CHAPTER II

AFFINE GEOMETRY

In the previous chapter we indicated how several basic ideas from geometry have natural interpretations in terms of vector spaces and linear algebra. This chapter continues the process of formulating basic geometric concepts in such terms. It begins with standard material, moves on to consider topics not covered in most courses on classical deductive geometry or analytic geometry, and it concludes by giving an abstract formulation of the concept of geometrical incidence and closely related issues.

1. Synthetic affine geometry

In this section we shall consider some properties of Euclidean spaces which only depend upon the axioms of incidence and parallelism

Definition. A *three-dimensional incidence space* is a triple $(S, \mathcal{L}, \mathcal{P})$ consisting of a nonempty set S (whose elements are called *points*) and two nonempty disjoint families of proper subsets of S denoted by $\mathcal{L}$ (*lines*) and $\mathcal{P}$ (*planes*) respectively, which satisfy the following conditions:

(I-1) Every line (element of $\mathcal{L}$) contains at least two points, and every plane (element of $\mathcal{P}$) contains at least three points.

(I-2) If **x** and **y** are distinct points of S, then there is a unique line L such that **x**, **y** $\in L$.

Notation. The line given by **(I-2)** is called the *line joining* **x** *and* **y** and denoted by **xy**.

(I-3) If **x**, **y** and **z** are distinct points of S and **z** $\notin$ **xy**, then there is a unique plane P such that **x**, **y**, **z** $\in P$.

(I-4) If a plane P contains the distinct points **x** and **y**, then it also contains the line **xy**.

(I-5) If P and Q are planes with a nonempty intersection, then $P \cap Q$ contains at least two points.

Of course, the standard example in $\mathbb{R}^3$ with lines and planes defined by the formulas in Chapter I (we shall verify a more general statement later in this chapter). A list of other simple examples appears in Prenowitz and Jordan, *Basic Concepts of Geometry*, pp. 141–146.

A few theorems in Euclidean geometry are true for every three-dimensional incidence space. The proofs of these results provide an easy introduction to the synthetic techniques of these notes. In the first six results, the triple $(S, \mathcal{L}, \mathcal{P})$ denotes a fixed three-dimensional incidence space.

Definition. A set B of points in S is *collinear* if there is some line L in S such that $B \subset L$, and it is *noncollinear* otherwise. A set A of points in S is *coplanar* if there is some plane P in S such that $A \subset P$, and it is *noncoplanar* otherwise. — Frequently we say that the points **x**, **y**, $\cdots$ (*etc.*) are collinear or coplanar if the set with these elements is collinear or coplanar respectively.

THEOREM II.1. *Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be distinct points of S such that $\mathbf{z} \notin \mathbf{xy}$. Then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a noncollinear set.*

Proof.   Suppose that L is a line containing the given three points. Since $\mathbf{x}$ and $\mathbf{y}$ are distinct, by **(I-2)** we know that $L = \mathbf{xy}$. By our assumption on L it follows that $\mathbf{z} \in L$; however, this contradicts the hypothesis $\mathbf{z} \notin \mathbf{xy}$. Therefore there is no line containing $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.∎

THEOREM II.2. *There is a subset of four noncoplanar points in S.*

Proof.   Let P be a plane in S. We claim that P contains three noncollinear points. By **(I-1)** we know that P contains three distinct points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}_0$. If these three points are noncollinear, let $\mathbf{c} = \mathbf{c}_0$. If they are collinear, then the line L containing them is a subset of P by **(I-4)**, and since $\mathcal{L}$ and $\mathcal{P}$ are disjoint it follows that L must be a proper subset of P; therefore there is some point $\mathbf{c} \in P$ such that $\mathbf{c} \notin L$, and by the preceding result the set $\{\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}\}$ is noncollinear. Thus in any case we know that P contains three noncollinear points.

Since P is a proper subset of S, there is a point $\mathbf{d} \in S$ such that $\mathbf{d} \notin P$. We claim that $\{\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}\ \mathbf{d}\}$ is noncoplanