A(\eta,\zeta) \;=\; k \cdot \Psi_B(\eta,\zeta)$. Since $c_{i,j}$ is the value of $\Psi_C(\mathbf{e}_i,\mathbf{e}_j)$ if $\mathbf{e}_i$ and $\mathbf{e}_j$ are the standard unit vectors (the kth coordinate of $\mathbf{e}_k$ is 1 and the rest are 0), we see that $a_{i,j} \;=\; k \cdot b_{i,j}$ for all i and j, and hence we see that $B = k \cdot A.\blacksquare$

Intersections of two conics

Earlier in this section we noted that a line and a quadric intersect in at most two points. This may be viewed as a generalization of a standard fact from elementary algebra; namely, if we are given a system of two equations in two unknowns, with one linear and the other quadratic, then the system of two equations has at most two solutions. There is a similar principle which states that two quadratic equations in two unknowns have at most four solutions. In terms of analytic geometry, this means that two conics have at most four points in common. A geometrical derivation of the latter result for projective conics appears in Exercise II.6.11, and in the note following this exercise the statement about solutions to systems of quadratic equations is also discussed.

EXERCISES

In all these exercises $\mathbb{F}$ denotes a (commutative) field in which $1 + 1 \neq 0$.

1. Find the singular points (if any) of the projective conics given in Exercise 3 of the previous section.

2. Find the equations of the tangent lines to the following conics in $\mathbb{RP}^2$ at the indicated points:

(*i*) The conic defined by $x_1^2 + 2x_1x_2 + 4x_1x_3 + 3x_2^2 - 12x_1x_3 + 2x_3^2 \;=\; 0$ at the points

$$ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}. $$

(*ii*) The conic defined by $x_1^2 - 2x_1x_2 + 4x_2^2 - 4x_3^2 = 0$ at the points

$$\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

Definition. Let Σ be a hyperquadric in $\mathbb{FP}^n$ defined by the $(n+1) \times (n+1)$ matrix A such that Σ has at least one nonsingular point. Two points X and Y in $\mathbb{FP}^n$ are said to be *conjugate* with respect to Σ if they have homogeneous coordinates ξ and η respectively such that $^{\mathbf{T}}\xi A \eta = 0$. By Theorem 6, this definition does not depend upon any of the choices (including A). Moreover, a point is self-conjugate if and only if it lies on Σ.

3. In the setting above, assume that $X \notin \Sigma$ and Y is conjugate to P with respect to Σ. Suppose that $XY \cap \Sigma$ consists of two points, say A and B. Prove that $\mathsf{XR}(X, Y, A, B) = -1$.

Note. If Σ is nonsingular and nonempty (hence A is invertible by Theorem 5) and $X \in \mathbb{FP}^n$, then a hyperplane with homogeneous coordinates $^{\mathbf{T}}\xi A$ is called the *polar hyperplane* of X with respect to Σ. The map $\mathbf{P}$ sending X to its polar hyperplane is a collineation from $\mathbb{FP}^n$ to its dual $\left(\mathbb{FP}^n\right)^*$ is called a **polarity**, and it has the property that the composite

$$\mathbb{FP}^n \xrightarrow[\cong]{\mathbf{P}} \left(\mathbb{FP}^n\right)^* \xrightarrow[\cong]{\mathbf{P}^*} \left(\mathbb{FP}^n\right)^{**}$$

is the identity.

4. Let Σ be an affine hyperquadric in $\mathbb{F}^n$, where $n \geq 3$, and suppose that L is a line in $\mathbb{F}^n$ such that $L \subset \Sigma$. Denote the projective extension of Σ by Σ^*. Prove that the ideal point L_∞, and in fact the entire projective line

$$\mathsf{J}[L] \cup \{L_\infty\}$$

is contained in Σ^*. [*Hint:* The field $\mathbb{F}$ contains at least three elements. What does this imply about the number of points on L, and how does this lead to the desired conclusion?]

5. Prove the determinant identity stated in the proof of Theorem 2 (when each $y_i = 1$ this is the classical Vandermonde determinant).[6]

6. Consider the conic in $\mathbb{FP}^2$ defined by the equation $x_1x_2 - x_3^2 = 0$. What are the tangent lines to this curve at its points of intersection with the line at infinity defined by $x_3 = 0$? [*Hint:* If $\mathbb{F} = \mathbb{R}$ then this conic is the projective extension of the hyperbola with equation $xy = 1$.]

7. Answer the corresponding question for the conic defined by the equation $x_1^2 - x_2x_3 = 0$. [*Hint:* If $\mathbb{F} = \mathbb{R}$ then this conic is the projective extension of the parabola with equation $y = x^2$.]

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[6]See `http://www.math.duke.edu/~johnt/math107/vandermonde.pdf` for more on this topic. ALEXANDRE THÉOPHILE VANDERMONDE (1735–1796) is known for his work on the roots of polynomial equations and his fundamental results on determinants.

3. Bilinear forms

At this point it is convenient to discuss a topic in linear algebra which is generally not covered in first courses on the subject. For the time being, $\mathbb{F}$ will be a (commutative field with no assumption on whether or not $1 + 1 = 0$ or $1 + 1 \neq 0$.

Definition.   Let V be a vector space over $\mathbb{F}$. A *bilinear form* on $\mathbb{F}$ is a function

$$\Phi : V \times V \longrightarrow \mathbb{F}$$

with the following properties:

(Bi–1) $\Phi(\mathbf{v} + \mathbf{v}', \mathbf{w}) \; = \; \Phi(\mathbf{v}, \mathbf{w}) \; + \; \Phi(\mathbf{v}', \mathbf{w})$ for all $\mathbf{v}$, $\mathbf{v}'$, $\mathbf{w} \; \in \; V$.

(Bi–2) $\Phi(\mathbf{v}, \mathbf{w} + \mathbf{w}') \; = \; \Phi(\mathbf{v}, \mathbf{w}) \; + \; \Phi(\mathbf{v}, \mathbf{w}')$ for all $\mathbf{v}$, $\mathbf{w}$, $\mathbf{w}' \; \in \; V$.

(Bi–3) $\Phi(c \cdot \mathbf{v}, \mathbf{w}) \; = \; c \cdot \Phi(\mathbf{v}, \mathbf{w}) \; = \; \Phi(\mathbf{v}, c \cdot \mathbf{w})$ for all $\mathbf{v}$, , $\mathbf{w} \; \in \; V$ and $c \; \in \; \mathbb{F}$.

The reader will notice the similarities between the identities for Φ and the identities defining the dot product on $\mathbb{R}^n$. Both are scalar valued, distributive in both variables, and homogeneous (of degree 1) with respect to scalars. However, we are not assuming that Φ is commutative — in other words, we make no assumption about the difference between $\Phi(\mathbf{v}, \mathbf{w})$ and $\Phi(\mathbf{w}, \mathbf{v})$ — and we can have $\Phi(\mathbf{x}, \mathbf{x}) \; = \; 0$ even if $\mathbf{x}$ is nonzero.

EXAMPLES. **1.**   Let $\mathbb{F} = \mathbb{R}$ and $V = \mathbb{R}^2$, and let $\Phi(\mathbf{x}, \mathbf{y}) \; = \; x_1 y_2 - x_2 y_1$, where by convention $\mathbf{a} \in \mathbb{R}^2$ can be written in coordinate form as (a_1, a_2). Then $\Phi(\mathbf{y}, \mathbf{x}) \; = \; -\Phi(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}$ and $\mathbf{y}$ and we also have $\Phi(\mathbf{z}, \mathbf{z}) \; = \; 0$ for all $\mathbf{z} \in \mathbb{R}^2$.

2.   Let $\mathbb{F}$ and V be as above, and $\Phi(\mathbf{x}, \mathbf{y}) \; = \; x_1 y_1 - x_2 y_2$. In this case we have the commutativity identity $\Phi(\mathbf{y}, \mathbf{x}) \; = \; \Phi(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}$ and $\mathbf{y}$, but if $\mathbf{z} = (1, 1)$, or any multiple of the latter, then $\Phi(\mathbf{z}, \mathbf{z}) \; = \; 0$.

3.   Let A be an $n \times n$ matrix over $\mathbb{F}$, and let V be the vector space of all $n \times 1$ column matrices. Define a bilinear form Φ_A on V by the formula

$$\Phi_A(\mathbf{x}, \mathbf{y}) \; = \; {}^{\mathrm{T}}\mathbf{x} A \mathbf{y} \; .$$

Examples of this sort appeared frequently in the preceding section (see also Appendix E). Actually, the first two examples are special cases of this construction in which A is given as follows:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In fact, the following theorem shows that, in principle, the preceding construction gives **all** possible bilinear forms on finite-dimensional vector spaces.

THEOREM VII.7. *Let v be an$n$-dimensional vector space over $\mathbb{F}$, and let $\mathcal{A} \; = \; \{\mathbf{a}_1, \; \cdots \; , \mathbf{a}_n\}$ be an ordered basis for V. If Φ is a bilinear form over $\mathbb{F}$, let $[\Phi]_{\mathcal{A}}$ be the $n \times n$ matrix whose (i, j) entry is equal to $\Phi(\mathbf{a}_i, \mathbf{a}_j)$. Then the map sending Φ to $[\Phi]_{\mathcal{A}}$ defines a $1 - 1$ correspondence between bilinear forms over V and $n \times n$ matrices over $\mathbb{F}$.*

The matrix $[\Phi]_{\mathcal{A}}$ is called *the matrix of Φ with respect to the ordered basis $\mathcal{A}$.*

Proof. *The mapping is $1-1$.* Suppose that we are given two bilinear forms Φ and Ψ such that $\Phi(\mathbf{a}_i, \mathbf{a}_j) = \Psi(\mathbf{a}_i, \mathbf{a}_j)$ for all i and j (this is the condition for $[\Phi]_{\mathcal{A}}$ and $[\Psi]_{\mathcal{A}}$ to be equal). If $\mathbf{v}, \mathbf{w} \in V$, express these vectors as linear combinations of the basis vectors as follows:

$$\mathbf{v} = \sum_i x_i \mathbf{a}_i \qquad \mathbf{w} = \sum_j y_j \mathbf{b}_j$$

Then by **(Bi–1) — (Bi–3)** we have

$$\Phi(\mathbf{v}, \mathbf{w}) = \sum_{i,j} x_i y_j \Phi(\mathbf{a}_i, \mathbf{a}_j) = \sum_{i,j} x_i y_j \Psi(\mathbf{a}_i, \mathbf{a}_j) = \Psi(\mathbf{v}, \mathbf{w})$$

and since $\mathbf{v}$ and $\mathbf{w}$ are arbitrary we have $\Phi = \Psi$.$\square$

The mapping is onto. If B is an $n \times n$ matrix and $\mathbf{v}, \mathbf{w} \in V$ are as in the preceding paragraph, define

$$\mathbf{f}_{B,\mathcal{A}} = \sum_{i,j} x_i y_j b_{i,j} .$$

This is well-defined because the coefficients of $\mathbf{v}$ and $\mathbf{w}$ with respect to $\mathcal{A}$ are uniquely determined. The proof that $\mathbf{f}_{B,\mathcal{A}}$ satisfies **(Bi–1) — (Bi–3)** is a sequence of routine but slightly messy calculations, and it is left as an exercise. Given this, it follows immediately that B is equal to $[\mathbf{f}_{B\mathcal{A}}]_{\mathcal{A}}$.■

CHANGE OF BASIS FORMULA. Suppose we are given a bilinear form Φ on an n-dimensional vector space V over $\mathbb{F}$, and let $\mathcal{A}$ and $\mathcal{B}$ be ordered basis for V. In several contexts it is useful to understand the relationship between the matrices $[\Phi]_{\mathcal{A}}$ and $[\Phi]_{\mathcal{B}}$. The equation relating these matrices are given by the following result:

THEOREM VII.8. *Given two ordered bases $\mathcal{A}$ and $\mathcal{B}$, define a transition matrix by the form*

$$\mathbf{b}_j = \sum_i p_{i,j} \mathbf{a}_i .$$

If Φ is a bilinear form on V as above, then we have

$$[\Phi]_{\mathcal{B}} = {}^{\mathbf{T}}P [\Phi]_{\mathcal{A}} P .$$

Proof. We only need to calculate $\Phi(\mathbf{b}_i, \mathbf{b}_j)$; by the equations above, we have

$$\Phi(\mathbf{b}_i, \mathbf{b}_j) = \Phi\left(\sum_k p_{k,i} \mathbf{a}_k, \sum_m p_{m,j} \mathbf{a}_m \right) =$$

$$\sum_k \left(p_{k,i} \left(\sum_m p_{m,k} \Phi(\mathbf{a}_k, \mathbf{a}_m) \right) \right) .$$

However, the coefficient of $p_{k,i}$ is just the (k,j) entry of $[\Phi]_{\mathcal{A}} P$, and hence the entire summation is just the (i,j) entry of ${}^{\mathbf{T}}P [\Phi]_{\mathcal{A}} P$, as claimed.■

Definition. A bilinear form Φ is *symmetric* if $\Phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}$ and $\mathbf{y}$.

THEOREM VII.9. *Let Φ and $\mathcal{A}$ be as in Theorem 7. Then Φ is symmetric if and only if $[\Phi]_{\mathcal{A}}$ is a symmetric matrix.*

Proof.   Suppose that Φ is symmetric. Then $\Phi(\mathbf{a}_i, \mathbf{a}_j) \; = \; \Phi(\mathbf{a}_j, \mathbf{a}_i)$ for all i and j, and this implies that $[\Phi]_{\mathcal{A}}$ is a symmetric matrix.

Conversely, if $[\Phi]_{\mathcal{A}}$ is symmetric and $\mathbf{v}, \; \mathbf{w} \in V$ (the same notation as in Theorem 7), then by Theorem 7 we have

$$\Phi(\mathbf{v}, \mathbf{w}) \; = \; \sum_{i,j} \left([\Phi]_{\mathcal{A}}\right)_{i,j} x_i y_j \qquad\qquad \Phi(\mathbf{w}, \mathbf{v}) \; = \; \sum_{i,j} \left([\Phi]_{\mathcal{A}}\right)_{j,i} x_i y_j \; .$$

Since $[\Phi]_{\mathcal{A}}$ is symmetric, the two summations are equal, and therefore we must have

$$\Phi(\mathbf{y}, \mathbf{x}) \; = \; \Phi(\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x}$ and $\mathbf{y}$.∎

We have introduced all of the preceding algebraic machinery in order to prove the following result:

THEOREM VII.10. *Let $\mathbb{F}$ be a field in which $1 + 1 \neq 0$, and let A be a symmetric $n \times n$ matrix over $\mathbb{F}$. Then there is an invertible matrix P such that $^{\mathbf{T}}PAP$ is a diagonal matrix.*

This will be a consequence of the next result:

THEOREM VII.11. *Let Φ be a symmetric bilinear form on an n-dimensional vector space V over a field $\mathbb{F}$ for which $1 + 1 \neq 0$. Then there is an ordered basis $\mathbf{v}_1, \; \cdot \; , \mathbf{v}_n$ of V such that $\Phi(\mathbf{v}_i, \mathbf{v}_j) \; = \; 0$ if $i \neq j$ and $\Phi(\mathbf{v}_i, \mathbf{v}_i) \; = \; d_i$ for suitable scalars $d_i \in \mathbb{F}$.*

Proof that Theorem 11 implies Theorem 10.   Define a bilinear form Φ_A as in Example 1 above. By construction $[\Phi_A]_{\mathcal{U}} \; = \; A$, where $\mathcal{U}$ is the ordered basis obtained of standard unit vectors. On the other hand, if $\mathcal{V}$ is the ordered basis obtained from Theorem 11, then $[\Phi_A]_{\mathcal{V}}$ is a diagonal matrix. Apply Theorem 8 with $\Phi \; = \; \Phi_A$, $\mathcal{A} \; = \; \mathcal{U}$, and $\mathcal{B} \; = \; \mathcal{V}$.∎

Proof of Theorem 11.   If $\dim V \; = \; 1$, the result is trivial. Assume by induction that the result holds for vector spaces of dimension $n - 1$.

CASE 1.   Suppose that $\Phi(\mathbf{x}, \mathbf{x}) \; = \; 0$ for all $\mathbf{x}$. Then $\Phi(\mathbf{x}, \mathbf{y}) \; = \; 0$ for all $\mathbf{x}$ and $\mathbf{y}$ because we have

$$\Phi(\mathbf{x}, \mathbf{y}) \; = \; \tfrac{1}{2}\Phi(\mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y}) \; - \; \Phi(\mathbf{x}, \mathbf{x}) \; - \; \Phi(\mathbf{y}, \mathbf{y})$$

and consequently $[\Phi]_{\mathcal{A}} \; = \; \mathbf{0}$ for *every* ordered basis $\mathcal{A}$.□

CASE 2.   Suppose that $\Phi(\mathbf{v}, \mathbf{v}) \; \neq \; 0$ for some $\mathbf{v}$. Let W be the set of all $\mathbf{x} \in V$ such that $\Phi(\mathbf{x}, \mathbf{v}) \; = \; 0$.[7] We claim that $W + \mathbb{F} \cdot \mathbf{v} \; = \; V$ and $W \cap \mathbb{F} \cdot \mathbf{v} \; = \; \{\mathbf{0}\}$. — The second assertion is trivial because $\Phi(\mathbf{v}, c \cdot \mathbf{v}) \; = \; 0$ implies that $c \cdot \Phi(\mathbf{v}, \mathbf{v}) \; = \; 0$. Since $\Phi(\mathbf{v}, \mathbf{v}) \; \neq \; 0$, this can only happen if $c \; = \; 0$, so that $c \cdot \mathbf{v} \; = \; \mathbf{0}$. To prove the first assertion, we must observe that for arbitrary $\mathbf{v} \in V$ the vector

$$\Pi(\mathbf{x}) \; = \; \mathbf{x} \; - \; \frac{\Phi(\mathbf{x}, \mathbf{v})}{\Phi(\mathbf{v}, \mathbf{v})} \, \mathbf{v}$$

---

[7]If Φ is the usual dot product on $\mathbb{R}^n$, then this is the hyperplane through $\mathbf{0}$ that is perpendicular to the line $\mathbf{0v}$.

lies in W (to verify this, compute $\Phi\big(\Pi(\mathbf{x}), \mathbf{v}\big)$ explicitly).[8] The conditions on W and $\mathbb{F} \cdot \mathbf{v}$ together with the dimension formulas imply that $\dim W = n - 1$.

Consider the form Ψ obtained by restricting Φ to W; it follows immediately that Ψ is also symmetric. By the induction hypothesis there is a basis $\mathbf{w}_1, \cdots, \mathbf{w}_{n-1}$ for W such that $\Phi(\mathbf{w}_i, \mathbf{w}_j) = 0$ if $i \neq j$. If we adjoin $\mathbf{v}$ to this set, then by the conditions on W and $\mathbb{F} \cdot \mathbf{v}$ we obtain a basis for V. Since $\Phi(\mathbf{v}, \mathbf{w}_j)$ is zero for all j by the definition of W, it follows that the basis for V given by $\mathbf{v}$ together with $\mathbf{w}_1, \cdots, \mathbf{w}_{n-1}$ will have the desired properties.∎

The proof above actually gives and explicit method for finding a basis with the required properties: Specifically, start with a basis $\mathbf{v}_1, \cdots, \mathbf{v}_n$ for V. If some $\mathbf{v}_i$ has the property $\Phi(\mathbf{v}_i \mathbf{v}_i) \neq 0$, rearrange the vectors so that the first one has this property. If $\Phi(\mathbf{v}_i, \mathbf{v}_i) = 0$ for all i, then either $\Phi = \mathbf{0}$ or else some value $\Phi(\mathbf{v}_i, \mathbf{v}_j)$ is nonzero (otherwise $\Phi = \mathbf{0}$ by Theorem 10). Rearrange the basis so that $\Phi(\mathbf{v}_1, \mathbf{v}_2) \neq 0$, and take a new basis $\{\, \mathbf{v}_i \,\}$ with $\mathbf{v}_1' = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_i' = \mathbf{v}_i$ otherwise. Then $\Phi(\mathbf{v}_1', \mathbf{v}_1') \neq 0$, and thus in all cases we have modified the original basis to one having this property.

Now we modify $\mathbf{v}_i'$ such that $\mathbf{v}_1'' = \mathbf{v}_1'$ and $\Phi(\mathbf{v}_i'', \mathbf{v}_1'') = 0$ if $i > 1$. Specifically, if $i \geq 2$ let

$$\mathbf{v}_i'' = \mathbf{v}_i' - \frac{\Phi(\mathbf{v}_i', \mathbf{v}_1')}{\Phi(\mathbf{v}_1', \mathbf{v}_1')} \, \mathbf{v}_1' \, .$$

Having done this, we repeat the construction for $\mathbf{w}_1, \cdots, \mathbf{w}_{n-1}$ for W with $\mathbf{w}_i = \mathbf{v}_{i+1}''$. When computing explicit numerical examples, it is often convenient to "clear the denominator of fractions" and multiply $\mathbf{v}_i''$ by $\Phi(\mathbf{v}_1', \mathbf{v}_1')$. This is particularly true when the matrix $\Phi(\mathbf{v}_i, \mathbf{v}_j)$ are integers (as in Exercise 2 below).

EXERCISES

1. Prove that the map sending bilinear forms to matrices in Theorem 7 is surjective.

2. Find an invertible matrix P such that ${}^{\mathrm{T}}PAP$ is diagonal, where A is the each of the following matrices with real entries:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} .$$

3. A symmetric bilinear form Φ on an n-dimensional vector space V over a field $\mathbb{F}$ is said to be *nondegenerate* if for each nonzero $\mathbf{x} \in V$ there is some $\mathbf{y} \in V$ such that $\Phi(\mathbf{x}, \mathbf{y}) \neq 0$. Given an ordered basis $\mathcal{A}$ for V, show that Φ is nondegenerate if and only if the matrix $[\Phi]_{\mathcal{A}}$ is invertible. [*Hint:* Suppose that $\mathbf{x}$ satisfies $B\mathbf{x} = \mathbf{0}$, where B is the matrix in the previous sentence, and let $\mathbf{v} = \sum_i x_i \mathbf{a}_i$. If $\mathbf{w} = \sum_j y_j \mathbf{z}_j$, explain why ${}^{\mathrm{T}}\mathbf{y}B\mathbf{x} = \Phi(\mathbf{x}, \mathbf{y})$ and how this is relevant.]

---

[8]If Φ is the ordinary dot product, then $\Pi(\mathbf{x})$ is the foot of the perpendicular dropped from $\mathbf{x}$ to the plane determined by W, and hence $\mathbf{0}\Pi(\mathbf{x})$ is perpendicular to W.

4. Projective classification of hyperquadrics

A standard exercise in plane and solid analytic geometry is the classification of conics and quadrics up to changes of coordinates given by rotations, reflections and translations. Stated differently, the preceding is the classification up to finding a rigid motion sending on to the other. An account of the classification for arbitrary dimensions appears on pages 257–262 of Birkhoff and MacLane, *Survey of Modern Algebra* (3rd Ed.). A related classification (up to finding an affine transformation instead of merely a rigid motion) is discussed in Exercise 4 below. In this section we are interested in the corresponding projective problem involving projective hyperquadrics and (projective) collineations.

Throughout this section we assume that $\mathbb{F}$ is a field in which $1+1 \neq 0$. Furthermore, if $\Sigma \subset \mathbb{FP}^n$ is a hyperquadric, then we shall use $\mathsf{SingSet}(\Sigma)$ to denote its subset of singular points.

We shall begin with an important observation.

THEOREM VII.12. *Let g be a projective collineation of $\mathbb{FP}^n$. Then a subset $\Sigma \subset \mathbb{FP}^n$ is a hyperquadric if and only if $g[\Sigma]$ is. Furthermore, the singular sets of these hyperquadrics satisfy*

$$g\left[\, \mathsf{SingSet}(\Sigma) \,\right] \;=\; \mathsf{SingSet}\left(\, T[\Sigma] \,\right)$$

and if $\mathrm{Tang}_X(\Sigma)$ denotes the tangent hyperplane to Σ at a nonsingular point X, then

$$g\left[\, \mathrm{Tang}_X(\Sigma) \,\right] \;=\; \mathrm{Tang}_{g(X)}\left(\, T[\Sigma] \,\right) \ .$$

Proof.   Let A be a symmetric $(n+1) \times (n+1)$ matrix which defines the hyperquadric Σ.

According to Theorem VI.14, there is an invertible linear transformation C of $\mathbb{F}^{n+1,1}$ such that $T\left(\mathbb{F} \cdot \xi\right) \;=\; \mathbb{F} \cdot C(\xi)$ for all nonzero vectors $\xi \in \mathbb{F}^{n+1,1}$. Let B be the matrix of C in the standard basis. Then X lies in $T[\Sigma]$ if and only if $T^{-1}(X)$ lies in Σ. If ξ is a set of homogeneous coordinates for X, then the conditions in the preceding sentence are equivalent to

$$^{\mathbf{T}}\xi\, ^{\mathbf{T}}B^{-1}AB^{-1}\xi \;-\; 0$$

and the displayed equation is equivalent to saying that X lies on the hyperquadric associated to the (symmetric) matrix $^{\mathbf{T}}B^{-1}AB^{-1}$.

To check the statement about singular points, note that a point X lies on $\mathsf{SingSet}(\Sigma)$ if and only if X has homogeneous coordinates ξ such that $^{\mathbf{T}}\xi A \;=\; \mathbf{0}$, and the latter is equivalent to

$$^{\mathbf{T}}\xi\, ^{\mathbf{T}}B\, ^{\mathbf{T}}B^{-1}AB^{-1} \;=\; \mathbf{0}$$

which in turn is equivalent to

$$^{\mathbf{T}}(B\xi) \cdot \left(^{\mathbf{T}}B^{-1}AB^{-1}\right) \;=\; \mathbf{0} \ .$$

To check the statement on tangent hyperplanes, note that Y lies on the tangent hyperplane to Σ at X if and only if there are homogeneous coordinates ξ for X and η for Y such that $^{\mathbf{T}}\xi A\eta \;=\; \mathbf{0}$, and the latter is equivalent to

$$^{\mathbf{T}}\xi\, ^{\mathbf{T}}B\, ^{\mathbf{T}}B^{-1}AB^{-1}B\eta \;=\; \mathbf{0}$$

which in turn is equivalent to

$$^{\mathbf{T}}(B\xi) \cdot \left(^{\mathbf{T}}B^{-1}AB^{-1}\right)\eta) \;=\; \mathbf{0} \ .$$

The latter is equivalent to saying that $T(Y)$ is in the tangent hyperplane to $T[\Sigma]$ at $T(X)$.∎

Definition.   Two hypequadrics Σ and Σ' are *projectively equivalent* if there is a projective collineation T such that $T[\Sigma] = \Sigma'$. We sometimes write this relation as $\Sigma \sim \Sigma'$. It is clearly an equivalence relation, and the main goal of this section is to understand this relation when $\mathbb{F}$ is the real or complex numbers.

We shall first describe some necessary and sufficient conditions for the projective equivalence of hyperquadrics.

THEOREM VII.13. *Let Σ be a hyperquadric in $\mathbb{FP}^n$, and let T be a projective collineation of $\mathbb{FP}^n$. Then the following hold:*

(i) *The dimensions of the geometrical subspaces of singular points of Σ and $T[\Sigma]$ must be equal.*

(ii) *If Σ contains no geometrical subspace of dimension r, then neither does $T[\Sigma]$.*

Proof.   (i) By definition, $\mathsf{SingSet}(\Sigma)$ is the set of all X whose homogeneous coordinates ξ satisfy $\mathbf{T}\xi A = \mathbf{0}$, and hence $\mathsf{SingSet}(\Sigma)$ is a geometrical subspace. Now Theorem 12 implies that $T[\mathsf{SingSet}(\Sigma)] = \mathsf{SingSet}T[\Sigma]$, and hence

$$\dim \mathsf{SingSet}(\Sigma) = \dim T[\mathsf{SingSet}(\Sigma)] = \dim\big(\mathsf{SingSet}\,T[\Sigma]\big) .\square$$

(ii) Suppose $Q \subset T[\Sigma]$ is an r-dimensional geometrical subspace. Since T^{-1} is also a projective collineation, the set

$$T^{-1}[Q] \subset T^{-1}\big[T[\Sigma]\big] = \Sigma$$

is also an r-plane.∎

THEOREM VII.14. *Suppose that Σ and Σ' are hyperquadrics which are defined by the symmetric matrices A and B respectively. Assume that there is an invertible matrix C and a nonzero constant k such that $B = \mathbf{T}CAC$. Then Σ and Σ' are projectively equivalent.*

Proof.   Let T be the projective collineation defined by C^{-1}, and if $X \in \mathbb{FP}^n$ let ξ be a set of homogeneous coordinates for X. Then by Theorem 12 we have

$$T[\Sigma] = \{\, \mathbf{x} \mid \mathbf{T}\xi\mathbf{T}CAC\xi = 0 \,\} = \{\, \mathbf{x} \mid \mathbf{T}\xi(k^{-1}B)\xi = 0 \,\} = \{\, \mathbf{x} \mid k^{-1}\big(\mathbf{T}\xi B\xi\big) = 0 \,\} = \Sigma' .∎$$

NOTATION.   Let D_r be the $n \times n$ diagonal matrix $(n \geq r)$ with ones in the first r entries and zeros elsewhere, and let $D_{p,q}$ denote the $n \times n$ diagonal matrix $(n \geq p+q)$ with ones in the first entries, (-1)'s in the next q entries, and zeros elsewhere.

REMARKS. 1.   If A is a symmetric matrix over the complex numbers, then for some invertible matrix P the product $\mathbf{T}PAP$ is D_r for some r. For Theorem 10 guarantees the existence of an invertible matrix P_0 such that $A_1 = \mathbf{T}P_0AP_0$ is diagonal. Let P_1 be the diagonal matrix whose entries are square roots of the corresponding nonzero diagonal entries of A_1, and ones in the places where A_1 has zero diagonal entries. Then the product $P = P_0P_1^{-1}$ has the desired properties. This uses the fact that every element of the complex numbers $\mathbb{C}$ has a square root in $\mathbb{C}$, and in fact the same argument works in every field $\mathbb{F}$ which is closed under taking square roots.

2.   If A is a symmetric matrix over the **complex** numbers, then for some invertible matrix P the product $^{\mathbf{T}}PAP$ is $D_{p,q}$ for some p and q. As in the preceding example, choose an invertible matrix P_0 such that $A_1 = {}^{\mathbf{T}}P_0AP_0$ is diagonal. Let P_1 be the diagonal matrix whose entries are square roots of the **absolute values** of the corresponding nonzero diagonal entries of A_1, and ones in the places where A_1 has zero diagonal entries. Then the product $P = P_0 P_1^{-1}$ has the desired properties. The need for more complicated matrices arises because over $\mathbb{R}$ one only has square roots of nonnegative numbers, and if $x \in \mathbb{R}$ then either x or $-x$ is nonnegative.

The preceding remarks and Theorems 12–14 yield a complete classification of hyperquadrics in $\mathbb{FP}^n$ up to projective equivalence if $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. We shall start with the complex case, which is easier.

THEOREM VII.15.   *Let $\Gamma_r \subset \mathbb{CP}^n$ defined by the matrix D_r described above. Then every nonempty hyperquadric in $\mathbb{CP}^n$ is projectively equivalent to Γ_r for some uniquely determined value of r.*

Proof.   By Remark 1 above and Theorem 14, we know that Σ is projectively equivalent to Γ_r for some r. It suffices to show that if Γ_r and Γ_s are projectively equivalent then $r = s$.

By the preceding results we know that $\dim \mathsf{SingSet}(\Gamma_r)$ is the dimension of the subspace of all X whose homogeneous coordinates ξ satisfy $^{\mathbf{T}}\xi D_r = \mathbf{0}$, and the dimension of that subspace is equal to $n - r + 1$. Therefore, if Γ_r and Γ_s are projectively equivalent then we must have $n - r + 1 = n - s + 1$, which implies that $r = s$, so there is only one such hyperquadric that can be equivalent to Σ and thus uniqueness follows.∎

The preceding argument goes through if $\mathbb{C}$ is replaced by an arbitrary field $\mathbb{F}$ which is closed under taking square roots.

Over the real numbers, the classification is somewhat more complicated but still relatively simple.

THEOREM VII.16.   *Let $\Gamma_{p,q} \subset \mathbb{RP}^n$ defined by the diagonal matrix $D_{p,q}$ described above. Then every nonempty hyperquadric in $\mathbb{RP}^n$ is projectively equivalent to $\Gamma_{p,q}$ for some uniquely determined values of p and q such that $p \geq q$.*

Proof.   As in the proof of the preceding theorem, by Theorem 14 and Remark 1 we know that an arbitrary projective hyperquadric is projectively equivalent to $\Gamma_{p,q}$ for some p and q. This hyperquadric is represented by $D_{p,q}$; if we permute the homogeneous coordinates, we see that $\Gamma_{p,q}$ is projectively equivalent to the hyperquadric defined by the matrix $-D_{q,p}$, and since the negative of this matrix defines the same hyperquadric it follows that $\Gamma_{p,q}$ is projectively equivalent to $\Gamma_{q,p}$. Since either $p \geq q$ or $q \geq p$, it follows that every hyperquadric is projectively equivalent to $\Gamma_{u,v}$ for some $u \geq v$.

To complete the proof, it will suffice to show that if $\Gamma_{p,q}$ is projectively equivalent to $\Gamma_{u,v}$ where $p \geq q$ and $u \geq v$, then $p + q = u + v$ and $p = u$. To see the first equality, note that the dimension of $\mathsf{SingSet}(\Gamma_{a,b})$ is equal to $n - (a + b) + 1$ by the argument in the preceding theorem, and as in that proof we conclude that $p + q = u + v$.

To see the second equality, we shall characterize the integer p in $\Gamma_{p,q}$ as follows.

($\ddagger$) *The hyperquadric $\Gamma_{p,q}$ contains a geometric subspace of dimension $n - p$ but no such subspace of higher dimension.*

This and the second part of Theorem 13 will combine to prove that if $\Gamma_{p,q}$ is projectively equivalent to $\Gamma_{u,v}$ where $p \geq q$ and $u \geq v$, then we also have $p = u$.

An explicit geometrical subspace S of dimension $N - p$ is given by the equations
$$
\begin{aligned}
x_i - x_{p+i} &= 0 & 1 \leq i \leq q, \\
x_i &= 0 & q \leq i \leq p \;.
\end{aligned}
$$
Consider the geometrical subspace T defined by
$$
x_{p+1} = x_{p+2} = \cdots = x_{n+1} = 0 \;.
$$
This geometrical subspace is $(p-1)$-dimensional. Furthermore, if $X \in T \cap \Sigma$ has homogeneous coordinates $(x_1, \cdots, x_{n+1})$ we have $x_i = 0$ for $i > p$, so that
$$
\sum_{i \leq p} x_i^2 = 0 \;.
$$
The latter implies that $x_i = 0$ for $i \leq p$, and hence it follows that $x_i = 0$ for **all** i; this means that *the intersection $T \cap \Sigma$ is the empty set.*

Suppose now that $S' \subset \Sigma$ is a geometrical subspace of dimension $\geq n - p + 1$. Then the addition law for dimensions combined with $\dim(S' \star T) \leq n$ shows that $S' \cap T \neq \varnothing$, and since $S' \subset \Sigma$ we would also have $\Sigma \cap T \neq \varnothing$. But we have shown that the latter intersection is empty, and hence it follows that Σ cannot contain a geometrical subspace of dimension greater than $(n-p)$, which is what we needed to show in order to complete the proof.∎

COMPUTATIONAL TECHNIQUES. Over the real numbers, there is another standard method for finding an equivalent hyperquadric defined by a diagonal matrix. Specifically, one can use the following diagonalization theorem for symmetric matrices to help find a projective collineation which takes a given hyperquadric to one of the given type:

> *Let A be a symmetric matrix over the real numbers. Then there is an orthogonal matrix P (one for which $^{\mathbf{T}}P = P^{-1}$) such that $^{\mathbf{T}}PAP$ is a diagonal matrix. Furthermore, if λ_i is the ith entry of the diagonal matrix, then the ith column of P is an eigenvector of A whose associated eigenvalue is equal to λ_i.∎*

This statement is often called the **Fundamental Theorem on Real Symmetric Matrices**, and further discussion appears on pages 51–52 of the following online notes:

<div align="center">http://math.ucr.edu/~res/math132/linalgnotes.pdf</div>

If we combine the Fundamental Theorem on Real Symmetric Matrices with other material from this section, we see that the construction of a projective collineation taking the hyperquadric Σ_A defined by A to a hyperquadric defined by an equation of the form
$$
\Sigma_i \; d_i x_i^2 = 0
$$
reduces to finding the eigenvalues and eigenvectors of A. This approach is probably the most effective general method for solving problems like those in Exercise 3 below.

SPECIALIZATION TO THE REAL PROJECTIVE PLANE. We shall conclude this section by restating a special case of Theorem 16 that plays a crucial role in Section 6.

THEOREM VII.17. *All nonempty nonsingular conics in $\mathbb{RP}^2$ are projectively equivalent. In fact, they are equivalent to the affine unit circle which is defined by the homogeneous coordinate equation $x_1^2 + x_2^2 - x_3^2 = 0$.*

Proof.   We must consider all $\Gamma_{p,q}$ with $p \geq q$ and $p + q = 3$ (this is the condition for the singular set to be empty). The only possibilities for (p, q) are $(2, 1)$ and $(3, 0)$. However, $\Gamma_{3,0}$ — the set of points whose homogeneous coordinates satisfy $x_1^2 + x_2^2 + x_3^2 = 0$ — is empty, so there is a unique possibility and it is given by $\Gamma_{2,1}$, which is the affine unit circle.■

EXERCISES

1.   For each projective quadric in Exercise VII.1.3, determine the quadric in $\mathbb{RP}^3$ to which it is projectively equivalent.

2.   Show that the number of projective equivalence classes of hypequadrics in $\mathbb{RP}^n$ is equal to $\frac{1}{4}(n + 2)(n + 4)$ if n is even and $\frac{1}{4}(n + 3)^2$ if n is odd.

3.   For each example below, find a projective collineation of $\mathbb{RP}^2$ that takes the projectivizations of the following affine conics into the unit circle (with affine equation $x^2 + y^2 = 1$).

(i) The hyperbola $xy = 4$.

(ii) The parabola $y = x^2$.

(iii) The ellipse $4x^2 + 9y^2 = 36$.

(iv) The hyperbola $4x^2 - 9y^2 = 36$.

4.   (a) What should it mean for two **affine** hyperquadrics in $\mathbb{R}^n$ to be affinely equivalent?

(b) Prove that every affine hyperquadric in $\mathbb{R}^n$ is equivalent to one defined by an equation from the following list:

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2 = 0 \qquad (r \leq n)$$

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2 + 1 = 0 \qquad (r \leq n)$$

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2 + x_{r+1} = 0 \qquad (r < n)$$

See Birkhoff and MacLane, pp. 261–264, or Section V.2 of the online notes

http://math.ucr.edu/~res/math132/linalgnotes.pdf

for further information on the classification of affine hyperquadrics.

5.   Let Γ be a nonempty nonsingular conic in $\mathbb{RP}^2$. Prove that there are infinitely many points X such that no line through X is tangent to Γ, but if a point $X \notin \Gamma$ lies on a tangent line to Γ then there is also a second tangent line to Γ through X, and there is also a line through X which does not meet Γ. [*Hint:*   Why does it suffice to prove these for the unit circle, and why are the statements true in that case? Note that for each point at infinity there is a tangent line to the unit circle.]

5. Duality and projective hyperquadrics

In this section we shall show that the duality properties for geometrical subspaces of coordinate projective spaces are part of a larger pattern of duality which includes hyperquadrics. As in most other sections of these notes, $\mathbb{F}$ will denote a (commutative) field in which $1 + 1 \neq 0$.

Definition. A *hypersurface of the second class* in $(\mathbb{F}\mathbb{P}^n)^*$ is the set $\mathfrak{F}$ of all hyperplanes H whose homogeneous coordinates θ satisfy a homogeneous quadratic equation
$$\theta B^{\mathbf{T}}\theta \;=\; 0.$$
If we write $\theta \;=\; (u_1, \cdots, u_{n+1})$ and B has entries $b_{i,j}$, this is equivalent to the scalar equation $\sum_{i,j} b_{i,j} u_i u_j \;=\; 0$.

The dual of a tangent line is a *cotangent $(n-2)$-subspace V* contained in a hyperplane H which belongs to the hypersurface $\mathfrak{F}$. The cotangent condition becomes an assertion that *exactly one of the following two statements is valid*:

  (i) H is the only hyperplane in $\mathfrak{F}$ containing V.

  (ii) Every hyperplane in $\mathfrak{F}$ contains V.

In the first case we say that H is a *nonsingular* hyperplane in $\mathfrak{F}$, and in the second we say that H is a *singular* hyperplane in $\mathfrak{F}$.

By duality the set of all cotangent $(n-2)$-spaces at a nonsingular hyperplane is the set of all $(n-2)$-spaces through a point called the *point of contact* of $\mathfrak{F}$ at H. This point has homogeneous coordinates $B^{\mathbf{T}}\theta$.

Nonsingular hyperquadrics and nonsingular hypersurfaces of the second class satisfy the following useful interrelationship:

THEOREM VII.18. (i) *The set of all hyperplanes $\mathbf{T}\Sigma$ tangent to a nonsingular hyperquadric Σ is a hypersurface of the second class.*

(ii) *The set of all points of contact $\mathbf{K}\mathfrak{F}$ to a nonsingular hypersurface $\mathfrak{F}$ of the second class is a nonsingular hyperquadric.*

(iii) *In the setting of the preceding two statements, we have $\mathbf{K}\mathbf{T}\Sigma \;=\; \Sigma$ and $\mathbf{T}\mathbf{K}\mathfrak{F} \;=\; \mathfrak{F}$.*

Proof. (i) Suppose Σ is defined as the set of all X whose homogeneous coordinates satisfy $^{\mathbf{T}}\xi A \xi \;=\; 0$. We claim that H lies in $\mathbf{T}\Sigma$ if and only if its homogeneous coordinates θ satisfy $\theta A^{-1}\,^{\mathbf{T}}\theta \;=\; 0$.

Suppose that $H \in \mathbf{T}\Sigma$. Let X be a point such that H is the tangent hyperplane to X, and let θ and ξ be homogeneous coordinates for H and X respectively. Then we have $\theta \;=\; \,^{\mathbf{T}}\xi A$, and hence
$$\theta A^{-1}\,^{\mathbf{T}}\theta \;=\; \left(^{\mathbf{T}}\xi A\right) \theta A^{-1}(A\xi) \;=\; \,^{\mathbf{T}}\xi A \xi \;=\; 0$$
which is what we wanted to prove.

Conversely, suppose that homogeneous coordinates θ for H satisfy the equation
$$\theta A^{-1}\,^{\mathbf{T}}\theta \;=\; 0 \;.$$

Let $\xi = A^{-1}\,{}^{\mathbf{T}}\theta$, and let X have homogeneous coordinates ξ. Then $\theta = {}^{\mathbf{T}}\xi A$ and
$$ {}^{\mathbf{T}}\xi A \xi = {}^{\mathbf{T}}\xi A\, A^{-1}\, A\xi = \theta A^{-1}\,{}^{\mathbf{T}}\theta = 0 $$
so that H lies in $\mathbf{T}\Sigma$.

Finally, note that $\mathbf{T}\Sigma$ is nonsingular because it is defined by the invertible matrix A^{-1}.$\Box$

(ii) The statement about $\mathbf{K}\mathfrak{F}$ follows by duality. It remains to show that $\mathbf{KT}\Sigma = \Sigma$. Howver, if Σ is defined by the invertible matrix A, then $\mathbf{T}\Sigma$ is defined by the inverse matrix A^{-1}, and therefore by duality it follows that $\mathbf{KT}\Sigma$ is defined by the matrix
$$ \left(A^{-1}\right)^{-1} = A $$
and hence it must be equal to Σ.

Finally, the assertion $\mathbf{TK}\mathfrak{F} = \mathfrak{F}$ follows by duality.$\blacksquare$

Extending duality to nonsingular hyperquadrics.   The preceding theorem implies the following general principle:

AUGMENTED PRINCIPLE OF DUALITY.   *A statement about coordinate projective n-spaces over fields remains true if — in addition to the previously specified interchanges involving geometrical subspaces — one interchanges the phrases* point on a nonsingular hyperquadric *and* tangent hyperplane to a nonsingular hyperquadric.$\blacksquare$

Important examples of this extended dualization will be given in the next (and final) section of these notes.

EXERCISES

1.   Find the equations defining the tangent lines to the projectivizations of the following affine conics:

(i) The parabola $y^2 = 4ax$.

(ii) The ellipse $a^2x^2 + b^2y^2 = a^2b^2$.

(iii) The hyperbola $a^2x^2 - b^2y^2 = a^2b^2$.

(iv) The hyperbola $xy = a$.

2.   Find the equation defining the conic in $\mathbb{RP}^2$ whose tangent lines satisfy the equation
$$ u_1^2 - 2u_1u_2 + u_2^2\,2u_2u_3 + 2u_1u_3 + u_3^2 = 0 \ . $$
[*Hint:* Look at the proof of Theorem 18.]

3.   Write out the plane dual to the following statements about conics in the projective plane $\mathbf{P}$:

(i) At the points X and Y on the nonsingular conic Γ, the respective tangent lines L and M meet at a point Z.

(ii) No three points of the nonsingular conic Γ are collinear.
(iii) There are two lines in the (projective) plane $\mathbf{P}$ that are tangent to both of the nonsingular conics Γ_1 and Γ_2.

6. Conics in the real projective plane

Projective conics have a great many interesting properties, the most famous of which is Pascal's Theorem (see Theorem 24 below). A thorough discussion of projective conics appears in Coolidge, *A History of the Conic Sections and Quadric Surfaces*. In this section we shall limit ourselves to proving a few of the more important and representative theorems in the subject.

Throughout this section we shall be considering coordinate projective planes over a fixed field $\mathbb{F}$ in which $1 + 1 \neq 0$. We shall also assume that $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_3$ after Theorem 22. Of course, this means that all the results in this section are valid in the real and complex projective planes.

THEOREM VII.19. *Given any five points in $\mathbb{FP}^2$, no three of which are collinear, there is a unique conic containing them. Furthermore, this conic is nonsingular.*

Proof. Let A, B, C, D, E, V be five points, no three of which are collinear. We shall first prove the result in a special case and then prove that it holds more generally.

CASE 1. Suppose that homogeneous coordinates α, β, γ, δ for A, B, C, D are given by standard values:

$$\alpha \;=\; \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \beta \;=\; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \gamma \;=\; \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \delta \;=\; \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Since no three of the five points are collinear, homogeneous coordinates for V are given by $a_0\alpha + b_0\beta + c_0\gamma$, where none of a_0, b_0, c_0 is equal to zero. Dividing by a_0, we can find homogeneous coordinates ψ for V such that $\psi \;=\; \alpha + b\beta + c\gamma$. Since $V \notin AD$, the scalars b and c must be distinct.

Suppose there is a conic Γ containing A, B, C, D, E, V and that it is defined by the symmetric 3×3 matrix P. We need to determine the entries $p_{i,j}$ from the equations ${}^{\mathbf{T}}\xi P\xi \;=\; 0$, which should hold for $\xi \;=\; \alpha$, β, γ, δ, ψ. If $\xi = \alpha$, then direct substitutions implies that $p_{1,1} \;=\; 0$. Likewise, if we make the substitutions $\xi = \beta$ and $\xi = \gamma$ we find that $p_{2,2} = p_{3,3} = 0$. Continuing in this manner, if we make the substitution $\xi = \delta$ and use the previously derived values for the diagonal entries, we find that

$$2 \cdot (p_{1,2} \;+\; p_{2,3} \;+\; p_{1,3}) \;=\; 0$$

and if we follow this with the substitution $\xi = \psi$ we also obtain the equation

$$2 \cdot (b\,p_{1,2} \;+\; bc\,p_{2,3} \;+\; c\,p_{1,3}) \;=\; 0 \;.$$

Thus the entries of the symmetric matrix P satisfy the following conditions:

$$p_{1,1} \;=\; p_{2,2} \;=\; p_{3,3} \;=\; 0$$

$$p_{1,2} \;=\; \frac{(1-b)c}{b-c}\,p_{2,3} \qquad\qquad p_{1,3} \;=\; \frac{(1-c)b}{c-b}\,p_{2,3}$$

Therefore the coefficients of P are uniquely determined up to a scalar multiple, and it follows that there is at most one conic containing the given five points. On the other hand, if we set

$$P = \begin{pmatrix} 0 & (1-b)c & b(c-1) \\ (1-b)c & 0 & b-c \\ b(c-1) & b-c & 0 \end{pmatrix}$$

then the preceding calculations show that the given five points lie in the conic defined by P.

To prove that the conic is nonsingular, it suffices to show that the determinant of the matrix P defined above is nonzero. First note that $V \notin CD$ and $V \notin BD$ imply $b \neq 1$ and $c \neq 1$ respectively. Thus the determinant of P, which is

$$2\,(1-b)c(b-c)b(b-1)$$

must be nonzero, proving that the conic is nonsingular.□

CASE 2.   Suppose that A, B, C, D, E, V are arbitrary. By the Fundamental Theorem of Projective Geometry, there is a projective collineation Φ such that

$$\Phi(A),\ \Phi(B),\ \Phi(C),\ \Phi(D),\ \Phi(E),\ \Phi(V)$$

satisfy the conditions of Case 1. Let Γ_0 be the unique nonsingular conic given by Case 1; then $\Gamma = \Phi^{-1}[\Gamma_0]$ is a nonsingular conic containing A, B, C, D, E, V by Theorem 12, proving existence. To show uniqueness, suppose that Γ' is another conic containing the given five points; then $\Phi[\Gamma']$ is a conic containing $\Phi(A)$, $\Phi(B)$, $\Phi(C)$, $\Phi(D)$, $\Phi(E)$, $\Phi(V)$ and therefore by Case 1 we have $\Phi[\Gamma'] = \Gamma_0$. Consequently, we have $\Gamma' = \Phi^{-1}\circ\Phi[\Gamma'] = \Phi^{-1}[\Gamma_0] = \Gamma$.■

If $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_3$ then there is a converse to the preceding theorem; namely, every nonsingular conic contains at least five points (see Theorem 21). In fact, no three of these points can be collinear, for the noncollinearity of three arbitrary points on a nonsingular conic will be a consequence of the next result.

THEOREM VII.20. *No three points on a nonsingular conic are collinear.*

Proof.   Let A, B, C be three collinear points. We claim that every conic containing all three of them is singular. By the Fundamental Theorem of Projective Geometry and Theorem 12, it suffices to consider the case in which homogeneous coordinates α and β for A and B are the first two unit vectors in $\mathbb{F}^{3,1}$.

By Theorem V.7, homogeneous coordinates γ for C may be chosen so that $\gamma = \alpha + c\gamma$, where $c \neq 0$. If the conic Γ is defined by the symmetric 3×3 matrix P, then computations like those of Theorem 19 imply that $p_{1,1} = p_{2,2} = 0$ and $2c\,p_{1,2} = 0$. Thus P has the following form:

$$P = \begin{pmatrix} 0 & 0 & p_{1,3} \\ 0 & 0 & p_{2,3} \\ p_{1,3} & p_{2,3} & p_{3,3} \end{pmatrix}$$

However, direct computation shows that such a matrix is not invertible, and therefore the conic Γ is singular by Theorem 5.■

Here is the other result we need to establish a converse to Theorem 19:

THEOREM VII.21. *Let Γ be a nonempty conic in $\mathbb{F}\mathbb{P}^2$ containing at least one nonsingular point, and assume that the field $\mathbb{F}$ contains at least n distinct elements. Then Γ contains at least $(n+1)$ distinct points.*

In particular, if $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_3$, then Γ contains at least five distinct points (note that $\mathbb{F}$ cannot be isomorphic to $\mathbb{Z}_2$ because we are assuming that $1 + 1 \neq 0$ in $\mathbb{F}$).

Proof. Let $X \in \Gamma$ be a nonsingular point, and let L be the tangent line through X. Then there are at least n other lines through X, say $L_1, \cdots, L_n$. Since each L_i is not a tangent line and $X \in L_i \cap \Gamma$, there must be a second point $X_i \in L_i \cap \Gamma$.
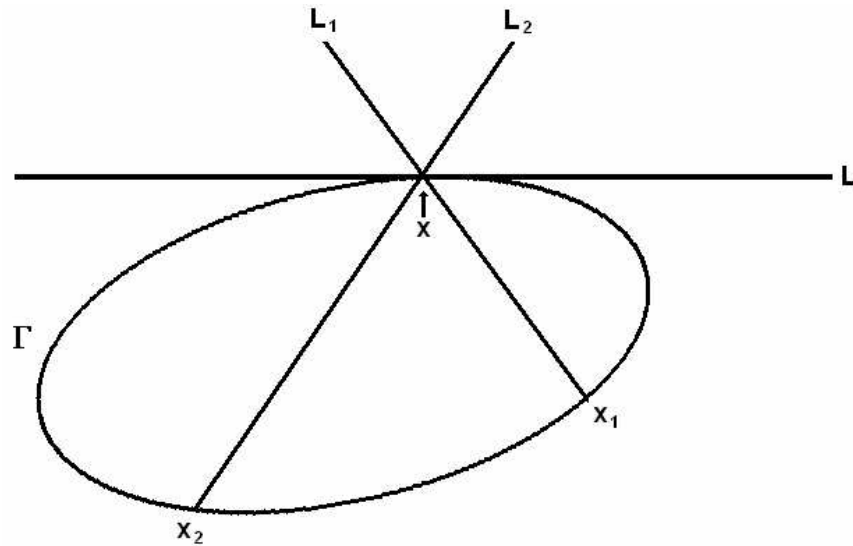

Figure VII.1

If $i \neq j$, then $X_i \neq X_j$ because otherwise L_i and L_j would have two points in common and we know these lines are distinct. Therefore the points $X, X_1, \cdots, X_n$ must be distinct points of Γ.■

A synthetic approach to conics

The theorem above give an incidence-theoretic characterization of nonsingular conics and suggest that synthetic methods might be useful in the study of conics. The next two theorems give a completely synthetic characterization of nonsingular conics due to J. Steiner.[9]

From this point on, unless stated otherwise, we shall assume that the field $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_3$.

THEOREM VII.22. *Let A and B be distinct points in $\mathbb{F}\mathbb{P}^2$, and let Φ be a projective collineation of $\mathbb{F}\mathbb{P}^2$ sending A to B. Then*

$$\mathbf{K} = \{X \in \mathbb{F}\mathbb{P}^2 \mid X = A \quad \text{or} \quad X \in \Phi(L) \cap L \text{ for some line } L \text{ through } \quad A\}$$

is a conic. (Notice that $B \in \mathbf{K}$, for we may take $L = AB$ in the definition).

_____

[9]JAKOB STEINER (1796–1863) is known for his work on projective geometry from a strongly synthetic viewpoint and for results in other branches of geometry.

Proof. Let P be an invertible 3×3 such that if ξ is a set of homogeneous coordinates for X, then $\Phi(X) = \mathbb{F} \cdot P\xi$. Also, let α be a set of homogeneous coordinates for A, and let β be a set of homogeneous coordinates for B such that $\beta = P \cdot \alpha$.

We need to find a homogeneous quadratic equation which defines $\mathbf{K}$. By Exercise V.1.5, if L is a line in $\mathbb{FP}^2$ and has homogeneous coordinates λ, then the line $\Phi[L]$ has homogeneous coordinates λP^{-1}. Thus $X \in \mathbf{K}$ if and only if $X = A$ or its homogeneous coordinates ξ satisfy

$$\xi = {}^{\mathbf{T}}\!\left(\lambda P^{-1}\right) \times {}^{\mathbf{T}}\!\lambda$$

for some line L whose homogeneous coordinates λ satisfy $\lambda \cdot \alpha = 0$. Equivalently, we have $X \in \mathbf{K}$ if and only if $X = A$ or

$$\lambda P^{-1}\xi = \lambda \cdot \xi = \lambda \cdot \alpha = 0 .$$

It follows that $X \in \mathbf{K}$ if and only if α, ξ and $p^{-1}\xi$ are linearly dependent (the case $X \neq A$ is immediate from the preceding three equations, while the case $X = A$ is trivial). Since P defines an invertible linear transformation, the vectors α, ξ and $P^{-1}\xi$ are linearly independent if and only if $\beta = P \cdot \alpha$, $P \cdot \xi$ and $\xi = P P^{-1}\xi$ are linearly independent. The linear dependence of the latter is in turn equivalent to the vanishing of the determinant $[\xi, P\xi, \beta]$. But the latter expression is a homogeneous quadratic polynomial in the entries of ξ and hence it is the defining equation of a conic.∎

Conversely, every nonsingular conic is defined by a projective collineation as in Theorem 22.

THEOREM VII.23. (Steiner)   *Let Γ be a nonsingular conic in $\mathbb{FP}^2$ containing at least five distinct points, and let A and B be distinct points of Γ. Then there is a projective collineation Φ of $\mathbb{FP}^2$ sending A to B such that*

$$\Gamma = \{X \in \mathbb{FP}^2 \mid X = A \ \ \text{or} \ \ X \in \Phi(L) \cap L \text{ for some line } L \text{ through} \ \ A\} .$$

Proof.   Let X, Y, Z be three points of Γ which are distinct from A and B. By Theorem 20, no three of the points A, B, X, Y, Z are collinear. Thus there is a unique projective collineation Φ sending A to B and X, Y, Z to themselves. By Theorem 22, the points A, B and the collineation Φ determine a conic Γ' defined by the formula above. By construction the three points X, Y, Z lie on Γ', and therefore $\Gamma = \Gamma'$ by Theorem 12.∎

NOTATION. If Γ is a conic and A, $B \in \Gamma$, then the collineation Φ of Theorem 23 is called a *Steiner collineation* associated to A, B and Γ. We note that this collineation is **not** unique, for different choices of the three points X, Y, Z yield different collineations.

Conics and inscribed polygons

Definition.   Let $P_1, \cdots, P_n$ be $n \geq 3$ points in $\mathbb{FP}^2$ such that no three are collinear. The **simple (projective) n-gon** $P_1 \cdots P_n$ is defined to be

$$P_1P_2 \cup \cdots \cup P_{n-1}P_n \cup P_nP_1 .$$

Dually, if $L_1, \cdots, L_n$ is a set of $n \geq 3$ lines such that no three are concurrent, the dual of a simple n-gon is the finite set of points determined by the intersections $L_i \cap L_{i+1}$ and $L_n \cap L_1$ (*i.e.*, a set of n points such that no three are collinear), and the union of the lines is the simple n-gon determined by these n points.

The following result due to B. Pascal[10] is one of the most celebrated theorems in projective geometry:

THEOREM VII.24. (Pascal's Theorem)   *Suppose that Γ is a nonsingular conic in $\mathbb{FP}^2$ and let the simple hexagon $A_1 \cdots A_6$ be* inscribed *in Γ (in other words, $A_i \in \Gamma$ for all i). Let*

$$X \ \in \ A_1 A_2 \ \cap \ A_4 A_5 \ , \qquad Y \ \in \ A_2 A_3 \ \cap \ A_5 A_6 \ , \qquad Z \ \in \ A_3 A_4 \ \cap \ A_6 A_1 \ .$$

Then X, Y and Z are collinear.

The line containing these three points is called the *Pascal line* of the hexagon.
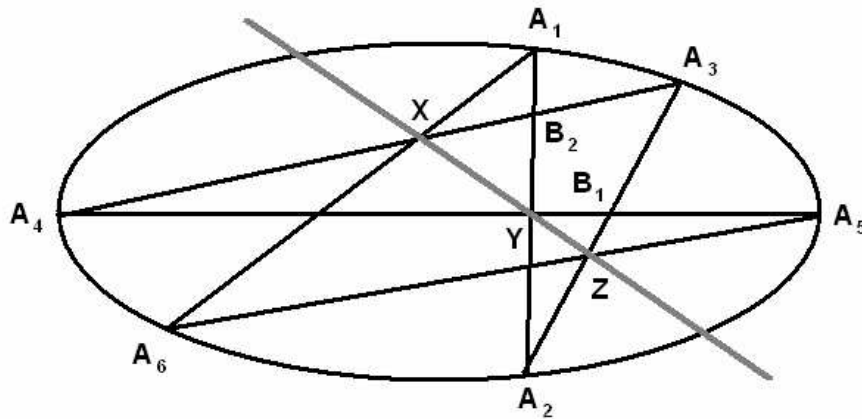

Figure VII.2

We have stated Pascal's Theorem for nonsingular conics, but a version of the result is also true for singular conics given by the union of two lines, provided the hexagon is degenerate in the sense that $\{A_1, A_3, A_5\}$ lie on one line and $\{A_2, A_4, A_6\}$ lie on the other. In such a situation, *the conclusion of Pascal's Theorem reduces to the conclusion of Pappus' Theorem*, and hence one can view Pappus' Theorem as a special case of Pascal's Theorem.[11]

SPECIAL CASE. Suppose that Γ in $\mathbb{RP}^2$ is given by the ordinary unit circle and $A_1 \cdots A_6$ is a regular hexagon which is inscribed in Γ. Then it is clear that $A_1 A_2 || A_4 A_5$, $A_2 A_3 || A_5 A_6$ and $A_3 A_4 || A_6 A_1$ (see the illustration below — note that A_1, A_2, A_3, A_4, A_5, A_6 correspond to A, B, C, D, E, F in the drawing), so that X, Y and Z are all ideal points and the Pascal line is equal to the line at infinity.

---

[10]BLAISE PASCAL (1623–1662) is known for contributions to a wide range of areas in the mathematical and physical sciences as well as philosophy. Aside from the theorem appearing here, he is particularly recognized for scientific work on fluid mechanics, probability theory, a counting machine which was the prototype for devices like mechanical odometers, as well as the philosophy of science. Most of his philosophical writings were highly religious in nature.

[11]And this is why French and German writers often use phrases translating to "Pascal's Theorem" when referring to the result known as Pappus' (Hexagon) Theorem in the English language.
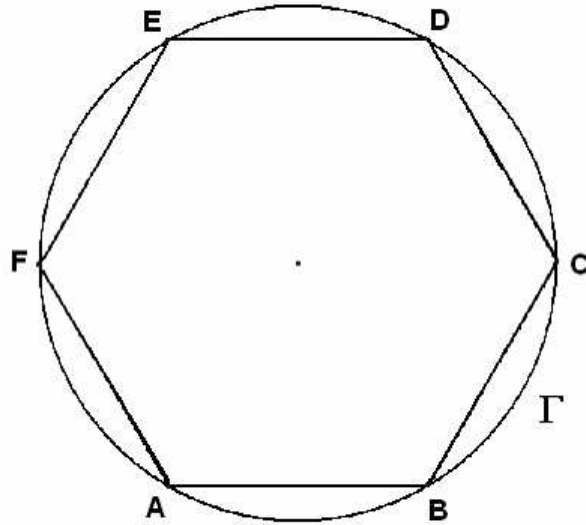
Figure VII.3

Proof of Pascal's Theorem (= Theorem 24).   By Steiner's Theorem there is a projective collineation Φ such that $\Phi(A_1) = A_5$ and Φ also has the following properties:

$$\Phi[A_1A_4] = A_5A_4 \qquad \Phi[A_1A_2] = A_5A_2 \qquad \Phi[A_1A_3] = A_5A_3$$

$$\Phi[A_1Z = A_1A_6] = A_5A_6 = A_5Y$$

As suggested by Figure VII.2, we define B_1 to be the point at which A_2A_3 meets A_4A_5, and we define B_2 to be the point at which A_3A_4 meets A_1A_2. Since Φ is a projective collineation, by Exercise VI.3.3 we have the following cross ratio equations:

$$\mathsf{XR}\,(A_1A_4,\ A_1A_3,\ A_1B_2,\ A_1Z) = \mathsf{XR}\,(\ \Phi[A_1A_4],\ \Phi[A_1A_3],\ \Phi[A_1B_2],\ \Phi[A_1Z]\ ) =$$

$$\mathsf{XR}\,(A_5B_1,\ A_5A_3,\ A_5A_4,\ A_5Y)$$

By construction, the points Z and B_2 are on A_3A_4, and the points Y and B_1 are on $A-2A_3$. Therefore Theorem V.14 implies that the first cross ratio in the displayed equation is equal to $\mathsf{XR}(A_4, A_3, B_2, Z)$ and the second is equal to $\mathsf{XR}(B_1, A_3, A_2, Y)$, so that

$$\mathsf{XR}(A_4, A_3, B_2, Z) = \mathsf{XR}(B_1, A_3, A_2, Y)\ .$$

Since $A_4B_1 = A_4A_5$ and $B_2A_2 = A_1A_2$ it follows that $X \in A_4B_1 \cap A_3X \cap B_2A_2$. Thus we also have

$$\mathsf{XR}(B_1, A_3, A_2, Y) = \mathsf{XR}(A_4, A_3, B_2, W)$$

where $W \in A_3A_4 \cap XY$. But the right hand side of the equation is also equal to the cross ratio $\mathsf{XR}(A_4, A_3, B_2, W)$, and therefore $W = Z$ by Theorem V.10. In particular, this implies that

$$Z \in A_3A_4 \cap XY$$

so that X, Y and Z are collinear.∎

If we now apply the Augmented Principle of Duality formulated in Section V, we immediately obtain the following result (Brianchon's Theorem),[12] which was originally established without using duality:

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[12]CHARLES JULIEN BRIANCHON (1783–1864) worked in mathematics and chemistry; in mathematics he is known for rediscovering Pascal's Theorem and proving the result which bears his name.

THEOREM VII.25. (Brianchon's Theorem)   *Suppose that Γ is a nonsingular conic in $\mathbb{FP}^2$ and let the simple hexagon $A_1 \cdots A_6$ be* circumscribed *about Γ (in other words, the lines $A_i A_{i+1}$ are tangent to Gamma for all i, and likewise for $A_6 A_1$). Then the lines $A_1 A_4$, $A_2 A_3$ and $A_2 A_3$ are concurrent.*■

The point of concurrency is called the *Brianchon point.*

SPECIAL CASE. Suppose that Γ in $\mathbb{RP}^2$ is given by the ordinary unit circle and $A_1 \cdots A_6$ is a regular hexagon which is inscribed in Γ. Then the Brianchon point is the center of the circle.
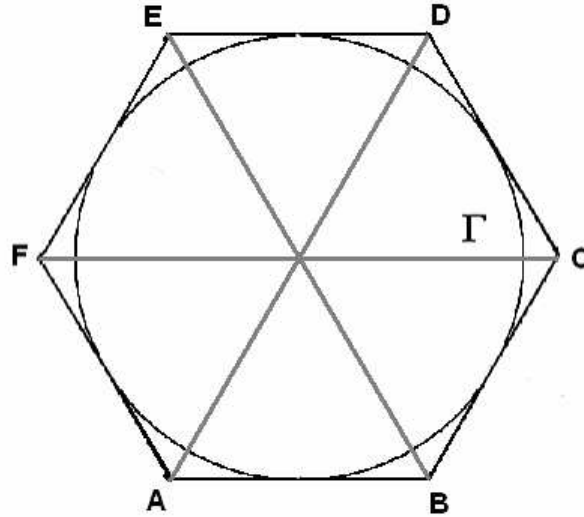

Figure VII.4

There is also a converse to Pascal's Theorem (and hence, by duality, there is also a converse to Brianchon's Theorem).

THEOREM VII.26. *Let $A_1 \cdots A_6$ be a simple hexagon, and let X, Y, Z be defined as in Pascal's Theorem. If these three points are collinear, then there is a conic Γ such that $A_i \in \Gamma$ for all i.*

By Theorem 19, if there is a conic containing the given six points, then it must be nonsingular.

Proof. Let Γ be the unique nonsingular conic containing the first five points $A_1, \cdots, A_5$ and let Φ be the Steiner collineation for Γ with $\Phi(A_1) = A_5$ and $\Phi[A_1 A_j] = A_5 A_j$ for $j = 2, 3, 4$. It will suffice to prove that $\Phi[A_1 A_6] = A_5 A_6$. Let B_1 and B_2 be defined as in the proof of

Pascal's Theorem. Since $\{A_4, A_3, B_2, Z\}$ and $\{B_1, A_3, A_2, Y\}$ are quadruples of collinear points and

$$X \in A_4 B_1 \cap A_3 \cap A_2 B_2 \cap YZ$$

Theorem V.14 implies that

$$\mathsf{XR}(A_4, A_3, B_2, Z) = \mathsf{XR}(B_1, A_3, A_2, Y) \ .$$

This in turn implies the following equation:

$$\mathsf{XR}(A_1 A_4, A_1 A_3, A_1 B_2, A_1 Z) = \mathsf{XR}(A_5 B_1, A_5 A_3, A_5 A_2, A_5 Y)$$

Since $A_1B_2 = A_1A_2$, $A_1Z = A_1A_6$, $A_5B_1 = A_5A_4$ and $A_5Y = A_5A_6$, the equation above may be rewritten as follows:

$$\mathsf{XR}(A_1A_4, A_1A_3, A_1A_2, A_1A_6) = \mathsf{XR}(A_5A_4, A_5A_3, A_5A_2, A_5A_6)$$

On the other hand, since Φ is a projective collineation, the right hand side is equal to the following:

$$\mathsf{XR}\big(\Phi[A_1A_4], \Phi[A_1A_3], \Phi[A_1A_2], \Phi[A_1A_6]\big) = \mathsf{XR}(A_5A_4, A_5A_3, A_5A_2, \Phi[A_1A_6])$$

Therefore it follows that $\Phi[A_1A_6] = A_5A_6$, which is what we needed to verify in order to complete the proof.■

The statement of the dual theorem to Theorem 26 is left to the reader (see the exercises).■

<div align="center">Degenerate cases of Pascal's Theorem</div>

There are analogs of Pascal's Theorem for inscribed simple n-gons where $n = 3, 4, 5$ (and by duality there are similar analogs of Brianchon's Theorem). Roughly speaking, these are limiting cases in which two consecutive vertices merge into a single point and the line joining the two points converges to the tangent line at the common point. The proofs of these theorems require a simple observation about Steiner collineations.

THEOREM VII.27. *Let Γ be a nonsingular conic, let A and B be points of Γ, and let Φ be a Steiner collineation for Γ such that $\Phi(A) = B$. If $\mathbf{T}_A$ is the tangent line to Γ at A, then $\Phi[\mathbf{T}_A] = AB$; if $\mathbf{T}_B$ is the tangent line to Γ at B, then $\Phi[AB] = \mathbf{T}_B$.*

Proof.   Since $B \in \Phi[\mathbf{T}_A]$, we know that $\Phi[\mathbf{T}_A] = BC$ for some point C. If $D \in \Phi[\mathbf{T}_A] \cap BC$, then $D \in \Gamma$ by construction. But the only point in $\mathbf{T}_A \cap \Gamma$ is A itself, and therefore we must have $BC = BA$. Since Φ^{-1} is a Steiner collineation for Γ taking B to X, it follows that $\Phi^{-1}[\mathbf{T}_B] = AB$, which is equivalent to the desired equation $\Phi[AB] = \mathbf{T}_B$.■

Here are the analogs of Pascal's Theorem for inscribed pentagons and quadrilaterals; note that there are two separate analogs for quadrilaterals.

THEOREM VII.28. *Suppose that Γ is a nonsingular conic in $\mathbb{FP}^2$ and let the simple pentagon $A_1 \cdots A_5$ be inscribed in Γ. Let*

$$X \in A_1A_2 \cap A_4A_5\,, \qquad Y \in A_2A_3 \cap A_5A_1\,, \qquad Z \in A_3A_4 \cap \mathbf{T}_{A_1}\,.$$

Then X, Y and Z are collinear.

THEOREM VII.29. *Suppose that Γ is a nonsingular conic in $\mathbb{FP}^2$ and let the simple quadrilateral $A_1 \cdots A_4$ be inscribed in Γ. Let*

$$X \in \mathbf{T}_{A_1} \cap A_2A_4\,, \qquad Y \in A_1A_2 \cap A_3A_4\,, \qquad Z \in \mathbf{T}_{A_2} \cap A_1A_3\,.$$

Then X, Y and Z are collinear.

THEOREM VII.30. *Suppose that Γ is a nonsingular conic in $\mathbb{FP}^2$ and let the simple quadrilateral $A_1 \cdots A_4$ be inscribed in Γ. Let*

$$D \in A_1A_3 \cap A_2A_4\,, \qquad E \in A_1A_4 \cap A_2A_3\,, \qquad F \in \mathbf{T}_{A_1} \cap \mathbf{T}_{A_2}\,.$$

Then D, E and F are collinear.

The proofs of these theorems are easy variants of the proofs of Pascal's Theorem and are left to the reader as exercises.■

Similarly, formulations and proofs of the duals to all these results are left to the reader as exercises.■

The final degenerate case of Pascal's Theorem requires a special argument. As noted in Appendix A, the cross product of vectors in $\mathbb{F}^3$ satisfies the following condition known as the *Jacobi Identity* (see Theorem A.21):

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \;+\; \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) \;+\; \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \;=\; \mathbf{0} \; .$$

THEOREM VII.31. *Let ABC be a (projective) triangle inscribed in the nonsingular conic Γ. Let*

$$D \;\in\; \mathbf{T}_A \cap BC \; , \qquad E \;\in\; \mathbf{T}_B \cap AC \; , \qquad F \;\in\; \mathbf{T}_C \cap AB \; .$$

Then D, E and F are collinear.

Proof. By Theorems 5, 10 and 14, the nonsingular conic Γ is projectively equivalent to one defined by an equation of the form $ax^2 + by^2 + cz^2 \;=\; 0$ where none of the coefficients a, b, c is equal to zero. Dividing these by a suitable constant, we may assume $c = -1$. Therefore it suffices to prove the result for conics defined by equations of the form $ax^2 + by^2 - z^2 = 0$.

Let ρ be the 1×3 matrix $(0\ 0\ -2)$. If $X \in \Gamma$ and has homogeneous coordinates

$$\xi \;=\; {}^{\mathbf{T}}(x_1 \quad x_2 \quad x_3)$$

then homogeneous coordinates for the tangent line $\mathbf{T}_X$ to Γ at X are given by $\xi^{\#} \;=\; {}^{\mathbf{T}}\xi \;+\; x_3\,\rho$.

Let α, β, γ denote homogeneous coordinates for A, B, C, and let $\alpha^{\#}$, $\beta^{\#}$, $\gamma^{\#}$ denote corresponding homogeneous coordinates for the tangent lines $\mathbf{T}_A$, $\mathbf{T}_B$ and $\mathbf{T}_C$. It will suffice to show that the vectors

$$\alpha^{\#} \;\times\; (\beta \times \gamma) \; , \qquad \beta^{\#} \;\times\; (\gamma \times \alpha) \; , \qquad \gamma^{\#} \;\times\; (\alpha \times \beta)$$

are linearly dependent. However, their sum is equal to

$$\big[\, \alpha \;\times\; (\beta \times \gamma) \;+\; \beta \;\times\; (\gamma \times \alpha) \;+\; \gamma \;\times\; (\alpha \times \beta) \,\big] \;+$$
$$ {}^{\mathbf{T}}\rho \;\times\; \big(\, a_3\,(\beta \times \gamma) \;+\; b_3\,(\gamma \times \alpha) \;+\; c_3\,(\alpha \times \beta) \,\big)$$

and we claim that this sum vanishes. The term in square brackets vanishes by the Jacobi Identity; to analyze the remaining term(s), we may use the "back–cab formula"

$$ {}^{\mathbf{T}}\rho \;\times\; (\eta \times \zeta) \;=\; (\rho \cdot \zeta)\eta \;-\; (\rho \cdot \eta)\zeta \;=\; 2(z_3\,\eta \;-\; y_3\zeta)$$

to see that the expression

$$ {}^{\mathbf{T}}\rho \;\times\; \big(\, a_3\,(\beta \times \gamma) \;+\; b_3\,(\gamma \times \alpha) \;+\; c_3\,(\alpha \times \beta) \,\big)$$

is a sum of six terms that cancel each other in pairs.■

As before, the formulation of the dual theorem is left to the reader as an exercise.■

EXERCISES

1. Prove that the conclusion of Theorem 21 is still valid if Γ is completely singular, provided it contains at least two points. [*Hint:* The set of singular points is a geometrical subspace.]

2. Find the equations of the conics in $\mathbb{RP}^2$ which pass through the following five points: with the following homogeneous coordinates:

(*i*) The five points with the following homogeneous coordinates:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

(*ii*) The five points with the following homogeneous coordinates:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

3. Let Φ be the collineation of $\mathbb{RP}^2$ sending the point with homogeneous coordinates $^{\mathbf{T}}(1 \quad 1 \quad 2)$ to $^{\mathbf{T}}(2 \quad 2 \quad 1)$, and the lines with homogeneous coordinates

$$(2 \quad 0 \quad -1) \qquad (1 \quad -1 \quad 0) \qquad (1 \quad 1 \quad -1)$$

to the lines with homogeneous coordinates

$$(1 \quad 0 \quad -2) \qquad (1 \quad 1 \quad -4) \qquad (1 \quad 2 \quad -6)$$

respectively. Find the equation of the conic Γ for which Φ is a Steiner collineation (using Theorem 22).

4. Let Γ be the ellipse defined by the affine equation $x^2 + 3y^2 = 4$ in $\mathbb{R}^2$ (hence its projectivization has no ideal points). If T is the unique projective collineation of $\mathbb{RP}^2$ sending J$(\pm 2, 0)$ and J$(0, -\frac{2}{3}\sqrt{3})$ to themselves, and sending J$(-1, 1)$ to J$(1, 1)$, then T is a Steiner collineation for Γ. Likewise, if S is the unique projective collineation of $\mathbb{RP}^2$ sending J$(\pm 2, 0)$ and J$(0, \frac{2}{3}\sqrt{3})$ to themselves, and sending J$(-1, 1)$ to J$(1, 1)$, then S is also a Steiner collineation for Γ. Show that S and T must be distinct projective collineations. [*Hint:* If $S = T$, then this map fixes the four points on Γ where it meets the x- and y-axes. What does the Fundamental Theorem of Projective Geometry imply about $S = T$ in this case?]

5. State the duals of Theorems 19, 21 and 27–31 (the duality principle implies that these dual results are automatically valid).

6. Prove Theorems 27–30 and their converses.

7. In each of Theorems 27–31, find the line containing the given intersection points if the simple polygon is a regular polygon in the Euclidean plane. Similarly, in the duals of Theorems 27–31, find the points of concurrency for the same polygons.

8. Let Γ be a nonsingular conic in $\mathbb{FP}^2$, and let $\{A, B, C\}$ and $\{A', B', C'\}$ be two disjoint noncollinear sets of points on Γ. Prove that the lines AA', BB' and CC' are concurrent. — A drawing and hints appear on the next page.
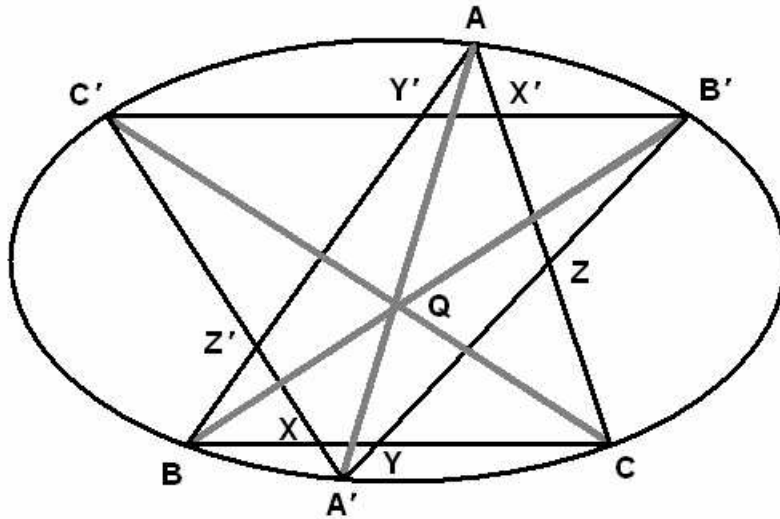
Figure VII.5

[*Hint:* Define X, Y, Z and X', Y', Z' as suggested in the figure. First prove that AA', XX' and YY' are concurrent using Pascal's Theorem. Clearly corresponding results hold for BB' and CC'. Let $U \in C'Y \cap BX'$, and prove that Z', U and Q are collinear by Pappus' Theorem. Also show that ZU, BB' and CC' are concurrent using Pappus' Theorem for $\{C', X', B\}$ and $\{B, Y, C\}$. Finally, apply Pascal's Theorem to $AB'C'A'BC$ to show that BB', CC' and ZZ' are concurrent. Using similar results for XX' and YY' and the previous concurrency relations involving AA', BB' and CC', prove that all six lines are concurrent.]

What is the dual of the preceding result?

9. Let Γ, $\{A, B, C\}$ and $\{A', B', C'\}$ be as in Exercise 8. Prove that the six lines determined by the triangles ABC and $A'B'C'$ form the sides of a hexagon that is tangent to another conic. [*Hint:* The hexagon is $Y'X'ZYXZ'$. Apply the converse to Brianchon's Theorem.]

10. Show that a set of six points on a nonsingular conic Γ determines sixty simple hexagons (in general these sixty hexagons have distinct Pascal lines, and the footnote on page 152 of Fishback contains further information on the totality of all such configurations).

> A conic in $\mathbb{FP}^2$ is said to be *nondegenerate* if it is not equal to a line or a point and degenerate otherwise.

11. Suppose that the field $\mathbb{F}$ satsifies $1 + 1 \neq 0$ and contains at least four elements, and let Γ_1 and Γ_2 be distinct nondegenerate conics in $\mathbb{FP}^2$. Prove that Γ_1 and Γ_2 consists of at most four points. [*Hint:* If both are nonsingular, explain why the conclusion follows from Theorems 19 and 20. On the other hand, if a conic is singular, why is it equivalent to a conic defined by a homogeneous quadratic equation of the form $x_1^2 + dx_2^2 = 0$? Explain why such a conic is nondegenerate if and only if $-d$ is a nonzero perfect square in $\mathbb{F}$, and in this case it is a pair of intersecting lines. What is the maximum number of points in which a nonsingular conic intersects a union of two lines, and what is the maximum number of points in $(L_1 \cup L_2) \cap (L_3 \cup L_4)$ if the sets L_i are all lines?]

NOTE. Among other things, this justifies a basic principle in high school algebra regarding solutions of systems of two quadratic equations in two unknowns; namely, systems of two independent equations have at most four simultaneous solutions.

Generalizations of the preceding exercise. There are generalizations in two directions. One can consider n independent polynomial equations in n unknowns, and one can consider polynomials of arbitrary positive degrees. Furthermore, it is also possible to consider projectivized versions of the question with n independent equations that are *homogeneous* polynomials of $n + 1$ variables (in other words, the degrees of all the monomial summands are the same); the previously discussed situations may be viewed as specializations of the projective equations in which the last homogeneous coordinate is set equal to 1. A general version of a result called Bézout's Theorem [13] implies that the number of solutions for such a system is at most $d_1 \cdots d_n$, where d_i is the degree of the ith polynomial. Our result for two quadratic equations required an assumption that $1 + 1 \neq 0$ in the field $\mathbb{F}$, and for equations of higher degree it is generally necessary to make an assumption on the *characteristic* of $\mathbb{F}$, which is either the least positive integer k such that a sum of k copies of 1 with itself is zero or else it is 0 if no such sum is equal to zero (as is the case for the rationals, real numbers and complex numbers). Further information on Bézout's Theorem is contained in the following online reference:

$$\texttt{http://en.wikipedia./org/wiki/B\%C3\%A9ezout's\_theorem}$$

The classical version of Bézout's Theorem is also covered book by Bumcrot in the bibliography, and the general case is covered in the book by Blum, Cucker, Shub and Smale.[14]

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[13]The name is pronounced "bay-zoo." ÉTIENNE BÉZOUT (1730–1783) is mainly known for the theorem bearing his name, and he also wrote several texts that were widely used.

[14]STEPHEN SMALE (1930–), pronounced "smail," is known for fundamental work in geometric topology, the qualitative behavior of solutions to ordinary differential equations, and the theory of computability for solutions to mathematical problems.

APPENDIX A

REVIEW OF LINEAR ALGEBRA

The following is a summary (almost entirely without proofs) of the linear algebra needed in the projective geometry notes. All the material is standard, but there are some relatively minor differences in viewpoint compared to standard undergraduate courses and texts.

1. There is more emphasis on certain results which are particularly important in the use of linear algebra to study geometric questions.

2. In most cases, the system of scalars is taken to be as general as possible. — This has both advantages and disadvantages. Maximum generality is needed to show that systems which satisfy the short list of axioms for an affine 3-space always have somewhat reasonable coordinate systems. However, especially at the beginning, the need to consider abstract mathematical systems of scalars and their specific properties (for example, whether or not multiplication of scalars satisfies $ab = ba$) may be distracting or even worse. A reader who is only interested in scalars for which $ab = ba$ may substitute "field" for "skew-field" throughout and ignore the adjectives "left" and "right" in discussions of vector spaces. If a reader wishes to go further and only consider scalars that are real or complex numbers, this can be done by assuming that "field" refers to one or both of these systems.

More complete treatments of linear algebra appear in the appropriate books listed in the Bibliography.

1. Vector spaces

The basic operations involving ordinary 2- and 3-dimensional vectors are vector addition and scalar multiplication (in this context *scalar* means real number). We begin by listing the relevant properties of the real numbers below. For each statement, it is assumed that a, b, c are arbitrary elements of the real numbers $\mathbb{R}$.

(F-1) (*Commutative Law of Addition*). $a + b = b + a$.
(F-2) (*Associative Law of Addition*). $a + (b+c) = (a+b) + c$.
(F-3) (*Existence of Zero Element*). There is a unique $0 \in \mathbb{R}$ such that $a + 0 = a$.
(F-4) (*Existence of Negative Elements*). If $a \in \mathbb{R}$, then there is a unique $(-a) \in \mathbb{R}$ such that $a + (-a) = 0$.
(F-5) (*Left Distributive Law*). $a \cdot (b+c) = (ab) + (ac)$.
(F-6) (*Right Distributive Law*). $(a+b) \cdot c = (ac) + (bc)$.
(F-7) (*Existence of Unit Element*). There is a unique $1 \in \mathbb{R}$ such that $a \cdot 1 = 1 \cdot a = a$.
(F-8) (*Existence of Nonzero Reciprocals*). If $a \in \mathbb{R}$ is not equal to zero, then there is a unique $a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.
(F-9) (*Associative Law of Multiplication*). $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
(F-10) (*Commutative Law of Multiplication*). $a \cdot b = b \cdot a$.

We call any triple $(\mathbb{F}, +, \cdot)$ consisting of a set with binary operations "+" and "·" satisfying **(F-1)**–**(F-10)** a (commutative) **field**. Many undergraduate texts on linear algebra begin by allowing the scalars to belong to an arbitrary field. In fact, a great deal of the theory does not depend upon property **(F-10)**, and for the purposes of projective geometry it is especially useful to drop this assumption. Thus we shall call a triple which only satisfies the first nine assumptions a **skew-field** (such a system is also called a **division ring** in the literature). If the final property is also satisfied, then frequently we shall say that the system is **commutative**.

THE MAIN EXAMPLES. Three basic number systems for arithmetic and algebra are fields: The *real numbers*, which we denote by $\mathbb{R}$, the *complex numbers*, which we denote by $\mathbb{C}$, and the *rational numbers*, which we denote by $\mathbb{Q}$. In each case the binary operations are given by ordinary addition and multiplication. — Another important class of examples are given by the finite fields of *integers mod(ulo) p*, where p is a prime number; these fields are denoted by $\mathbb{Z}_p$.

An example of a skew-field which is **not** commutative is given by the *quaternions*, which were first described by W. R. Hamilton (1805–1865) in the 19th century and will be denoted by $\mathbb{K}$ in these notes.[1] These consist of all formal expressions

$$\mathbf{x} = x_0 \mathbf{1} + x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k}$$

with real coefficients x_m, and two such expressions are taken to be equal if and only if all four coefficients are equal. The *sum* of two quaternions $\mathbf{x} + \mathbf{y}$ is the expression whose coeffients are $x_m + y_m$ for $m = 0, 1, 2, 3$. The general formula for the *product* is much less concise, and it is probably easier to think of the product as uniquely defined by the basic assumptions — particularly the left and right distributive laws — and the following multiplication formulas:

$$\mathbf{1} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{1} = \mathbf{x} \quad (\text{all } \mathbf{x}), \qquad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$\mathbf{i} \cdot \mathbf{j} = -\mathbf{j} \cdot \mathbf{i} = \mathbf{k}, \quad \mathbf{j} \cdot \mathbf{k} = -\mathbf{k} \cdot \mathbf{j} = \mathbf{i}, \quad \mathbf{k} \cdot \mathbf{i} = -\mathbf{i} \cdot \mathbf{k} = \mathbf{j}$$

The verification that $\mathbb{K}$ satisfies the first seven properties is essentially a simple exercise in bookkeeping, and the verification of **(F-9)** is similar but requires more work. To prove **(F-8)**, one first defines the *conjugate* of $\mathbf{x}$ by

$$\overline{\mathbf{x}} = x_0 \mathbf{1} - x_1 \mathbf{i} - x_2 \mathbf{j} - x_3 \mathbf{k}$$

and then verifies that

$$\left(x_0^2 + x_1^2 + x_2^2 + x_3^2 \right)^{-1} \cdot \overline{\mathbf{x}}$$

satisfies the properties one wants for $\mathbf{x}^{-1}$ (more elementary but somewhat lengthy bookkeeping — note the similarity between this formula and a standard formula for the reciprocal of a nonzero complex number). Of course, the relation $\mathbf{i} \cdot \mathbf{j} = -\mathbf{j} \cdot \mathbf{i}$ shows that **(F-10)** does not hold in $\mathbb{K}$. In all the examples we have discussed, we have $\mathbf{x} \cdot \mathbf{y} = \pm \mathbf{y} \cdot \mathbf{x}$, but it is also possible (and not really difficult) to construct examples for which $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{y} \cdot \mathbf{x}$ are not real multiples of each other.

VECTOR OPERATIONS. Having described the basic properties of addition and multiplication that we shall need, the next step is to list the properties of vector addition and scalar multiplication that will be important for our purposes. In each case, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$ are arbitrary elements of the space of vectors V, and a and b denote arbitrary scalars.

(V-1) (*Commutative Law of Addition*). $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
(V-2) (*Associative Law of Addition*). $\mathbf{v} + (\mathbf{w} + \mathbf{x}) = (\mathbf{v} + \mathbf{w}) + \mathbf{x}$.

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[1]The material in this paragraph may be skipped if the reader is only interested in commutative fields or the basic number systems described above.

(V-3) (*Existence of Zero Element*).   There is some $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
(V-4) (*Existence of Negative Elements*).   If $\mathbf{v} \in \mathbb{R}$, then there is some $(-\mathbf{v}) \in V$ such that
$\quad\quad \mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
(V-5) (*Left Distributive Law*). $a \cdot (\mathbf{v} + \mathbf{w}) = (a\mathbf{v}) + (a\mathbf{w})$.
(V-6) (*Right Distributive Law*). $(a + b) \cdot \mathbf{v} = (a\mathbf{v}) + (b\mathbf{v})$.
(V-7) (*Unit Multiplication Property*). $1 \cdot \mathbf{v} = \mathbf{v}$.
(V-8) (*Mixed Associative Law*). $a \cdot (b\mathbf{v}) = (ab) \cdot \mathbf{v}$.

We call any triple $(V, +, \cdot)$ consisting of a set with a binary operation "+" and *left scalar multiplication* "$\cdot$" by elements of a *skew-field* $\mathbb{F}$ satisfying **(V-1)**–**(V-8)** a **left vector space** over $\mathbb{F}$.

STANDARD EXAMPLE. Let $\mathbb{F}^n$ be the set of all ordered n-tuples of elements in $\mathbb{F}$ (if $n = 2$ or 3 and $\mathbb{F} = \mathbb{R}$ then these are ordinary vectors). Define addition and scalar multiplication by

$$(\mathbf{x}_1, \cdots, \mathbf{x}_n) + (\mathbf{y}_1, \cdots, \mathbf{y}_n) = (\mathbf{x}_1 + \mathbf{y}_1, \cdots, \mathbf{x}_n + \mathbf{y}_n)$$

$$c \cdot (\mathbf{x}_1, \cdots, \mathbf{x}_n) = (c\,\mathbf{x}_1, \cdots, c\,\mathbf{x}_n).$$

Verification of **(V-1)** – **(V-8)** is yet another exercise in bookkeeping.

The above definition has a "right handed" counterpart in which scalars are multiplied on the **right**. As one might expect, such systems are called right vector spaces. They all satisfy **(V-1)** – **(V-4)** and analogs of **(V-5)** – **(V-8)** in which the order of vectors and scalars in all multiplications is reversed. The correct analog of the Mixed Associative Law is $\mathbf{v}\cdot(ab) = (\mathbf{v}\cdot a)\cdot b$. We may make $\mathbb{F}^n$ into a right vector space by defining addition as before and setting

$$(\mathbf{x}_1, \cdots, \mathbf{x}_n) \cdot c = (\mathbf{x}_1 c, \cdots, \mathbf{x}_n c).$$

If $\mathbb{F}$ is commutative, this right multiplication differs from the left multiplication only in the order of writing vectors and scalars, with both operations yielding the same vector. However, if $\mathbb{F}$ is not commutative then the two multiplications may differ considerably.

Henceforth, we shall do everything for left vector spaces almost exclusively. In all cases the corresponding right-handed results may be obtained by making a few obvious transpositions.

Vector subspaces

Definition.   A nonempty subset W of a (left) vector space V is called a (left) *vector subspace* provided the following hold:

(i) If $\mathbf{x}$ and $\mathbf{y}$ are in W, then so is $\mathbf{x} + \mathbf{y}$.

(ii) If $\mathbf{x} \in W$ and $c \in \mathbb{F}$, then $c\mathbf{x} \in W$.

The vector space identity $0 \cdot \mathbf{v} = \mathbf{0}$ is a direct consequence of **(V-1)**–**(V-8)**, and thus it follows that $\mathbf{0} \in W$. Furthermore, it follows immediately that W also satisfies **(V-1)**–**(V-8)**, and therefore W is a vector space in its own right with respect to the addition and scalar multiplication operations it inherits from V.

Theorem A.1. *Let V be a vector space, and let W_α be a family of subspaces of V, where α runs over all elements of some indexing set A. Then the intersection*

$$\bigcap_{\alpha \in A} W_\alpha$$

is also a vector subspace of V.

Theorem A.2. *Let V be a vector space, and let W_1 and W_2 be vector subspaces of V. Then the set $W_1 + W_2$ given by*

$$\{\, \mathbf{x} \in V \mid \mathbf{x} = \mathbf{v}_1 + \mathbf{v}_2, \text{ where } \mathbf{x}_1 \in W_1 \text{ and } \mathbf{x}_2 \in W_2 \,\} \, .$$

is a subspace of V.

Besides addition and multiplication, the real numbers have a concept of *order*; given two unequal real numbers a and b, we know that one is greater than the other. Since $a < b$ if and only if $b - a$ is *positive*, it follows that the ordering is determined by the set of positive elements $\mathbb{R}^+$. This set has the following properties:

(O-1) If x, $y \in \mathbb{R}^+$, then $x + y$, $xy \in \mathbb{R}^+$.

(O-2) If $x \in \mathbb{R}$, then **exactly one** of the statements $x \in \mathbb{R}^+$, $x = 0$, and $-x \in \mathbb{R}^+$ is true.

Given a skew-field $\mathbb{F}$ and a nonempty set of elements $\mathbb{F}^+$ satisfying these two properties, we shall call the quadruple $(\mathbb{F}, +, \cdot, \mathbb{F}^+)$ an *ordered skew-field*, and we call $\mathbb{F}^+$ its set of *positive elements*. The basic properties of ordered fields in the commutative case are presented in Sections I.3 and II.4 of Birkhoff and MacLane. One additional property suffices to give a complete characterization of the real numbers. Since its presentation is a bit lengthy, the reader is referred to Chapter IV of Birkhoff and MacLane for further information.

2. Dimension

Definition. Let V be a vector space over $\mathbb{F}$, and let $A \subset V$ be a nonempty subset. A vector $\mathbf{x} \in V$ is said to be a *linear combination of the vectors in A* if there exist $\mathbf{v}_1, \cdots, \mathbf{x}_n \in A$ and $c_1, \cdots, c_n \in \mathbb{F}$ such that $\mathbf{x} = \sum_i c_i \mathbf{x}_i$. We say that A is a *spanning set for V* (or the vectors in A *span the vector space V*) if every vector in V is a linear combination of the vectors in A. In general, if $A \subset V$ then $\mathsf{Span}(A)$ is defined to be the set of all linear combinations of vectors in A. It is immediate that $\mathsf{Span}(A)$ is a subspace of V.

EXAMPLE. If $\mathbf{e}_i \in \mathbb{F}^n$ is the n-tuple whose i^{th} entry is 1 and whose other entries are zero, then the vectors in $\{\, \mathbf{e}_1, \cdots, \mathbf{e}_n \,\}$ span $\mathbb{F}^n$ because $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{F}^n$ implies

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i \, .$$

Definition. A nonempty set of vectors A with $A \neq \{\, \mathbf{0} \,\}$ is *linearly independent* if for every $\mathbf{x} \in A$ the vectors $\mathbf{x}$ is not a linear combination of vectors in $A - \{\, \mathbf{x} \,\}$. A set is *linearly dependent* if it is not linearly independent. Thus the set $\{\, \mathbf{0} \,\}$ is linearly dependent by convention.

EXAMPLE. The set $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is linearly independent; for each i, the ith coordinate of $\mathbf{e}_i$ is equal to 1, but every linear combination the remaining vectors has an ith coordinate equal to zero.

The next result is not always emphasized in linear algebra texts or courses, but the proof is fairly straightforward and the conclusion is important for geometrical purposes.

Theorem A.3. *Let $A \subset V$ is linearly independent and $\mathbf{y} \notin \mathsf{Span}(A)$. Then $A \cup \{\mathbf{y}\}$ is linearly independent.*

Theorem A.4. *The set $A \subset V$ is linearly indepnedent if and only if every nonzero vector in $\mathsf{Span}(A)$ has an essentially unique expression as a linear combination as a linear combination of the vectors in A.*

More precisely, the condition in Theorem A.4 means that if the expressions

$$\sum_{i=1}^{r} c_i \mathbf{v}_i \qquad \sum_{j=1}^{s} d_j \mathbf{w}_j$$

(with all $c_i,\ d_j \neq 0$) yield the same vector, then $r = s$, the ordered lists of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_r$ and $\mathbf{w}_1, \cdots, \mathbf{w}_r$ are rearrangements of each other, and if $j(1), \cdots, j(r)$ is the rearrangement such that $\mathbf{w}_{j(i)} = \mathbf{v}_i$ for all i, then the corresponding coefficients are equal, so that $d_{j(i)} = c_i$.[2]

Remark. Actually, a set of vectors is linearly independent if and only if the zero vector has only one expression as a linear combination of vectors in the set; namely, the expression in which all the coefficients are zero.

Definition. A *basis* for V is a subset A which is linearly independent and spans V. If $V = \mathbb{F}^n$, the preceding discussion shows that the set of unit vectors $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is a basis.

Theorem A.5. *Suppose that V is a vector space having a basis with n elements, where n is some positive integer. Then the following hold:*

(i) Every other basis for V has exactly n elements.

(ii) Every subset of V containing $(n+1)$ vectors is linearly dependent.

(iii) Every linearly independent subset $A \subset V$ is contained in a subset $B \subset V$ which is a basis.

Definition. If V has a basis with n (finitely many) elements, then V is said to be *finite-dimensional* and n is called the *dimension* of V, written $\dim V$. By the preceding theorem, the number n is independent of the choice of basis. If V has no finite basis, we shall say that V is *infinite-dimensional*.

The following theorems are a natural continuation of Theorem A.5.

---

[2]The reason for the qualifying term "essentially" reflects the lack of a finiteness assumption on the spanning set A. In our setting, only finite sums are meaningful, and if A is infinite then any finite linear combination $\sum_i c_i \mathbf{a}_i$ can be rewritten as $\sum_i c_i \mathbf{a}_i + \sum_j 0 \, \mathbf{b}_j$, where $\mathbf{b}_1, \cdots, \mathbf{b}_q$ is a subset of A which is disjoint from $\mathbf{a}_1, \cdots, \mathbf{a}_n$. Of course, these two expressions are essentially the same even if they might not be completely identical.

Theorem A.6. *Suppose that V has a spanning set with finitely many (say n) elements. Then V is finite-dimensional and $\dim V \leq n$. In fact, given a finite spanning set A, there is a subset $B \subset A$ which is a basis for V.*

Theorem A.7. *If W is a vector subspace of V and V is finite-dimensional, then so is W and $\dim W \leq \dim V$. In fact, every basis for W is contained in a basis for V.*

Theorem A.8. *Suppose that W_1 and W_2 are subspaces of the finite-dimensional vector space V such that $W_1 \subset W_2$ and $\dim W_1 = \dim W_2$. Then $W_1 = W_2$.*

Finally, we have the following important formula relating the discussions of the dimensions of two subspaces with those of their intersection and sum (see Theorems A.1 – A.2).

Theorem A.9. *If V is a vector space and W_1 and W_2 are finite-dimensional subspaces, then $W_1 \cap W_2$ and $W_1 + W_2$ are also finite-dimensional and*

$$\dim W_1 \;+\; \dim W_2 \;=\; \dim(W_1 \cap W_2) \;+\; \dim(W_1 + W_2) \;.$$

3. Systems of linear equations

Since we are considering skew-fields in which multiplication need not be commutative, we shall need to distinguish between the left and right scalar multiplications in this section. We remark that the entire discussion remains valid if one interchanges right and left and reverses the order of factors occurring in all products.

Suppose we are given a system of m homogeneous linear equations in n unknowns:

$$
\begin{aligned}
a_{1,1}x_1 \;+\; \cdots \;+\; a_{1,n}x_n \;&=\; 0 \\
a_{2,1}x_1 \;+\; \cdots \;+\; a_{2,n}x_n \;&=\; 0 \\
&\cdots \\
&\cdots \\
a_{m,1}x_1 \;+\; \cdots \;+\; a_{m,n}x_n \;&=\; 0
\end{aligned}
$$

The *solution space* of this system is defined to be the set of all $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{F}^n$ for which the equations above are satisfied. It is easy to see that this is a *right* subspace of $\mathbb{F}^n$. We shall describe how the coefficients $a_{i,j}$ determine its dimension.

Recall that an $m \times n$ matrix with entries in a set Y (or over a set Y) is a rectangular array of indexed elements in Y with m rows and n columns. We may view the rows as elements of the n-fold product Y^n.

Clearly the coefficients of the given system of linear equations describe and $m \times n$ matrix over $\mathbb{F}$, and we may view the rows of the matrix as vectors in $\mathbb{F}^n$. Define the (left) *row space* to be the (left) subspace spanned by the rows, and set the (left) *row rank* equal to the dimension of this subspace. By the results of the previous section, the row rank does not exceed the smaller of m and n.

Theorem A.10. *Let $A = (a_{i,j})$ be the $m \times n$ matrix defined by the system of equations displayed above, and assume that its row rank is r. Then the dimension of the solution space for the original system of linear equations is $n - r$.*

Since there is a basically mechanical method for finding the row rank (the *Gauss-Jordan elimination procedure*), this result gives an effective method for finding the dimension of the solution space. In fact, one can push the method a little further and use it to obtain an explicit basis for the solution space.

HYPOTHESIS FOR THE REST OF THIS SECTION. From this point until the end of the current section, we shall assume that $\mathbb{F}$ is a field (so that multiplication is commutative).

If $\mathbb{F}$ is a field, there is an alternate characterization of the row rank in terms of *determinants*. The determinant is a function assigning to each square matrix A an element of $\mathbb{F}$ denoted by $\det A$ (the determinant of A), and it has the following properties:

(D-1) If A is $n \times n$, then $\det A \neq 0$ if and only if the row rank of A is equal to n.

(D-2) (*Expansion by minors*) For each all i, j such that $1 \leq i$, $j \leq n$, let $A_{i,j}$ be the matrix obtained by deleting the ith row and j^{th} column. Then for all p and q between 1 and n we have

$$\det A \; = \; \sum_{i=1}^{n} (-1)^{i+q} a_{i,q} \cdot \det A_{i,q} \; = \; \sum_{j=1}^{n} (-1)^{p+j} a_{p,j} \cdot \det A_{p,j} \; .$$

The first expression for the determinant is called its *expansion by minors along the qth column* and the second is called its *expansion by minors along the pth row*.

(D-3) If $A = (a)$ is 1×1, then $\det A = a$.

In particular, from these rules one can derive the standard formulas for 2×2 and 3×3 determinants that appear in high school algebra (and likewise for the more complicated rules which hold for all n and are discussed in undergraduate linear algebra courses).

Definition. If A is a matrix, then a *square submatrix* is a square matrix obtained from a by deleting certain rows and columns; for example, if $k \leq m$, n then we obtain a $k \times k$ submatrix by deleting exactly $(m - k)$ rows and $(n - k)$ columns. The *determinant rank* of a (not necessarily square) matrix A is the largest d such that A contains an $r \times r$ submatrix with nonvanishing determinant.

Theorem A.11. *The row rank and determinant rank of a matrix are equal.*

4. LINEAR TRANSFORMATIONS

Definition.   Let $\mathbb{F}$ be a skew-field, and let V and W be (left) vector space over $\mathbb{F}$. A *linear transformation* $T : V \to W$ is a function from V to W satisfying the following conditions for all $\mathbf{x}$, $\mathbf{y} \in V$ and $c \in \mathbb{F}$:

(1) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$.

(2) $T(c \cdot \mathbf{x}) = c \cdot T(\mathbf{x})$.

Theorem A.12. *Let V be an n-dimensional (left) vector space over the skew-field $\mathbb{F}$ with basis $\{\, \mathbf{v}_1, \, \cdots , \mathbf{v}_n \,\}$, and let W be a (left) vector space over $\mathbb{F}$ with $\mathbf{w}_1, \, \cdots , \mathbf{w}_n \in W$. Then there is a unique linear transformation $T : V \to W$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$, for all i.*

IMPORTANT SPECIAL CASE OF THEOREM A.12.   If we view $\mathbb{F}$ as a vector space over itself using the given operations of addition and multiplication, then every linear transformation from $\mathbb{F}$ to itself is multiplication by an element of $\mathbb{F}$.

Definition.   Let $T : V \to W$ be a linear transformation as above. The *kernel* of T, written Kernel(T), is the set of all $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{0}$, and the *image* of T, written Image(T), is the set of all $\mathbf{w} \in W$ such that $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v} \in V$. These sets are (left) vector subspaces of V and W respectively. If T_1 and T_2 are linear transformations from V to W, their *sum* is the linear transformation defined by

$$\bigl[T_1 + T_2\bigr](\mathbf{v}) = T_1(\mathbf{v}) + T_2(\mathbf{v})$$

for all $\mathbf{v} \in V$. If $T : V \to W$ and $U : W \to X$ are linear transformations, their *product* is the composite linear transformations $U \circ T : V \to X$ defined by

$$\bigl[U \circ T\bigr](\mathbf{v}) = T_1(\mathbf{v}) + T_2(\mathbf{v})$$

for all $\mathbf{v} \in V$.

Theorem A.13. *With respect to the above definitions of addition and multiplication, linear transformations from V satisfy properties* **(F-1)** *–* **(F-7)** *and* **(F-9)** *for skew-fields* PROVIDED *the relevant sums and products are defined.*

Remarks.   **1.**   The zero linear transformation $V \to W$ (where V and W are arbitrary) is the one sending every vector of V to zero. If $T : V \to W$ is linear, then so is its negative $-T : V \to W$ defined by $[-T](\mathbf{v}) = -\bigl[T(\mathbf{v})\bigr]$. If V is a vector space, then the identity map $I_V : V \to V$ defined by $I_V(\mathbf{v}) = \mathbf{v}$ is also a linear transformation.

2.   Both **(F-8)** and **(F-10)** fail completely for sums and products of linear transformations. A linear transformation $T : V \to W$ for which there is a transformation $S : W \to V$ such that $S \circ T = I_V$ and $T \circ S = I_W$ is said to be *invertible*. If there is a linear transformation S with these properties, then there is only one such linear transformation, and it satisfies the inverse function relationship $\mathbf{v} = S(\mathbf{w})$ if and only if $\mathbf{w} = T(\mathbf{v})$. Thus S is the inverse mapping T^{-1}.

Theorem A.14. (*i*) *A linear transformation* $T : V \to W$ *is invertible if and only if it is one-to-one (equivalently, its kernel is zero) and onto (equivalently, its image is W).*

(*ii*) *If V and W are finite-dimensional, an invertible linear transformation $T : V \to W$ exists if and only if* $\dim V = \dim W$.

(*iii*) *If $T : V \to W$ and $U : W \to X$ are invertible, then so is $U \circ T$ and $[U \circ T]^{-1} = T^{-1} \circ U^{-1}$.*

(*iv*) *If $T : V \to W$ is an invertible linear transformation and V_0 is a k-dimensional vector subspace of V, then $T[V_0]$ is a k-dimensional vector subspace of W. Conversely if $T[V_0]$ is a k-dimensional vector subspace of W, then V_0 is a k-dimensional vector subspace of V.*

(*v*) *Let $\{\, \mathbf{v}_1, \; \cdots \; , \mathbf{v}_n \,\}$ and $\{\, \mathbf{w}_1, \; \cdots \; , \mathbf{w}_n \,\}$ be ordered bases for V. Then there is a unique invertible linear transformation $T : V \to V$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$, for all i.*

There are several basic criteria for recognizing invertible linear transformations of finite-dimensional vector spaces.

Theorem A.15. *Suppose that V and W are finite-dimensional vector spaces such that $\dim V = \dim W$ and $T : V \to W$ is linear. Then the following are equivalent:*

(*i*) *The linear transformation T is invertible.*

(*ii*) *If $\{\, \mathbf{v}_1, \; \cdots \; , \mathbf{v}_n \,\}$ is a basis for V, then $\{\, T(\mathbf{v}_1), \; \cdots \; , T(\mathbf{v}_n) \,\}$ is a basis for W.*

(*iii*) *The linear transformation T is one-to-one.*

(*iv*) *The linear transformation T is onto.*

The reader should compare the equivalences (*i*) $\Leftrightarrow$ (*iii*) and (*i*) $\Leftrightarrow$ (*iv*) with Theorem A.13(*i*).

If $\mathbb{F}$ is *commutative* (hence a field), then one can also define the scalar product of a linear transformation by the rule $[cT](\mathbf{v}) = c \cdot T(\mathbf{v})$. If we let $\mathbf{Lin}_{\mathbb{F}}(V, W)$ denote the set of $\mathbb{F}$-linear transformations from one vector space V into another vector space W, this scalar product and the previously defined addition make $\mathbf{Lin}_{\mathbb{F}}(V, W)$ into a vector space over $\mathbb{F}$ in its own right. Scalar multiplication also has the following other properties:

$$
\begin{aligned}
c\,(U \circ T) \; &= \; (c\,U) \circ T \; = \; U \circ (c\,T) \\
(c\,I) \circ T \; &= \; c \cdot T \; = \; T \circ (c\,I)
\end{aligned}
$$

HYPOTHESIS. For the remainder of this section we shall assume that $\mathbb{F}$ is a (commutative) field.

Linear transformations and matrices

Linear transformations from an n-dimensional vector space to an m-dimensional vector space are closely related to $m \times n$ matrices over $\mathbb{F}$. Let $\mathcal{A} = \{\, \mathbf{v}_1, \; \cdots \; , \mathbf{v}_n \,\}$ be an ordered basis for V, and let $\mathcal{B} = \{\, \mathbf{w}_1, \; \cdots \; , \mathbf{w}_m \,\}$ be an ordered basis for W. Then we may define an $m \times n$ matrix whose entries $a_{i,j}$ are given by the formulas

$$
T(\mathbf{v}_j) \; = \; \sum_{i=1}^{m} a_{i,j}\, \mathbf{w}_i \; .
$$

We shall denote this matrix by $[T]_{\mathcal{A}}^{\mathcal{B}}$. The following relations are immediate consequences of this definition:

$$[T + S]_{\mathcal{A}}^{\mathcal{B}} = [T]_{\mathcal{A}}^{\mathcal{B}} + [S]_{\mathcal{A}}^{\mathcal{B}}$$

$$[cT]_{\mathcal{A}}^{\mathcal{B}} = c[T]_{\mathcal{A}}^{\mathcal{B}}$$

As usual, addition and scalar multiplication of matrices are defined entry by entry.

We define product matrices by the usual formula

$$C = A \cdot B \implies c_{r,s} = \sum_{t=1}^{n} a_{r,t} \, b_{t,s} \ .$$

Such products are defined only if the number of columns in A equals the number of rows in B; matrix products satisfy analogs of the associative and distributive laws for scalars (and also analogs of the scalar multiplication identities displayed above), but multiplication is not commutative, even if both AB and BA are defined and have the same numbers of rows and columns.

The definition of matrix product is set up so that the associated matrices satisfy a basic compatibility relation with respect to composites of linear transformations:

$$[U \circ T]_{\mathcal{A}}^{\mathcal{C}} = [U]_{\mathcal{B}}^{\mathcal{C}} \cdot [T]_{\mathcal{A}}^{\mathcal{B}}$$

Theorem A.16. *If* $\dim V = n$ *and* $\dim W = m$, *then the mapping sending a linear transformation* T *to the matrix* $[T]_{\mathcal{A}}^{\mathcal{B}}$ *defines an invertible linear transformation of vector spaces.*

The following **Change of Basis Theorem** is useful in several contexts.

Theorem A.17. *Let* $T : V \to V$ *be a linear transformation, where* V *is finite-dimensional. Given two ordered bases* $\mathcal{A} = \{\, \mathbf{v}_1, \ \cdots \ , \mathbf{v}_n \,\}$ *and* $\mathcal{B} = \{\, \mathbf{w}_1, \ \cdots \ , \mathbf{w}_m \,\}$ *for* V, *define an* $n \times n$ *matrix* P *by the equations*

$$\mathbf{w}_j = \sum_{i=1}^{n} p_{i,j} \, \mathbf{v}_i \ .$$

Then $[T]_{\mathcal{B}}^{\mathcal{B}} = P \cdot [T]_{\mathcal{A}}^{\mathcal{A}} \cdot P^{-1}$.

Remark. If I_V is the identity transformation, then $[I_V]_{\mathcal{A}}^{\mathcal{A}} = (\, \delta_{i,j} \,)$, where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq nj$ (the *Kronecker delta symbol*). This holds for every ordered basis $\mathcal{A}$. As usual, we denote this matrix by I_n or (more simply) I.

The transposition operation on matrices

Definition. If A is an $m \times n$ matrix over $\mathbb{F}$, its **transpose** is the $n \times m$ matrix $^{\mathbf{T}}A = (\, a'_{i,j} \,)$ defined by $a'_{i,j} = a_{j,i}$. In other words, the rows and columns A and $^{\mathbf{T}}A$ are the columns and rows (respectively) of A. The transposition construction on matrices has several important properties:

Theorem A.18. *If the relevant sums and products are defined, then the following hold:*

(i) ${}^{\mathbf{T}}\!(A + B) = {}^{\mathbf{T}}\!A + {}^{\mathbf{T}}\!B.$

(ii) ${}^{\mathbf{T}}\!(c\,A) = c\,{}^{\mathbf{T}}\!A.$

(iii) ${}^{\mathbf{T}}\!({}^{\mathbf{T}}\!A) = A.$

(iv) If A is a square matrix, then $\det {}^{\mathbf{T}}\!A = \det A.$

(v) ${}^{\mathbf{T}}\!(AB) = {}^{\mathbf{T}}\!B \cdot {}^{\mathbf{T}}\!A.$

(vi) ${}^{\mathbf{T}}\!I_n = I_n.$

Furthermore, the (row and determinant) ranks of A and ${}^{\mathbf{T}}\!A$ are equal.

Invertible matrices

We say that an $n \times n$ matrix A is *invertible* if there is another $n \times n$ matrix A^{-1} for which $A \cdot A^{-1} = A^{-1} \cdot A = I$. This matrix is unique, for if B and C satisfy $BA = AC = I$, then we have

$$ B = B \cdot I = B \cdot (AC) = (BA) \cdot C = I \cdot C = C . $$

Inverse matrices satisfy analogs of the product formula for inverse linear transformations in Theorem A.14(*ii*), and as in the case of inverse functions one has the identity

$$ \left(A^{-1} \right)^{-1} = A . $$

Finally, we note that $I^{-1} = I$ and if A has an inverse and $c \neq 0$, then (cA) also has an inverse, which is equal to $c^{-1} A^{-1}$.

Theorem A.19. *(i) Let V be an n-dimensional vector space over $\mathbb{F}$, and let $\mathcal{A}$ be an ordered basis for V. Then the linear transformation $T : V \to V$ is invertible if and only if $[T]_{\mathcal{A}}^{\mathcal{A}}$ is, in which case we have*

$$ \left[T^{-1}\right]_{\mathcal{A}}^{\mathcal{A}} = \left([T]_{\mathcal{A}}^{\mathcal{A}} \right)^{-1} . $$

(ii) If A is an invertible $n \times n$ matrix over $\mathbb{F}$, then so is ${}^{\mathbf{T}}\!A$ and

$$ {}^{\mathbf{T}}\!\left(A^{-1} \right) = \left({}^{\mathbf{T}}\!A \right)^{-1} . $$

EXERCISES

1. Suppose that V is an n-dimensional vector space over $\mathbb{F}$ and $S \subset V$ is a subset. Prove that S is an $(n-1)$-dimensional vector subspace if and only if there is a nonzero linear transformation $g : V \to \mathbb{F}$ such that S is the kernel of g. [*Hints:* If S is a vector subspace of the type described, take a basis $\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}$ for S, extend it to a basis for V by adding a suitable vector $\mathbf{v}_n$, and define a linear transformation g by $g(\mathbf{v}_i) = 0$ if $i < n$ and $g(\mathbf{v}_n) = 1$. Verify that V has the required properties. — Conversely, if we are given an arbitrary linear transformation $T : V \to W$, then we know that its kernel is a vector subspace. In particular, this applies to S. It remains to show that $\dim V = n$ implies $\dim S = (n-1)$; since g is nonzero, all we can say a *priori* is that $r = \dim S < n$. As before, choose a basis take a basis $\mathbf{v}_1, \cdots, \mathbf{v}_n$ for V such that

the first r vectors form a basis for S; we need to show that $r = (n - 1)$. If $\mathbf{x} \in V$ is arbitrary, explain why the assumptions on S and g imply

$$\mathbf{x} - \left(\frac{g(\mathbf{x})}{g(\mathbf{v}_{r+1})} \right) \cdot \mathbf{v}_{r+1} \in S$$

and using this show that $\dim V$ must be equal to $r + 1$, so that $r = \dim S = (n - 1)$.]

2. Let V and W be vector spaces over $\mathbb{F}$, and define a **direct sum** vector space structure $V \oplus W$ on the Cartesian product $V \times W$ coordinatewise. In other words, we have the following:

$$(\mathbf{v}_1, \mathbf{w}_1) + (\mathbf{v}_2, \mathbf{w}_2) = (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 + \mathbf{w}_2), \qquad c(\mathbf{v}, \mathbf{w}) = (c\mathbf{v}, c\mathbf{w})$$

Verify that these operations make $V \oplus W$ into a vector space. If V and W are finite-dimensional, show that the same is true for $V \oplus W$ and $\dim(V \oplus W) = \dim V + \dim W$. [*Hint:* If $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is a basis for V and $\mathbf{w}_1, \cdots, \mathbf{w}_m$ is a basis for W, show that the vectors $(\mathbf{v}_i, \mathbf{0})$ and $(\mathbf{0}, \mathbf{w}_j)$ combine to form a basis for $V \oplus W$.]

3. Prove that every linear transformation $T : V \oplus W \to X$ has the form

$$T(\mathbf{v}, \mathbf{w}) = T_1(\mathbf{v}) + T_2(\mathbf{w})$$

for some linear transformations $T_1 : V \to X$ and $T_2 : W \to X$. Conversely, show that every function T satisfying the displayed identity is a linear transformation (assuming that T_1 and T_2 are linear transformations).

5. Dot and cross products

Much of the theory of dot products in $\mathbb{R}^n$ and cross products in $\mathbb{R}^3$ can be extended to $\mathbb{F}^n$ and $\mathbb{F}^3$ for an arbitrary (commutative) field $\mathbb{F}$. We shall develop as much as possible in this degree of generality; additional properties which require the use of real numbers will be discussed at the end of this section.

The standard *dot product* (or *inner product*) in $\mathbb{R}^n$ is the scalar defined by the formula

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

where the coordinates of $\mathbf{v}$ are given by $v_1, \cdot, v_n$. One can define a *standard bilinear pairing* on $\mathbb{F}^n$ by the same formula, and it is elementary to verify that this pairing, which is a mapping from $\mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$, has the following properties (here $\mathbf{x} \, \mathbf{y}, \, \mathbf{z} \in \mathbb{F}^n$ and $c \in \mathbb{F}$ are arbitrary):

(B-1) (*Bilinearity in the right hand variable*) $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$.

(B-2) (*Homogeneity*) $\langle c\mathbf{x}, \mathbf{y} \rangle = c \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, c\mathbf{z} \rangle$.

(B-3) (*Symmetry in the variables*) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

(B-4) (*Nondegeneracy*) For each nonzero $\mathbf{x} \in \mathbb{F}^n$ there is some $\mathbf{y} \in \mathbb{F}^n$ such that $\langle \mathbf{x}, \mathbf{y} \rangle \neq 0$.

The verifications of the first three properties is once again a sequence of routine computations. Several further properties are immediate consequences of the given ones; one particular example is *bilinearity in the left hand variable*:

$$\langle\, \mathbf{x} + \mathbf{y}\, \mathbf{z}\, \rangle \;=\; \langle\, \mathbf{x},\, \mathbf{z}\, \rangle \;+\; \langle\, \mathbf{y},\, \mathbf{z}\, \rangle$$

Over the real numbers, one has a property which is stronger than **(B-4)**; namely, if $\mathbf{x} \neq 0$, then $\langle\, \mathbf{x},\, \mathbf{y}\, \rangle$ is the square of the length of the vector $\mathbf{x}$ (see the final portion of this section). If $\mathbb{F}$ is an arbitrary field, one can verify **(B-4)** as follows: If $\mathbf{x} \neq \mathbf{0}$, then some coordinate x_i is nonzero, and hence if $\mathbf{e}_i$ is the ith unit vector then it follows that

$$\langle\, \mathbf{x},\, \mathbf{e}_i\, \rangle \;=\; x_i \;\neq\; 0\,.$$

Similarly, the *cross product* in $\mathbb{F}^3$, where $\mathbb{F}$ is an arbitrary field, can be defined by the following standard formula:

$$\mathbf{x} \times \mathbf{y} \;=\; \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix},\; \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix},\; \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right)$$

If we adopt the standard unit vector notation

$$\begin{aligned} \mathbf{i} &= (1,0,0) \\ \mathbf{j} &= (0,1,0) \\ \mathbf{k} &= (0,0,1) \end{aligned}$$

then the formula above may be restated more concisely as the following "formal 3×3 determinant," which is expanded by minors along the first row:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

The cross product has the following properties, all of which are relatively easy to verify:

(B-1) (*Bilinearity in the right hand variable*) $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) \;=\; (\mathbf{x} \times \mathbf{y}) \;+\; (\mathbf{x} \times \mathbf{z})$.

(B-2) (*Homogeneity*) $(c\,\mathbf{x} \times \mathbf{y}) \;=\; c\,(\mathbf{x} \times \mathbf{y}) \;=\; (\mathbf{x} \times c\,\mathbf{z})$.

(B-3) (*Anticommutativity*) $\mathbf{x} \times \mathbf{y} \;=\; -(\mathbf{y} \times \mathbf{x})$.

(B-4) (*Triple Product Formula*) $\langle\, \mathbf{x},\, \mathbf{y} \times \mathbf{z}\, \rangle \;=\; \langle\, \mathbf{z},\, \mathbf{x} \times \mathbf{y}\, \rangle \;=\; \langle\, \mathbf{y},\, \mathbf{z} \times \mathbf{x}\, \rangle \;=\;$

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}\,.$$

Since $\{\, \mathbf{i},\, \mathbf{j},\, \mathbf{k}\, \}$ is a basis for $\mathbb{F}^3$, the cross product is completely determined by the first three properties and the standard multiplication formulas

$$\mathbf{i} \times \mathbf{j} \;=\; \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} \;=\; \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} \;=\; \mathbf{j}\,.$$

Several further properties are immediate consequences of the given ones; one particular example is *bilinearity in the left hand variable*:

$$(\mathbf{x} + \mathbf{y}) \times \mathbf{z} \;=\; (\mathbf{x} \times \mathbf{z}) \;+\; (\mathbf{y} \times \mathbf{z})$$

Another is the the following *Class Two nilpotence condition*:

$$\mathbf{y} \times \mathbf{y} \;=\; \mathbf{0} \qquad \text{for all } \mathbf{y}$$

NOTE. The similarity between the cross product formulas and the multiplication rules for quaternions when $\mathbb{F} = \mathbb{R}$ is not coincidental. If we identify $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ with its natural quaternionic counterpart, then we have

$$\mathbf{xy} \; (quaternion \; product) \;=\; \mathbf{x} \times \mathbf{y} \;-\; \langle\, \mathbf{x}, \mathbf{y} \,\rangle \cdot \mathbf{1} \;.$$

In fact, for several decades during the 19th century quaternions were used for many of the mathematical and physical formulas that are now expressed using 3-dimensional vector cross products and the language of vector analysis. The latter was developed by J. W. Gibbs (1839–1903) and O. Heaviside (1850–1925).

CAUTION. The cross product does **not** satisfy an associativity identity, and in general

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) \;\neq\; (\mathbf{x} \times \mathbf{y}) \times \mathbf{z} \;.$$

To see this, note that $\quad \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) \;=\; \mathbf{i} \times -\mathbf{k} \;=\; -\mathbf{j} \quad$ but $\quad (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} \;=\; \mathbf{0} \times \mathbf{j} \;=\; \mathbf{0}$. However, there is a formula which is helpful in manipulating large cross product expressions relatively well:

Theorem A.20. *For all* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *we have*

$$\mathbf{a} \;\times\; (\mathbf{b} \times \mathbf{c}) \;=\; \langle\, \mathbf{a}, \mathbf{c} \,\rangle \mathbf{b} \;-\; \langle\, \mathbf{a}, \mathbf{b} \,\rangle \mathbf{c} \;.$$

If we reverse the orders of scalars and vectors and use capital letters, the right hand side becomes

$$\mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) \;-\; \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B})$$

which leads to the name **Back–Cab Rule** for the threefold product.

The following result is obtained by combining three cases of the Back–Cab Rule:

Theorem A.21. (Jacobi identity[3]) *For all* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *we have*

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \;+\; \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) \;+\; \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \;=\; \mathbf{0} \;.$$

For the sake of completeness, we mention one more fundamental relation between the dot and cross products:

Theorem A.22. *For all* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *we have*

$$\langle\, \mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d} \,\rangle \;=\; \langle\, \mathbf{a}, \mathbf{c} \,\rangle \cdot \langle\, \mathbf{b}, \mathbf{d} \,\rangle \;-\; \langle\, \mathbf{a}, \mathbf{d} \,\rangle \cdot \langle\, \mathbf{b}, \mathbf{c} \,\rangle \;.$$

Geometric interpretations over $\mathbb{R}$

We have already mentioned that the dot product over the reals has the following property:

(B-5) (*Positive definiteness*) For each $\mathbf{x} \in \mathbb{R}^n$ we have $\langle\, \mathbf{x}, \mathbf{x} \,\rangle \;\geq\; 0$, with equality if and only if $\mathbf{x} = \mathbf{0}$.

---

[3]CARL GUSTAV JACOB JACOBI (1804–1851) made many important contributions to a wide range of mathematical areas. The identity bearing his name arose in his work on differential equations and mathematical physics.

In fact, $\langle \mathbf{x}, \mathbf{x} \rangle = |\mathbf{x}|^2$, where $|\mathbf{x}|$ is the length of $\mathbf{x}$ (equivalently, the distance between $\mathbf{x}$ and $\mathbf{0}$). More generally, we can interpret the dot product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$ geometrically as $|\mathbf{x}| \cdot |\mathbf{y}| \cdot \cos \alpha$, where α is the measurement of the angle $\angle \mathbf{x0y}$ determined by $\mathbf{x}$, $\mathbf{0}$, $\mathbf{y}$.
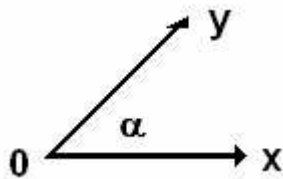

Figure A.1

It follows that the lines joining $\mathbf{0}$ to $\mathbf{x}$ and $\mathbf{0}$ to $\mathbf{y}$ are perpendicular if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

Similarly, the cross product may be viewed as a vector whose length is $|\mathbf{x}| \cdot |\mathbf{y}| \cdot \sin \alpha$. It is perpendicular to $\mathbf{x}$ and $\mathbf{y}$ by the triple product formula and basic properties of determinants (they vanish if two rows are the same), and its direction is specified by the *right hand rule*, which is illustrated below:
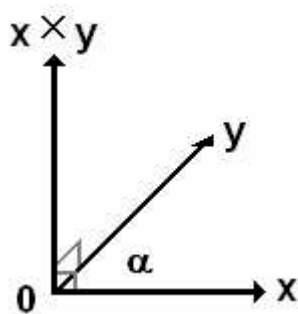

Figure A.2

ADDENDUM.   RIGID MOTIONS OF $\mathbb{R}^n$

As noted at the end of Section II.6, a *rigid motion* of $\mathbb{R}^n$ is a 1–1 correspondence of $\mathbb{R}^n$ with itself that preserves distances. In that discussion we made two unproved assertions:

(A) Rigid motions belong to the affine group $\mathrm{AFF}(\mathbb{R}^n)$.

(B) Rigid motions preserve angle measurements; more precisely, if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are noncollinear points and T is a rigid motion, then the cosine of $\angle \mathbf{abc}$ is equal to the cosine of $\angle\, T(\mathbf{a})T(\mathbf{b})T(\mathbf{c})$.

Both are fairly easy to prove, and arguments appear in some undergraduate linear algebra or geometry texts, but the proofs are not always easy to locate so we shall prove both assertions here. The following result disposes of **(A)** and is the key step in proving **(B)**.

Theorem A.23. *Let T be an isometry of $\mathbb{R}^n$. Then T may be expressed in the form*

$$T(\mathbf{x}) \;=\; \mathbf{b} \;+\; A(\mathbf{x})$$

where $\mathbf{b} \in \mathbb{R}^n$ is some fixed vector and A is an orthogonal linear tranformation of $\mathbb{R}^n$ (i.e., in matrix form we have that ${}^{\mathbf{T}}A = A^{-1}$).

REMARKS. It is an elementary exercise to verify that the composite of two isometries is an isometry (and the inverse of an isometry is an isometry). If A is orthogonal, then it is elementary to prove that $T(\mathbf{x}) \;=\; \mathbf{b} \;+\; A(\mathbf{x})$ is an isometry, and in fact this is done in most if not all undergraduate linear algebra texts. On the other hand, if $A = I$ then the map above reduces to a **translation** of the form $T(\mathbf{x}) = \mathbf{b} + \mathbf{x}$, and such maps are isometries because they satisfy the even stronger identity

$$T\big(\,\mathbf{x} \;-\; \mathbf{y}\,\big) \;=\; \mathbf{x} \;-\; \mathbf{y}\;.$$

Therefore every map of the form $T(\mathbf{x}) = \mathbf{b} + A(\mathbf{x})$, where $\mathbf{b} \in \mathbb{R}^n$ is some fixed vector and A is an orthogonal linear transformation of $\mathbb{R}^n$, is an isometry of the latter. Therefore the proposition gives a complete characterization of all isometries of $\mathbb{R}^n$.

Proof. This argument is often given in linear algebra texts, and if this is not done then hints are frequently given in the exercises, so we shall merely indicate the basic steps.

First of all, the set of all isometries of $\mathbb{R}^n$ is a group (sometimes called the *Galilean group* of $\mathbb{R}^n$). It contains both the subgroups of orthogonal matrices and the subgroup of translations (maps of the form $G(\mathbf{x}) = \mathbf{x} + \mathbf{c}$ for some fixed vector $\mathbf{c}$), which is isomorphic as an additive group to $\mathbb{R}^n$ with the vector addition operation. Given $\mathbf{b} \in \mathbb{R}^n$ let $\mathbf{S_b}$ be translation by $\mathbf{b}$, so that $A = \mathbf{S}_{-T(\mathbf{0})} \circ T$ is an isometry from $\mathbb{R}^n$ to itself satisfying $A(\mathbf{0}) = \mathbf{0}$. If we can show that A is linear, then it will follow that A is given by an orthogonal matrix and the proof will be complete.

Since A is an isometry it follows that

$$\big|A(\mathbf{x}) - A(\mathbf{y})\big|^2 \;=\; \big|\mathbf{x} - \mathbf{y}\big|^2$$

and since $A(0) = 0$ it also follows that A is length preserving. If we combine these special cases with the general formula displayed above we conclude that $\langle A(\mathbf{x}), A(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$ for all $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$. In particular, it follows that A sends orthonormal bases to orthonormal bases. Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be an orthonormal basis; then we have

$$\mathbf{x} \;=\; \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{u}_i\rangle \cdot \mathbf{u}_i$$

and likewise we have

$$A(\mathbf{x}) \;=\; \sum_{i=1}^{n} \langle A(\mathbf{x}), A(\mathbf{u}_i)\rangle \cdot A(\mathbf{u}_i)\;.$$

Since A preserves inner products we know that

$$\langle \mathbf{x}, \mathbf{u}_i\rangle \;=\; \langle A(\mathbf{x}), A(\mathbf{u}_i)\rangle$$

for all i, so that

$$A(\mathbf{x}) \;=\; \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{u}_i\rangle \cdot A(\mathbf{u}_i)\;.$$

and this implies that A is a linear transformation.∎

Isometries and angle measurements

We shall use the preceding result to prove **(B)**. Specifically, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, we need to prove that if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are noncollinear, then we have

$$\frac{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{b})}{|\mathbf{a} - \mathbf{b}|\,|\mathbf{c} - \mathbf{b}|} \;=\; \frac{(\,T(\mathbf{a}) - T(\mathbf{b})\,) \cdot (\,T(\mathbf{c}) - T(\mathbf{b})\,)}{|T(\mathbf{a}) - T(\mathbf{b})| \cdot |T(\mathbf{c}) - T(\mathbf{b})|} \;.$$

Note that if T_1 and T_2 are 1–1 correspondences of $\mathbb{R}^n$ to itself and both satisfy the equation above, then their composite $T_2 \circ T_1$ also has this property.

Since an arbitrary isometry is the composite of an orthogonal linear transformation and a translation, by the preceding sentence it suffices to know that both orthogonal linear transformations and translations satisfy the displayed equation. If A is an orthogonal linear transformation, then the identity $A(\mathbf{x} - \mathbf{y}) = A(\mathbf{x}) - A(\mathbf{y})$ implies

$$\frac{A(\mathbf{a} - \mathbf{b}) \cdot A(\mathbf{c} - \mathbf{b})}{|A(\mathbf{a} - \mathbf{b})| \cdot |A(\mathbf{c} - \mathbf{b})|} \;=\; \frac{(\,A(\mathbf{a}) - A(\mathbf{b})\,) \cdot (\,A(\mathbf{c}) - A(\mathbf{b})\,)}{|A(\mathbf{a}) - A(\mathbf{b})| \cdot |A(\mathbf{c}) - A(\mathbf{b})|} \;.$$

and the left hand side of this equation equals

$$\frac{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{b})}{|\mathbf{a} - \mathbf{b}|\,|\mathbf{c} - \mathbf{b}|}$$

because orthogonal linear transformations preserve inner products and vector lengths. Thus orthogonal linear transformations preserve angle measurements. Similarly, if G is a translation, then the identity $G(\mathbf{x} - \mathbf{y}) = G(\mathbf{x}) - G(\mathbf{y})$ implies

$$\frac{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{b})}{|\mathbf{a} - \mathbf{b}|\,|\mathbf{c} - \mathbf{b}|} \;=\; \frac{(\,G(\mathbf{a}) - G(\mathbf{b})\,) \cdot (\,G(\mathbf{c}) - G(\mathbf{b})\,)}{|G(\mathbf{a}) - G(\mathbf{b})| \cdot |G(\mathbf{c}) - G(\mathbf{b})|}$$

and hence translations also preserve angle measurements.■

Further topics

Additional information on isometries, classical geometric notions of congruence, and Euclidean similarity is contained in the following online document:

$$\texttt{http://math.ucr.edu/}\sim\texttt{res/math133/metgeom.pdf}$$

EXERCISES

Definition. An invertible square matrix P over the real numbers is said to be *conformal* if preserves angle measures in the sense described above; specifically, if $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are noncollinear points (hence their images under P are noncollinear), then the cosine of $\angle\,\mathbf{xyz}$ is equal to the cosine of $\angle\,P\mathbf{x}P\mathbf{y}P\mathbf{z}$. — By the linearity of P, this condition holds for all noncollinear $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ if and only if it holds for triples such that $\mathbf{y} \;=\; \mathbf{0}$. Every orthogonal matrix is conformal, and nonzero scalar multiples of the identity matrix are non-orthogonal conformal matrices if the scalar is not ± 1.

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1. Prove that the composite of two conformal matrices is conformal, and the inverse of a conformal matrix is conformal.

2. Prove that an $n \times n$ matrix is conformal if and only if if has the form cA, where c is a nonzero scalar and A is an orthogonal matrix. [*Hint:* Every matrix of the given form is conformal by the preceding exercise and the comments following the definition. Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be the usual orthonormal basis of unit vectors for the space of 3×1 column vectors, and explain why the image vectors $P\mathbf{e}_1, \cdots, P\mathbf{e}_n$ are pairwise orthogonal. Let $c_i > 0$ be the length of $P\mathbf{e}_i$, and consider the angles $\angle (\mathbf{e}_1 + \mathbf{e}_i)\mathbf{0}\,\mathbf{e}_1$ for $i > 1$. Show that the conformal property implies $c_i = c_1$ for all i by considering what will happen if $c_1 < c_i$ or $c_1 > c_i$, and using this show that $^{\mathbf{T}}P \cdot P = c_1^2\, I$. Why does this imply that $c_1^{-1} P$ is orthogonal?]

3. Prove that a 2×2 matrix P is conformal if and only if there are real numbers a and b which are not both zero such that
$$P = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} .$$
[*Hints:* Writing out all 2×2 orthogonal matrices is fairly straightforward; use this to check that the given conditions hold for conformal matrices. Conversely, if P is conformal, what is $^{\mathbf{T}}P \cdot P$, and how can one use this to show P is a multiple of an orthogonal matrix?]

> **Remark.** (*For readers who are somewhat familiar with the theory of functions of a complex variable.*) The conclusion of this exercise may be viewed as an abstract version of a basic principle in the theory of functions of a complex variable; namely, complex analytic functions define **conformal mappings** of the complex plane $\mathbb{C}$. — In other words, if α and β are two parametrized curves with the same value at zero and nonzero tangent vectors there, and f is an analytic function of a complex variable which is defined near that point and has a nonzero derivative there, then the angle at which α and β intersect equals the angle at which there images $f \circ \alpha$ and $f \circ \beta$ intersect. The condition on the matrix coefficients is essentially an abstract version of the *Cauchy-Riemann equations* for analytic functions.

THE JOIN IN AFFINE GEOMETRY

In Section II.5 we defined a notion of **join** for geometrical incidence spaces; specifically, if P and Q are geometrical subspaces of an incidence space S, then the join $P \star Q$ is the unique smallest geometrical subspace which contains them both. From an intuitive viewpoint, the name "join" is meant to suggest that $P \star Q$ consists of all points on lines of the form $\mathbf{xy}$, where $\mathbf{x} \in P$ and $\mathbf{y} \in Q$. If S is a projective n-space over some appropriate scalars $\mathbb{F}$, this is shown in Exercise 16 for Section III.4, and the purpose of this Appendix is to prove a similar result for an affine n-space over some $\mathbb{F}$.

Formally, we begin with a generalization of the idea described above.

Definition. Let (S, Π, d) be an abstract geometrical incidence n-space, and let $X \subset S$. Define $\mathbf{J}(X)$ to be the set
$$X \ \cup \ \{ \ \mathbf{y} \in S \mid \mathbf{y} \in \mathbf{uv} \text{ for some } \mathbf{u}, \ \mathbf{v} \in X \ \} \ .$$
Thus $\mathbf{J}(X)$ is X together with all points on lines joining two points of X. Note that the construction of $\mathbf{J}(X)$ from X can be iterated to yield a chain of subsets $X \subset \mathbf{J}(X) \subset \mathbf{J}\big(\mathbf{J}(X)\big) \ \cdots$. If X is a geometrical subspace of S, then the axioms for a geometrical incidence space imply that $\mathbf{J}(X) = X$, and by Theorem II.16 and Exercise II.2.1, a subset X of $\mathbb{F}^n$ satisfies $\mathbf{J}(X) = X$ if and only if X is an affine subspace V of $\mathbb{F}^n$, provided $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_2$.

The preceding discussion and definition lead naturally to the following:

QUESTION. *If S is a geometrical incidence n-space and P and Q are geometrical subspaces of S, what is the relationship between $P \star Q$ and $\mathbf{J}(P \cup Q)$? In particular, are they equal, at least if S satisfies some standard additional conditions?*

The exercise from Section III.4 shows that the two sets are equal if S is a standard projective n-space. In general, the next result implies that the two subsets need not be equal. but one is always contained in the other.

Theorem B.1. *In the setting above, we have $\mathbf{J}(P \cup Q) \subset P \star Q$. However, for each $n \geq 2$ there is an example of a* regular *geometrical incidence n-space such that, for some choices of P and Q, the set $\mathbf{J}(P \cup Q)$ is strictly contained in $P \star Q$.*

Proof. The inclusion relationship follows from **(G-2)** and the fact that $P \star Q$ is a geometrical subspace of S. On the other hand, if we take the affine incidence space structure associated to $\mathbb{Z}_2^n$ for $n \geq 2$, then for every subset $X \subset \mathbb{Z}^n$ we automatically have $\mathbf{J}(X) = X$ because every line consists of exactly two points. Thus if W and U are vector subspaces of $\mathbb{Z}_2^n$ such that neither contains the other, then $\mathbf{J}(W \cup U)$ is not a vector subspace. Since $\mathbf{0} \in W \cap U$, we know that $W \star U$ is the vector subspace $W + U$ by Theorem II.36, and it follows in this case that $\mathbf{J}(W \cup U)$ is strictly contained in $W \star U$.∎

Additional examples of regular indicence spaces for which $\mathbf{J}(P \cup Q)$ is strictly contained in $P \star Q$ may be constructed using Exercise II.5.7 in the notes. Specifically, let $D \subset \mathbb{R}^{n+1}$ be the set of all points $(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R} \cong \mathbb{R}^{n+1}$ such that $-1 < x_i < 1$ for all i, where $\mathbf{x} = (x_1, \ \cdots, x_n)$, so that D is the solid tower for which the base B is the open hypercube defined by the inequalities $|x_i| < 1$ and the tower is unbounded both above and below. By the previously cited exercise

we know that D is a regular geometrical incidence $(n+1)$-space if we define k-planes to be all nonempty intersections $P \cap D$, where P is a k-plane in $\mathbb{R}^{n+1}$. Furthermore, if S is $B \times \{0\}$ and T is given by the point $(\mathbf{0}, 1)$, then $S \star T = D$ but $\mathbf{J}(S \cup T)$ is a proper subset. For example, if $\mathbf{x} \in B$, $0 < t < 1$ and t is greater than all of the coordinate absolute values $|x_i|$, then $(\mathbf{0}, t)$ does not lie in $\mathbf{J}(S \cup T)$; if $n = 1$ this can be seen directly (try drawing a picture for motivation!) and one can extend everything directly to higher values of n. One can also construct many other such examples, but we shall stop here.

Note that the examples constructed in the proof of Theorem 1 are in fact *affine* incidence spaces. The main objective of this appendix is to prove that $\mathbf{J}(P \cup Q) = P \star Q$ if V is a vector space of dimension ≥ 2 over a field $\mathbb{F}$ which is not (isomorphic to) $\mathbb{Z}_2$.

Theorem B.2. *Let V be a vector space of dimension ≥ 2 over a field $\mathbb{F}$ which is not (isomorphic to) $\mathbb{Z}_2$, and suppose that $P = \mathbf{a} + U$ and $Q = \mathbf{b} + W$ are geometrical subspaces of V. Then the following hold:*

(i) The join $P \star Q$ is the affine span of $P \cup Q$.

(ii) $P \star Q = \mathbf{J}(P \cup Q)$.

Proof. FIRST STATEMENT. If R is the affine span of P and Q, then R is an affine subspace containing P and Q by Theorem II.19, Theorem II.16 and Exercise 1 for Section II.2 (this is where we use the assumption that $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_2$). Therefore it follows that R also contains $P \star Q$. On the other hand, if R' is a geometrical subspace containing P and Q, then by Theorem II.18 it contains all affine combinations of points in $P \cup Q$, and hence R' must contain R. Combining these observations, we conclude that R must be equal to $P \star Q$.

SECOND STATEMENT. By the previous theorem we know that $\mathbf{J}(P \cup Q) \subset P \star Q$, so it suffices to show that we also have the converse inclusion $P \star Q \subset \mathbf{J}(P \cup Q)$.

Let $\mathbf{x} \in P \star Q$, and let $\{\mathbf{d}_0, \cdots, \mathbf{d}_p\}$ and $\{\mathbf{c}_0, \cdots, \mathbf{c}_q\}$ be affine bases for P and Q respectively. Then by the conclusion of the first part of the theorem we may write

$$\mathbf{x} = \sum_{i=0}^{p} r_i \mathbf{d}_i + \sum_{j=0}^{p} s_j \mathbf{c}_j$$

where $\sum_i r_i + \sum_j s_j = 1$. Let $t = \sum_i r_i$, so that $\sum_j s_j = 1 - t$. There are now two cases, depending upon whether either or neither of the numbers t and $1 - t$ is equal to zero. If $t = 0$ or $1 - t = 0$ (hence $t = 1$), then we have $\mathbf{x} \in P \cup Q$. Suppose now that both t and $1 - t$ are nonzero. If we set

$$\alpha = \sum_{i=0}^{p} \frac{r_i}{t} \cdot \mathbf{d}_i \qquad \beta = \sum_{j=0}^{q} \frac{s_j}{(1-t)} \cdot \mathbf{c}_j$$

then $\alpha \in P$, $\beta \in Q$, and $\mathbf{x} = t\alpha + (1-t)\beta$; therefore it follows that $\mathbf{x} \in \mathbf{J}(P \cup Q)$.∎

APPENDIX C

REVERSAL OF MULTIPLICATION IN SKEW-FIELDS

If (S, Π, d) is a geometrical incidence n-space, then the methods and results of Section IV.4 show that there is a dual geometrical incidence n-space (S, Π, d) is a geometrical incidence whose points are the hyperplanes of S, and such that there is a canonical 1–1 correspondence between the k-planes of S^* and the $(n - k - 1)$-planes of S for all k such that $-1 \leq k \leq n$. If $\mathbb{F}$ is a (commutative) field and $S = \mathbb{FP}^n$, then the material on analytic projective geometry in Sections V.1 and VI.1 these notes shows that S^* is also isomorphic to $S = \mathbb{FP}^n$.

Isomorphisms between the projective n-space $\mathbb{FP}^n$ and its dual are called *polarities*, and they play important roles in projective geometry; one class of examples is discussed in Chapter VII of these notes. By the preceding discussion, we know that polarities exist if $\mathbb{F}$ is commutative, and it is natural to ask if this generalizes:

> Let $S = \mathbb{FP}^n$ be a projective n-space, where $n \geq 2$ and $\mathbb{F}$ is a skew-field which is *not necessarily commutative. Is there an incidence space isomorphism from S to its dual projective n space S^*?*

The purpose of this Appendix is to discuss this problem and the related algebraic questions. Unfortunately, the discussion ultimately involves algebraic input that goes far beyond the undergraduate level, so we shall merely sketch the main ideas and give references for additional information on the underlying algebra.

We begin by summarizing a point that is mentioned in Example 2 from Section IV.4 and is discussed further in Sections V.1 and VI.1. If $\mathbb{F}$ is a skew-field and $n \geq 2$ is an integer, then the dual projective n-space $(\mathbb{FP}^n)^*$ associated to $\mathbb{FP}^n$ is isomorphic to $(\mathbb{F}^{OP})\mathbb{P}^n$, where $\mathbb{F}^{OP}$ is the skew-field obtained from $\mathbb{F}$ by reversing the order of multiplication in $\mathbb{F}$. Of course, if $\mathbb{F}$ is commutative, then the multiplications in $\mathbb{F}$ and $\mathbb{F}^{OP}$ are identical, and therefore it follows that S^* is isomorphic to S in such cases. In order to handle the general situation, we must describe the relationship between the two skew-fields $\mathbb{F}$ and $\mathbb{F}^{OP}$ in the noncommutative case and derive its implications for the relationship between $\mathbb{FP}^n$ and the isomorphic dual objects $(\mathbb{F}^{OP})\mathbb{P}^n \cong (\mathbb{FP}^n)^*$.

In Appendix A we described a skew-field $\mathbb{K}$ called the *quaternions* which is not commutative, and thus it is natural to ask whether $\mathbb{KP}^n$ and $(\mathbb{K}^{OP})\mathbb{P}^n \cong (\mathbb{KP}^n)^*$ are isomorphic. By the general classification results, this will happen if and only if there is an algebraic isomorphism between $\mathbb{K}$ and $\mathbb{K}^{OP}$; note that since $\mathbb{K}$ is not commutative this map cannot be the identity.

In fact, there is an operation on $\mathbb{K}$ called *quaternionic conjugation* which is defined by

$$(a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k})^* = a_1 - a_2\mathbf{i} - a_3\mathbf{j} - a_4\mathbf{k}$$

(see Birkhoff and MacLane, pp. 222–225, especially Exercise 5, p. 224) which has the following properties:

$$\left(\mathbf{x}^*\right)^* = \mathbf{x} \qquad (\mathbf{x} + \mathbf{y})^* = \mathbf{x}^* + \mathbf{y}^* \qquad (\mathbf{x} \cdot \mathbf{y})^* = \mathbf{y}^* \cdot \mathbf{x}^*$$

Note that conjugation, like matrix transposition, reverses the order of multiplication. This conjugation map is 1–1 and onto by the first identity in the display, and the other two identities imply it defines an isomorphism from $\mathbb{K}$ to $\mathbb{K}^{OP}$.

More generally, by Theorem V.10 we have the following:

Theorem C.1. *Let $\mathbb{F}$ be a skew-field and let $S = \mathbb{FP}^n$, where $n \geq 2$. Then S is isomorphic to its projective dual space S^* if and only if $\mathbb{F}$ is isomorphic to $\mathbb{F}^{\mathsf{OP}}$.*

Therefore *the existence of projective incidence n-spaces that are not isomorphic to their projective duals* will follow **if** *there are skew-fields $\mathbb{F}$ such that $\mathbb{F}$ is not isomorphic to $\mathbb{F}^{\mathsf{OP}}$.*

Although there are many examples of skew-fields $\mathbb{F}$ for which $\mathbb{F}$ and $\mathbb{F}^{\mathsf{OP}}$ are not isomorphic, describing them is quite nontrivial. The simplest example of this type is given on pages 31–38 of the book by Blanchard listed below; in fact, the exercises on pages 37–38 yield a direct proof that $\mathbb{F}$ and $\mathbb{F}^{\mathsf{OP}}$ are not isomorphic in this case. However, we shall also outline a more comprehensive approach which yields this fact for Blanchard's example and allows one to find systematic infinite families of such skew-fields.

Given a skew-field $\mathbb{L}$, its *center* consists of all $a \in \mathbb{L}$ such that $a \cdot x = x \cdot a$ for all $x \in \mathbb{L}$. This center is always nontrivial, and in particular it contains the smallest subfield containing the unit element (which is isomorphic to either the rational numbers $\mathbb{Q}$ or one of the finite fields $\mathbb{Z}_p$). Furthermore, it is immediate that the center is a commutative subfield that we shall call $\mathbb{F}$, and it also follows that $\mathbb{L}$ is a vector space over $\mathbb{F}$. For example, if $\mathbb{L}$ is the quaternions, then $\mathbb{F}$ is the real numbers. Conversely, given a field $\mathbb{F}$ we can consider all skew-fields $\mathbb{L}$ with center $\mathbb{F}$ such that $\mathbb{L}$ is a *finite-dimensional* vector space over $\mathbb{F}$. Since each such $\mathbb{L}$ is isomorphic to a subring of a finite-dimensional matrix algebra over $\mathbb{F}$ (see Birkhoff and MacLane, pp. 226–227), the isomorphism classes of such objects form a set. A basic construction in algebra makes this set into an abelian group called the **Brauer group** of $\mathbb{F}$ and written $\mathrm{Br}(\mathbb{F})$. The details of the definition appear in the books by Adamson, Herstein, and Jacobson, and also in the papers by Serre (all of which are listed below), and the book by Gille and Szamuely covers the subject thoroughly (but at an extremely high level). The following online references might also be helpful:[1]

$$\texttt{http://en.wikipedia.org/wiki/Division\_ring}$$
$$\texttt{http://en.wikipedia.org/wiki/Field\_theory\_(mathematics)}$$
$$\texttt{http://en.wikipedia.org/wiki/Quaternion}$$
$$\texttt{http://en.wikipedia.org/wiki/Brauer\_group}$$

(Note that there are **two** underscore characters separating words in these links.)

The construction of $\mathrm{Br}(\mathbb{F})$ has the following immediate consequence:

Theorem C.2. *Suppose that the commutative field $\mathbb{F}$ admits no automorphisms other than the identity (for example, this holds if $\mathbb{F}$ is the rational numbers $\mathbb{Q}$ or the real numbers $\mathbb{R}$). Let $\mathbb{L}$ be a skew-field $\mathbb{L}$ which is finite-dimensional over its center $\mathbb{F}$, and assume further that $\mathbb{L}$ is isomorphic to $\mathbb{L}^{\mathsf{OP}}$. Then $\mathbb{L}$ determines an element of order 2 in the Brauer group $\mathrm{Br}(\mathbb{F})$.* ∎

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[1] In any citation of Wikipedia articles, it is important to recognize concerns about the accuracy of articles that are submitted by volunteers and subject to editing by a vast number of individuals whose views or understanding may be controversial or unreliable. This issue has been noted explicitly by Wikipedia in its article on itself, where the issue is discussed in some detail. However, for our purposes such questions do not cause problems because I have read the articles cited below and found them to be accurate.

Thus the proof that $\mathbb{L}$ is not isomorphic to $\mathbb{L}^{\mathsf{OP}}$ for Blanchard's example reduces to showing the following:

(i) The center of $\mathbb{L}$ is the rational numbers $\mathbb{Q}$.

(ii) The order of $\mathbb{L}$ in the Brauer group is greater than 2.

The first assertion is verified in Blanchard's book, and the second one follows from two easily stated facts:

(iii) By construction, the dimension of $\mathbb{L}$ over $\mathbb{Q}$ is equal to 9.

(iv) By the results on pages 93–95 of Artin, Nesbitt and Thrall, the order m of $\mathbb{L}$ in $\mathrm{Br}(\mathbb{F})$ satisfies $m^2 = \dim_{\mathbb{Q}} \mathbb{L}$, so that $m = 3$. In particular, $\mathbb{L}$ determines an element of odd order in the Brauer group of $\mathbb{Q}$ and hence cannot be isomorphic to $\mathbb{L}^{\mathsf{OP}}$.

In fact, very powerful methods exist for calculating Brauer groups, and these objects turn out to be fundamentally important to an algebraic subject called *class field theory*. Numerous references are given below. One consequence is that elements of order p exist in $\mathrm{Br}(\mathbb{Q})$ for every prime p (compare Adamson, pp. 220–221).

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http://www.numdam.org/numdam-bin/feuilleter?id=3DSHC_1950-1951_3_

J.-P. Serre. Local Class Field Theory. *Algebraic Number Theory* (Proceedings of the NATO Advanced Study Institute, 1965), pp. 129–163. Thompson Publishing, Washington, DC, 1967.

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[2]In this link, there are **two** underscore characters before the 3.

AUTOMORPHISMS OF THE COMPLEX NUMBERS

This appendix discusses an issue that arose in Section VI.3. It is written at a level somewhat higher than the notes, but not as high as the level of material in Appendix B. The necessary background can be found in the graduate algebra text by T. Hungerford and the introductory graduate level topology text by J. Munkres that are listed in the bibliography.

The Fundamental Theorem of Projective Geometry in Chapter VI (Theorem VI.14) implies that for each positive integer n there is a group homomorphism from the group $\text{COLL}(\mathbb{FP}^n)$ of geometric symmetries, or *collineations*, of $\mathbb{FP}^n$ to the automorphism group $\text{AUT}(\mathbb{F})$ of $\mathbb{F}$, and this mapping is onto. For some fields such as the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$, the only automorphism is the identity. On the other hand, in the notes we pointed out that the field $\mathbb{C}$ of complex numbers also has a second automorphism given by complex conjugation. This leads immediately to the following:

Quesion. *Are there any automorphisms of $\mathbb{C}$ other than the identity and complex conjugation?*

In fact, there are two reasonable ways of answering this question. If we restrict our attention to automorphisms that are *continuous* as mappings of $\mathbb{C} \cong \mathbb{R}^2$ to itself, then the identity and complex conjugation are the only possibilities. On the other hand, if we consider arbitrary automorphisms of $\mathbb{C}$ with no continuity requirement, then the automorphism group $\text{AUT}(\mathbb{C})$ of $\mathbb{C}$ is huge. One illustration of this is the formula for its cardinality given below.

Continuous automorphisms of $\mathbb{C}$

We shall first prove that there are only two continuous automorphisms.

Theorem D.1. *Let $\varphi : \mathbb{C} \to \mathbb{C}$ be an automorphism of the complex numbers that is also continuous as a function of two variables. Then φ is either equal to the identity or to complex conjugation.*

Proof. Every automorphism sends 0 and 1 to themselves, and from this it follows that every automorphism sends the rational numbers $\mathbb{Q} \subset \mathbb{C}$ to itself by the identity. Furthermore, if $a \in \mathbb{Q}$ is nonzero and $b \in \mathbb{C}$ satisfies $b^2 = a$, then we also have $\varphi(b)^2 = a$, and since $\pm b$ are the only two numbers such that $b^2 = a$ we must have $\varphi(b) = \pm b$.

Let $\mathbb{Q}[\mathbf{i}]$ be the set of all complex numbers of the form $a + b\mathbf{i}$, where $\mathbf{i}^2 = -1$ and $a, \, b \in \mathbb{Q}$. Since $\pm \mathbf{i}$ are the only two complex numbers whose squares are equal to -1, it follows that $\varphi(\mathbf{i}) = \pm \mathbf{i}$. Therefore φ agrees with either the identity or conjugation on $\mathbb{Q}[\mathbf{i}]$.

Just as the rationals are dense in $\mathbb{R}$, so also the set $\mathbb{Q}[\mathbf{i}]$ is dense in $\mathbb{C}$.[1] Since two continuous functions from $\mathbb{C}$ to itself are equal if they agree on a dense subset,[2] it follows that φ is equal to the identity or complex conjugation, depending upon whether $\varphi(\mathbf{i}) = \mathbf{i}$ or $\varphi[\mathbf{i}] = -\mathbf{i}$.■

<div align="center">Arbitrary automorphisms of $\mathbb{C}$</div>

This discussion will rely heavily on the existence of *transcendence bases* for the $\mathbb{C}$ with respect to its subfield $\mathbb{Q}$; more generally, this notion is defined for any pair of fields $\mathbb{F}$ and $\mathbb{E}$ such that $\mathbb{F}$ is a subfield of $\mathbb{E}$ (equivalently, $\mathbb{E}$ is an extension field of $\mathbb{F}$), at least if we have a field of characteristic zero which contains an isomorphic copy of $\mathbb{Q}$. Formal definitions and the basic properties of such subsets are contained in Section VI.1 of Hungerford's book. A brief summary of some important points is given in the Wikipedia reference:[3]

<div align="center">http://en.wikipedia.org/wiki/Transcendence_degree</div>

For our purposes the main thing to know is that if $\mathbb{E}$ is an extension field of $\mathbb{F}$, then different transcendence bases for $\mathbb{E}$ with respect to $\mathbb{F}$ have the same cardinality, and in fact one has the following basic result:

Theorem D.2. *Let $\mathbb{E}$ and $\mathbb{F}$ be as above, assume that $\mathbb{E}$ is algebraically closed, let B and B' be transcendence bases for $\mathbb{E}$ with respect to $\mathbb{F}$, and let $h : B \to B'$ be a $1-1$ correspondence. Then there is an automorphism φ of $\mathbb{E}$ such that φ maps itself to the identity and $\varphi(b) = h(b)$ for all $b \in B$.*■

EXAMPLE. In particular, if B is nonempty, then one can take B' to be the set of all negatives of elements in B and define $h(b) = -b$. Similarly, if we split B into two nonempty disjoint subsets $B_0 \cup B_1$, take B_1' to be the set of all negatives of elements in B_1, write $B' = B_0 \cup B_1'$, and take $h : B \to B'$ to be the identity on B_0 and multiplication by (-1) on B_1, then we obtain an automorphism that is different from the preceding one. In particular, if the cardinality of B is β, this yields a family of 2^β automorphisms of $\mathbb{E}$.

In order to apply the preceding theorem to the complex numbers, we need the following additional facts.

Theorem D.3. *Let $\mathbb{E}$ and $\mathbb{F}$ be as above, let B be a transcendence base for $\mathbb{E}$ over $\mathbb{F}$, let $\mathbb{L} \subset \mathbb{E}$ be the smallest subfield containing $\mathbb{F}$ and B, and suppose that $\mathbb{F}$ is a countably infinite set. Denote the cardinality of B by β. Then the cardinality of $\mathbb{L}$ is also equal to β.*■

Theorem D.4. *Let $\mathbb{L}$ be a field of infinite cardinality λ, and let $\mathbb{E}$ be the algebraic closure of $\mathbb{L}$. Then the cardinality of $\mathbb{E}$ is also equal to λ.*■

The applications to the complex numbers are now immediate.

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[1]See Munkres, Exercise 10, p. 194.

[2]Munkres, Exercise 5, p. 199, plus (*i*) the definition of dense subset on p. 191, (*ii*) the proof that metric spaces are Hausdorff on p. 129.

[3]Previous comments about Wikipedia articles also apply here.

Theorem D.5. *If B is a transcendence base for the complex numbers with respect to the rationals, then the cardinality of B is equal to $\mathfrak{c} = 2^{\aleph_0}$.*

Proof. Let $\mathbb{L}$ again be the smallest subfield containing both the rationals and B. By definition of transcendence bases, it follows that every element of $\mathbb{C}$ is algebraic over $\mathbb{L}$. Since $\mathbb{C}$ is algebraically closed, it contains an isomorphic copy of the algebraic closure $\mathbb{K}$ of $\mathbb{L}$. We claim that $\mathbb{K} = \mathbb{C}$; this is true because the elements of $\mathbb{C}$ are all algebraic over $\mathbb{L}$.

Applying the previous results, we see that the cardinalities of $\mathbb{C}$ and $\mathbb{L}$ must be the same, so the latter has cardinality $\mathfrak{c}$. By another of the theorems stated above, this in turn implies that the cardinality of the transcendence base B is also equal to $\mathfrak{c}$.$\blacksquare$

The existence of infinitely many distinct automorphisms of the complex numbers (in fact, $2^{\mathfrak{c}}$ of them) follows immediately, and hence the identity and complex conjugation are far from being the only ones. In fact, we can evaluate the cardinality of $\mathrm{AUT}(\mathbb{C})$ precisely:

Theorem D.6. *The cardinality of $\mathrm{AUT}(\mathbb{C})$ is equal to $2^{\mathfrak{c}}$.*

Proof. The cardinality γ of $\mathrm{AUT}(\mathbb{C})$ is certainly no greater than $\mathfrak{c}^{\mathfrak{c}}$, which is the cardinality of the set of (purely set-theoretic) functions from $\mathbb{C}$ to itself. On the other hand, we also have

$$2^{\mathfrak{c}} \;\leq\; \mathfrak{c}^{\mathfrak{c}} \;=\; \left(2^{\aleph_0}\right)^{\mathfrak{c}} \;=\; 2^{\aleph_0 \cdot \mathfrak{c}} \;=\; 2^{\mathfrak{c}}$$

(since $\mathfrak{c} \;\leq\; \aleph_0 \cdot \mathfrak{c} \;\leq\; \mathfrak{c} \cdot \mathfrak{c} \;=\; \mathfrak{c}$) so that

$$\mathfrak{c}^{\mathfrak{c}} \;=\; 2^{\mathfrak{c}}$$

and hence $\gamma \leq 2^{\mathfrak{c}}$. We had already established the reverse inequality earlier, and thus it follows that the equation in the theorem is correct.$\blacksquare$

FURTHER RESULTS. We should also mention one additional consequence of the preceding ideas which is mentioned in the Wikipedia article. Namely, there exist field homomorphisms from $\mathbb{C}$ to itself whose images are proper subfields of $\mathbb{C}$; in other words, $\mathbb{C}$ is algebraically isomorphic to proper subfields of itself.$\blacksquare$

ADDITIONAL MATERIAL ON HYPERQUADRICS

This Appendix discusses two issues that were mentioned in Section VII.2. The first relates the notion of tangent space in Section VII.2 to the standard concepts of tangent lines and planes that one encounters in single variable and multivariable calculus. In particular, we show that nonsingularity in the sense of the notes is equivalent to the usual nonsingularity conditions in calculus which are given by the nonvanishing of certain Jacobian determinants, and the tangent hyperplanes defined in these notes coincide with the notions of tangent hyperplanes that one sees in calculus courses, provided the points in question are nonsingular (there is some difference between the notion of tangent space in these notes and standard notions of tangents at singular points, but a discussion of such matters is beyond the scope of these notes). One reference for background material in multivariable calculus is the following standard textbook:

> **J. E. Marsden and A. J. Tromba.** Vector Calculus. (5$^{\text{th}}$ Ed.). *W. H. Freeman & Co., New York NY*, 2003. ISBN: 0–7147–4992–0.

The second topic in this Appendix concerns the determination of which matrices define the same hyperquadric. By Theorem VII.6, in many cases (including all nonsingular hyperquadrics) two symmetric matrices define the same projective hyperquadric if and only if one is a nonzero scalar multiple of the other. In Section E.2 we extend this theorem to some other cases and indicate how it fails in others.

DEFAULT HYPOTHESIS. As in Chapter VII, unless stated otherwise we assume that $\mathbb{F}$ is a (commutative) field such that $1 + 1 \neq 0$ in $\mathbb{F}$.

1. Tangent hyperplanes and differential calculus

We begin by discussing nonsingularity for affine hyperquadrics from the viewpoint of multivariable calculus. More generally, if we are given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ (*i.e.*, continuous partial derivatives), and $V \subset \mathbb{R}^n$ is the set of solutions to the equation $f(\mathbf{x}) = 0$, we want to formulate a mathematical condition which means that $\mathbf{x}$ is not a singular (or exceptional or special) point of V. For example, we expect this to mean that V has no corners or branches at $\mathbf{x}$.

EXAMPLE 1. Let $f(u,v) = v^3 - u^2$, and let $\mathbf{x} = (0,0)$. Then $\mathbf{x}$ has a 180° corner — or *cusp* — at $\mathbf{x}$
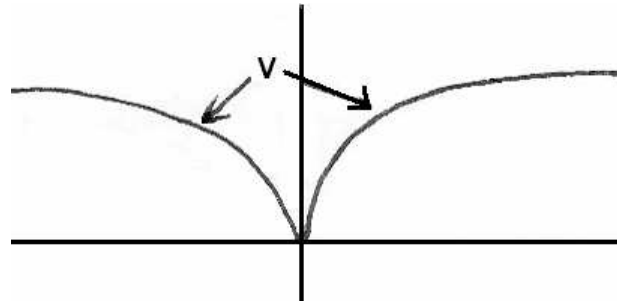

Figure E.1

EXAMPLE 2.   Let $f(u,v) \; = \; v^2 - u^2$, and let $\mathbf{x} \; = \; (0,0)$. Then f has two branches (*i.e.*, a **node**) at $\mathbf{x}$
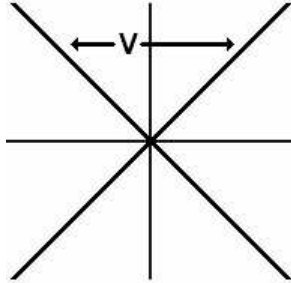

Figure E.2

In both examples, the curve is regular — or *nonsingular* — everywhere else.

In each of the preceding examples, the singular points are precisely the points $\mathbf{x} \in V$ for which the gradient $\nabla f(\mathbf{x})$ is equal to $\mathbf{0}$, and in fact the vanishing of the gradient turns out to be the condition for singularity. The reason of this comes from the Implicit Function Theorem of multivariable calculus (see Section 3.5 of the book by Marsden and Tromba mentioned at the beginning of this Appendix). By this theorem, if $\mathbf{x} \in V$ and $\nabla f(\mathbf{x}) \neq \mathbf{0}$, then for points close to $\mathbf{x}$ one of the coordinates u or v can be expressed as a smooth function of the other (in other words, one can solve for one of the coordinates in terms of the other). Thus if we restrict to points that are sufficiently close to $\mathbf{x}$ the curve V looks locally like the graph of a smooth function and hence is extremely regular. In the figure below we have

$$\frac{\partial F}{\partial v}(\mathbf{x}) \; \neq \; 0$$

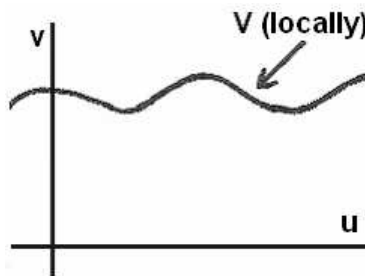and near $\mathbf{x}$ we can solve for v in terms of u.

Figure E.3

It is not always possible to solve *globally* for one coordinate in terms of the other. The simplest example is the circle Γ defined by the equation $f(u,v) = u^2 + v^2 - 1 = 0$. It is nonsingular at every point because $\mathbf{x} \in \Gamma$ imples that $\mathbf{x} \neq \mathbf{0}$ and $\nabla f(u,v) = (2u, 2v)$. However, since many vertical and horizontal lines meet the circle in two points, there is no way that we can view the circle globally as the graph of a reasonable (single valued) function.

Suppose now that $V \subset \mathbb{R}^n$ is the set of all $\mathbf{x}$ such that $f(\mathbf{x}) = 0$, where f is a function with continuous first partial derivatives. We shall say that $\mathbf{x}$ is an *analytically singular* point of V if $\nabla f(\mathbf{x}) = \mathbf{0}$ and $\mathbf{x}$ is an *analytically nonsingular* point otherwise.

Throughout this discussion, if $\mathbf{x}$ is a vector in $\mathbb{R}^n$ then we shall denote its coordinates by $x_1, \cdots, x_n$.

Theorem E.1. *Let Σ be the hyperquadric in $\mathbb{R}^n$ defined by $f(\mathbf{x}) = 0$, where*

$$f(\mathbf{x}) = \sum_{i,j} a_{i,j} x_i x_j + 2 \cdot \sum_k b_k x_k + c .$$

Then Σ is analytically singular at $\mathbf{x}$ if and only if

$$\begin{pmatrix} A & {}^{\mathbf{T}}\mathbf{b} \\ \mathbf{b} & c \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \mathbf{0}$$

where A is the matrix of second degree coefficients of f, $\mathbf{b}$ is the row vector of first degree coefficients, and c is the constant term.

Proof. Direct computation shows that the ith coordinate of $\nabla f(\mathbf{x})$ is equal to

$$\sum_j 2a_{i,j} x_j + 2b_i$$

so that $\mathbf{x}$ is an analytically singular point of Σ if and only if each of these expressions is equal to zero.

If the condition on matrix products

$$\begin{pmatrix} A & {}^{\mathbf{T}}\mathbf{b} \\ \mathbf{b} & c \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \mathbf{0}$$

is true, then by construction and the formula of the previous paragraph we know that $\nabla f(\mathbf{x})$ is twice the vector whose coordinates are the first n entries of the right hand side, and therefore we have $\nabla f(\mathbf{x}) = \mathbf{0}$, proving the "if" implication in the theorem.

Conversely, suppose that $\mathbf{x}$ is an analytically singular point. Then the same reasoning as before shows that the first n entries of

$$\begin{pmatrix} A & {}^{\mathbf{T}}\mathbf{b} \\ \mathbf{b} & c \end{pmatrix}$$

are zero, so the proof of this implication reduces to showing that the last entry equals zero, which translates to $\sum_i b_i x_i + c = 0$. We already know that

$$\sum_j a_{i,j} x_j + b_i = 0$$

for each i, and if we multiply each such equation by the associated coordinate x_i and sum over i we obtain

$$\sum_{i,j} a_{i,j} x_i x_j + b_i x_i = 0$$

and if we subtact this from the given equation $f(\mathbf{x}) = 0$ we obtain the desired equation $\sum_i b_i x_i + c = 0.\blacksquare$

Theorem E.2. *The affine hyperquadric in the previous theorem is totally analytically nonsingular if and only if the last column of the matrix*

$$\begin{pmatrix} A & {}^{\mathrm{T}}\mathbf{b} \\ \mathbf{b} & c \end{pmatrix}$$

is not expressible as a linear combination of the preceding ones.

Proof. Suppose first that the hyperquadric has a singular point. Let $\mathbf{x}$ be the singular point, and let $\mathbf{k}_i$ be the ith column of the matrix in the theorem. Then we have

$$\mathbf{k}_{n+1} = -\sum_i x_i \mathbf{k}_i \; .$$

Conversely, suppose that the last column is a linear combination of the others, so that (say) we have $\mathbf{k}_{n+1} = \sum_i y_i \mathbf{k}_i$. Then

$$\begin{pmatrix} A & {}^{\mathrm{T}}\mathbf{b} \\ \mathbf{b} & c \end{pmatrix} \cdot \begin{pmatrix} -\mathbf{y} \\ 1 \end{pmatrix} = \mathbf{0}$$

Explicitly, these imply that

$$
\begin{array}{ll}
(\mathbf{A}_i) & \sum a_{i,j}(-y_j) \, + \, b_i \; = \; 0, \\
(\mathbf{B}) & \sum b_j(-y_j) \, + \, c \; = \; 0 \; .
\end{array}
$$

In particular we have $0 = \sum (\mathbf{A}_i)(-y_i) + (\mathbf{B})$, which is just the equation

$$0 = \sum_{i,j} a_{i,j}(-y_i)(-y_j) \, + \, 2 \cdot \sum_k b_k(-y_k) \, + \, c \; .$$

Therefore $-\mathbf{y} \in \Sigma$. Since the ith coordinate of $\nabla f(-\mathbf{y})$ is the left hand side of $2(\mathbf{A}_i)$, it follows also that $\nabla f(-\mathbf{y}) = \mathbf{0}$, and therefore $-\mathbf{y}$ is a singular point of Σ.∎

Using the concepts of Section VII.2, we may reinterpret the preceding results as follows:

> Let Σ be an affine hyperquadric in $\mathbb{R}^n$, and let $\mathbb{P}(\Sigma)$ denote its projective extension. Then the set of singular points of $\Sigma^* \cap \mathrm{J}(\mathbb{R}^n)$ in the sense of Section VII.2 is equal to the set of analytically singular points of Σ as defined above.

EXAMPLE. There are nonsingular affine quadrics Σ such that Σ has no singular points but the projective extension Σ^* has singular points. In particular, the cylinder $\Sigma \subset \mathbb{R}^3$ defined by the equation $x^2 + y^2 = 1$ has no singular points, but its projective extension has a unique singular point; namely, the ideal point on the z-axis. For this example, the singular point is the intersection of Σ^* with the ideal plane.

Tangent spaces and tangents to curves

We begin by recalling the analytic definition of tangent line to a hyperquadric Σ in $\mathbb{R}^n$. Namely, it was given by $\mathbf{x} + \mathbb{R} \cdot \gamma'(t_0)$, where γ is a smooth curve lying totally in Σ such that $\gamma(t_0) = \mathbf{x}$.

Theorem E.3. *Let $\Sigma \subset \mathbb{R}^n$ be a nonsingular hyperquadric, let $\mathbf{x} \in \Sigma$, and let Σ be a line through $\mathbf{x}$. Then the following are equivalent:*

(i) The line L is tangent to Σ in the analytic sense.

(ii) The line L lines on the affine hyperplane of $\mathbb{R}^n$ defined by the equation

$$\sum_{i,j} a_{i,j} x_i u_j \ + \ \sum_k b_k(v_k + x_k) \ + \ c \ = \ 0 \ .$$

The second property tells us that the set of points on tangent lines in the analytic sense is a hyperplane whose projective extension has homogeneous coordinates $\sum a_{i,j} x_j \ + \ b_j$ and $\sum b_k x_k \ + \ c$; i.e., its homogeneous coordinates are given by the following vector:

$$\left({}^{\mathbf{T}}\mathbf{x} \quad 1 \right) \cdot \begin{pmatrix} A & {}^{\mathbf{T}}\mathbf{b} \\ \mathbf{b} & c \end{pmatrix}$$

Note that these are just the homogeneous coordinates for the tangent hyperplane to Σ^* at $J(\mathbf{x})$ as defined in Section VII.2.

Proof of the theorem. The fist condition implies the second. For each i let $x_i(t)$ denote the ith coordinate of $\gamma(t)$, where γ satisfies the conditions in (i). Then by our assumptions we have

$$\sum_{i,j} a_{i,j} x_i(t) x_j(t) \ + \ 2\sum_k b_k x_k(t) \ + \ c \ = \ 0 \ .$$

Differentiation with respect to t implies that

$$2\sum_{i,j} x_i(t) x_j'(t) \ + \ 2\sum_k x_k'(t) \ = \ 0 \ .$$

Let $\gamma'(t_0) \ = \ (u_1, \ \cdots \ , u_n)$ and evaluate the expression above at $t = t_0$. This implies the equation

$$2\sum a_{i,j} x_i u_j \ + \ \sum_k b_k u_k \ = \ 0 \ .$$

To show that $L \ = \ \mathbf{x} + \mathbb{R} \cdot \mathbf{u}$ lies in the set defined by the equation in (ii), it suffices to show that $\mathbf{v} = \mathbf{x} + \mathbf{u}$ lies in this subset. If we divide the equation above by 2 and add

$$f(\mathbf{x}) \ = \ \sum_{i,j} a_{i,j} x_i x_j \ + \ 2\cdot \sum_k b_k x_k \ + \ c$$

to it, the resulting equation is the one displayed in (ii). To see that this equation is a hyperplane, observe that $\nabla f(\mathbf{x}) \ = \ \mathbf{0}$ implies that $\sum a_{i,j} x_j \ + \ b_i \neq 0$ for some i.□

Proof that the second condition implies the first. This is a consequence of the following corollary to the Implicit Function Theorem:

> *If f is a smooth function of n variables with $f(\mathbf{x}) = 0$ but $\nabla f(\mathbf{x}) \neq \mathbf{0}$, then there is a smooth curve γ such that $\gamma(t)$ lies in the zero seto of f for all t, and we also have $\gamma(t_0) = \mathbf{x}$, $\gamma'(t_0) = \mathbf{v}$.*

(See Marsden and Tromba, pp. 248–250, for more about this.)

Since $\nabla f(\mathbf{x}) \neq \mathbf{0}$ if Σ is nonsingular, the statement above is applicable.■

2. Matrices defining the same singular hyperquadric

By Theorem VII.6 we know that if Σ is a nonempty nonsingular hyperquadric in $\mathbb{FP}^n$ which is defined by the nonzero symmetric matrices A and B, then A and B are nonzero scalar multiples of each other. In fact, we know this is also the case if Σ has at least one nonsingular point. It is natural to ask whether the existence of such a point is needed to prove such a result on symmetric matrices defining the same quadric. We shall show that the answer depends upon the algebraic properties of the field $\mathbb{F}$. In particular, one can drop the assumption about a nonsingular point if $\mathbb{F}$ is the complex numbers, but one cannot drop the assumption if $\mathbb{F}$ is the real numbers or the finite field $\mathbb{Z}_p$, where p is an odd prime.

At certain points in this section we shall use results from Section VII.4, and Theorem E.7 also uses some basic facts about fields.

The first result of this section shows that Theorem VII.6 generalizes if $\mathbb{F}$ is the field $\mathbb{C}$ of complex numbers.

Theorem E.4. *Let $\mathbb{F}$ be a field which satisfies the Default Hypothesis, and assume that $\mathbb{F}$ is closed under taking square roots, so that for each $a \in \mathbb{F}$ there is some $b \in \mathbb{F}$ such that $b^2 = a$. Let Σ be a nonempty quadric in $\mathbb{FP}^n$, and let A and B be nonzero symmetric matrices such that $\Sigma_A = \Sigma = \Sigma_B$. Then B is a nonzero scalar multiple of A.*

By Theorem VII.6, the proof of this theorem reduces to showing the following result.

Theorem E.5. *Let $\mathbb{F}$ be a field which satisfies the Default Hypothesis, assume that $\mathbb{F}$ is closed under taking square roots, and let Σ be a nonempty quadric in $\mathbb{FP}^n$. Then Σ has a nonsingular point.*

Proof. By the same argument employed to classify quadrics over the complex numbers in Section VII.4, there is a projective collineation T of $\mathbb{FP}^n$ such that $T[\Sigma]$ is defined by a homogeneous quadratic equation of the form

$$x_1^2 + \cdots + x_r^2 = 0$$

where $2 \leq r \leq n + 1$ (the proof of Theorem VII.15 only used the existence of square roots and $1 + 1 \neq 0$). If we can prove the theorem for these special examples of hyperquadrics, then the general case will follow because T defines a 1–1 correspondence between the nonsingular points of Σ and the nonsingular points of $T[\Sigma]$.

Finding a nonsingular point for one of the special hyperquadrics is elementary; specifically, take the point with $x_1 = 1$, $x_2 = \sqrt{-1}$, and all other coordinates equal to zero.■

Although the field of complex numbers is closed under taking square roots, many fields — including the real numbers — do not have this property, and for such fields the conclusion of the preceding theorem is almost never valid. More precisely, we have the following result:

Theorem E.6. *Let $\mathbb{F}$ be a field satisfying the Default Condition at the beginning of this section, and assume further that at least one element in $\mathbb{F}$ is not a perfect square but $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_3$. Then for each $n \geq 2$ there is a nonempty hyperquadric $\Sigma \subset \mathbb{FP}^n$ and symmetric nonzero $(n+1) \times (n+1)$ matrices A and B such that $\Sigma_A = \Sigma = \Sigma_B$ (i.e., Σ is defined by both A and B),but A and B are not multiples of each other.*

The proof of this theorem depends upon the following purely algebraic result:

Theorem E.7. *Let $\mathbb{F}$ be a field such that $\mathbb{F}$ contains exactly one element which is not a perfect square. Then $\mathbb{F}$ is isomorphic to $\mathbb{Z}_3$.*

Two comments about this theorem and its proof are worth noting: (i) The proof of the theorem also goes through if $1 + 1 = 0$ in $\mathbb{F}$. (ii) If $\mathbb{F}$ is isomorphic to $\mathbb{Z}_3$, then -1 is not a perfect square, and it is the only element with this property.

Proof of Theorem E.7. Let $\mathbb{F}^\times$ be the multiplicative abelian group of nonzero elements of $\mathbb{F}$, let Γ be the quotient of $\mathbb{F}^\times$ by the subgroup of nonzero elements that are perfect squares in $\mathbb{F}$, and let $\theta : \mathbb{F}^\times \to \Gamma$ be the canonical projection homomorphism. By construction, if $\alpha \in \Gamma$ then $\alpha^2 = 1$. Let K be the kernel of θ. By our assumption on $\mathbb{F}$, we know that K is a proper subgroup of $\mathbb{F}^\times$. Let $a \in \mathbb{F}^\times$ be the unique element which does not lie in K.

We claim that the subgroup K is trivial; if this were not the case and b^2 is a nontrivial element of K, then a and ab^2 would be distinct elements of $\mathbb{F}^\times$ that do not lie in K. Combining this with the discussion in the previous paragraph, we see that every element $c \in \mathbb{F}^\times$ satisfies $c^2 = 1$.

Standard results on roots of polynomials over fields imply that the nontrivial quadratic polynomial $x^2 - 1$ over a field $\mathbb{F}$ has at most two roots in $\mathbb{F}$. Since the preceding paragraph shows that the square of every element of $\mathbb{F}^\times$ is equal to 1, it follows that $\mathbb{F}^\times$ contains at most two elements and hence $\mathbb{F}$ contains either two or three elements. It is an elementary exercise to show that every field of this type is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$ depending upon whether $\mathbb{F}$ has two or three elements. If the field has two elements, then every element is a perfect square, so the only remaining possibility is that $\mathbb{F}$ is isomorphic to $\mathbb{Z}_3$.∎

Proof of Theorem E.6. By Theorem E.7, one can find distinct elements $c, d \in \mathbb{F}$ such that neither is a perfect square in $\mathbb{F}$. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & -c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & -d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-1} \end{pmatrix}$$

where $\mathbf{0}_{n-1}$ is an $(n-1) \times (n-1)$ zero matrix. It is straightforward to show that both Σ_A and Σ_B are equal to the $(n-2)$-plane in $\mathbb{FP}^n$ whose homogeneous coordinates satisfy $x_1 = x_2 = 0$. However, the matrices A and B are not multiples of each other.∎

EXAMPLES. **1.** If $\mathbb{F}$ is the real or rational numbers and a is a negative integer, then a is not a perfect square in $\mathbb{F}$.

2. More generally, suppose that $r < n$, and consider sequences α of positive (real or rational) numbers $a_2, \cdots, a_{r+1}$ in $\mathbb{F}$. For each such sequence α let $D(\alpha)$ be the $(n+1) \times (n+1)$ diagonal matrix with diagonal entries

$$1, \; a_2, \; \cdots, \; a_{r+1}, \; 0, \; \cdots, \; 0 \;.$$

For all possible choices of α the hyperquadric $\Sigma_{D(\alpha)}$ is the same subset of $\mathbb{FP}^n$ — namely the $(n-r)$-plane defined by $x_{r+1} = \cdots = x_{n+1} = 0$ — but no two of the matrices $D(\alpha)$ can be scalar multiples of each other (since the entries in the first row and first column are always 1, it follows that the only possible choice of scalar is 1, and this is impossible if the two sequences of numbers are different).

3. Also, if $\mathbb{F} = \mathbb{Z}_p$ where p is a prime not equal to 2 or 3, then there are $\frac{1}{2}(p-1)$ elements of $\mathbb{F}$ which are not perfect squares in $\mathbb{Z}_p$. For example, if $p = 5$ then 2 and 3 are not perfect squares, while if $p = 7$ then 3, 5 and 6 are not perfect squares, and if $p = 11$ then $2, 6, 7, 8$ and 10 are not perfect squares. Similarly, if $p = 13$ then $2, 5, 6, 7, 8$ and 11 are not perfect squares, and if $p = 17$ then $3, 5, 6, 7, 10, 11, 13$ and 14 are not perfect squares. Further discussion of the general case can be found in the books by Davenport (pp. 62–68) and LeVeque (pp. 45–46) cited at the end of this Appendix.

Final remarks

We shall conclude this section with three very loosely related observations. The first provides still further examples of real symmetric matrices which define the same hyperquadric but are not multiples of each other. The second shows that the conclusion to Theorem E.6 is also valid if $\mathbb{F}$ is isomorphic to $\mathbb{Z}_3$; this yields a converse to Theorem E.4: *If a field $\mathbb{F}$ is not closed under taking square roots, then some nonempty hyperquadric in $\mathbb{FP}^n$ ($n \geq 2$) is defined by two symmetric matrices that are not scalar multiples of each other.* In contrast, the third observation shows that if a hyperquadric is definable by two such matrices, then it is a "degenerate hyperquadric" which is equal to a k-plane for some positive integer k.

1. If $\mathbb{F} = \mathbb{R}$, then additional examples as in Theorem 6 are given by the matrix pairs

$$A = \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & \mathbf{0} \\ 1 & d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-1} \end{pmatrix}$$

where $\mathbf{0}_{n-1}$ is as above and $d > 1$. This is true because $x_1^2 + 2x_1x_2 + dx_2^2 = 0$ if and only if $x_1 = x_2 = 0$.

2. For similar reasons, if $\mathbb{F} = \mathbb{Z}_3$ the matrices given by

$$A = \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-1} \end{pmatrix} \qquad B_\pm = \begin{pmatrix} 1 & \pm 1 & \mathbf{0} \\ \pm 1 & -1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-1} \end{pmatrix}$$

define the same hyperquadric — namely, the $(n-2)$-plane in $\mathbb{Z}_3\mathbb{P}^n$ given by the linear equations $x_1 = x_2 = 0$ — even though none of these three matrices is a scalar multiple of another. In particular, this means that *Theorem 6 extends to the case where $\mathbb{F}$ is isomorphic to $\mathbb{Z}_3$*, and thus it is true for all fields $\mathbb{F}$ such that $1 + 1 \neq 0$ in $\mathbb{F}$ but $\mathbb{F}$ is not closed under taking square roots.∎

3. By Theorem VII.6, if Σ is a quadric defined by two matrices A and B which are not multiples of each other, then then every point of Σ is singular. The next to last result of this section implies that such quadrics are in fact (projective) geometrical subspaces of $\mathbb{FP}^n$.

Theorem E.8. *Suppose that the symmetric matrix A defines a hyperquadric $\Sigma \subset \mathbb{FP}^n$ such that every point of Σ is singular. Let $W \subset \mathbb{F}^{n+1,1}$ be the kernel of (left multiplication by) A. Then Σ is equal to $\mathcal{S}_1(W)$, and hence is a $(\dim W - 1)$-plane in $\mathbb{FP}^n$.*

Proof. Given $X \in \mathbb{FP}^n$, let ξ be a set of homogeneous coordinates for X. If $X \in \mathcal{S}_1(W)$, then $A\xi = \mathbf{0}$ and hence $^\mathsf{T}\xi A\xi = {}^\mathsf{T}\xi \cdot \mathbf{0} = 0$, so that $X \in \Sigma$. Conversely, if $X \in \Sigma$, then since every point of Σ is singular we must have $^\mathsf{T}\xi A = {}^\mathsf{T}\mathbf{0}$, and since A is symmetric this is equivalent to $A\xi = \mathbf{0}$, so that $\xi \in W$ and $X \in \mathcal{S}_1(W)$.∎

If we combine his result with Theorem VII.6, we obtain the following conclusion, which yields a fairly definitive statement on the relationship between hyperquadrics and their defining matrices:

Theorem E.9. *Let $\mathbb{F}$ be a field in which $1+1 \neq 0$, let $n \geq 2$, and let $\Sigma \subset \mathbb{F}\mathbb{P}^n$ be a hyperquadric which is* nondegenerate *in the sense that it is* **NOT** *a (projective) geometrical subspace of $\mathbb{F}\mathbb{P}^n$. Suppose that A and B are symmetric $(n+1) \times (n+1)$ matrices over $\mathbb{F}$ such that $\Sigma_A = \Sigma = \Sigma_B$. Then B and A are (nonzero) scalar multiples of each other.$\blacksquare$*

In other words, *every nondegenerate hyperquadric is defined by a matrix which is unique up to multiplication by a nonzero scalar.*

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Bibliography

This is certainly not meant to be comprehensive, but it does list numerous books and papers that influenced the writing of these notes as well as references to some additional topics that were mentioned but not discussed in much detail. Throughout the notes there are also references to World Wide Web sites for further information on various points; in all cases, the mathematical content of online sites has been checked for reliability and meets the standards of quality expected for regular printed publications.

1. BOOKS COVERING BACKGROUND MATERIAL

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R. Courant and H. Robbins. *What is Mathematics? An Elementary Approach to Ideas and Methods* (2nd Ed.). Oxford Univ. Press, New York, 1996.[3]

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[1]This has a classic book on problem solving for half a century. GEORGE PÓLYA (1887–1985) worked in probability, analysis, number theory, geometry, combinatorics and mathematical physics.

[2]This book is a fascinating piece of literature, but its historical scholarship is extremely inaccurate in many places.

[3]RICHARD COURANT (1888–1972) is known for his work on differential equations, for several outstanding books that have remained timely over many decades, and his establishment of a prestigious mathematical institute which now carries his name. HAROLD ROBBINS (1915–2001) made several major contributions to mathematical statistics and also did research in several other areas.

V. J. Katz. *A History of Mathematics: An Introduction* (2nd Ed.) Addison-Wesley, Boston, MA, 1998.

M. Kline.[4] *Mathematical Thought from Ancient to Modern Times.* Oxford Univ. Press, New York, 1972.

M. Kline. *Mathematics in Western Culture.* Oxford Univ. Press, New York, 1964.[5]

J. R. Newman (*ed.*). *The World of Mathematics, Volumes 1 – 4.* Dover, New York, 2003.

3. Similar material to the notes at comparable levels

(The referernces at the end of Appendix E also fit into this category.)

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M. K. Bennett. *Affine and Projective Geometry.* Wiley-Interscience, New York, 1995.

G. [Garrett] Birkhoff and S. MacLane. *A Survey of Modern Algebra.* (Reprint of the Third 1968 Edition.) Chelsea, New York, 1988.[6]

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[4]Morris Kline (1980–1972) is known for his books on the history and teaching of mathematics and his provocative views. A very brief description of his work and views appears in the online document `http://en.wikipedia.org/wiki/Morris_Kline`. Kline's books contain a great deal of information and many extremely well-written passages. However, his views on several topics are highly controversial, and frequently he makes sweeping, dramatic assertions which ignore key facts or the underlying complexities of certain issues. One must be aware of these when reading his strongly negative comments about 20th century mathematics or the mathematical legacies of certain ancient or Non-Western cultures.

[5]Chapters X and XI are particularly relevant to these notes.

[6]This book has been a classic reference for undergraduate abstract algebra since the publication of the First Edition in 1941, and it was the first book of its kind in English. Saunders MacLane (1909–2005) did noteworthy work in several areas of mathematics, and he is best known for his extremely influential work on **category theory**, which is the abstract study of functions or morphisms from one mathematically structured object to another.

[7]Irving Kaplansky (1917–2006) did important work in many different areas of algebra, including some topics with close ties to other major parts of mathematics.

4. Related material at comparable levels

(Most books under heading **3** also deserve a secondary classification here.)

J. N. Cederberg. *A Course in Modern Geometries.* (Corrected 2nd Printing of the 2nd Edition). Springer-Verlag, New York–(*etc.*), 2004.

H. S. M. Coxeter.[8] *The Real Projective Plane* (2nd Ed.). Cambridge Univ. Press, New York, 1955.

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(The referernces at the end of Appendix C also fit into this category.)

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E. Artin. *Geometric Algebra.* Wiley (Interscience), New York, 1957.

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G. [Garrett] Birkhoff. *Lattice theory.* Corrected reprint of the 1967 Third Edition. American Mathematical Society Colloquium Publications, Vol. 25. American Mathematical Society, Providence, RI, 1979.

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[8]HAROLD SMITH MACDONALD (Donald) COXETER (1907–2003) is mainly known for his research which establishes fundamental links between classical and modern themes in geometry, his influence which is reflected by numerous classic texts on various aspect of geometry, and his interactions with famous nonmathematicians whose work had substantial mathematical content, most notably the artist MAURITS C. ESCHER (1898–1972) and the architect/visionary R. BUCKMINSTER FULLER (1895–1983).

[9]Since the appearance of the First Edition in 1953, this book has been a classic, frequently cited reference for the theory of functions of real variables at the undergraduate level.

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J. C. Baez. *The Octonions.* Bull. Amer. Math. Soc. (2) **39** (2002), 145–205.

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R. H. Bruck and H. J. Ryser. *The nonexistence of certain finite projective planes.* Canad. J. Math. **1** (1949), 88–93.

A. Cronheim. *A proof of Hessenberg's Theorem.* Proc. Amer. Math. Soc. **4** (1953), 219–221.

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[10]GIAN-CARLO ROTA (1932–1999) is mainly known for his highly inflential work in combinatorics (the study of finite mathematical objects), and he also made noteworthy contributions to functional analysis and ergodic theory.

[11]WILLIAM VALLANCE DOUGLAS HODGE (1903–1975) discovered fundamentally important relationships involving algebraic geometry, differential geometry and partial differential equations, and his *Hodge Conjecture* on such relationships is regarded as one of the most important open questions in mathematics.

[12]NATHAN JACOBSON (1910–1999) made numerous fundamental contributions to the theory of abstract algebraic systems.

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8. Links to the World Wide Web

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http://en.wikipedia.org/wiki/Projective_geometry

http://www.nct.anth.org.uk/basics.htm

http://www.math.poly.edu/∼alvarez/teaching/projective-geometry/Exams/problems.html

http://www.math-mit.edu/∼kedlaya/geometryunbound/gu-060118.pdf

http://robotics.stanford.edu/∼birch/projective

http://www2.maths.ox.ac.uk/∼hitchin/hitchinnotes/hitchinnotes.html

http://www.stolaf.edu/people/cederj/Courses.dir/bib-356/index.html#beginning

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[13]Shiing-Shen Chern (1911–2004) was one of the leading figures in 20th century differential geometry, with research including several fundamental advances and wide ranging influence upon the development of the subject.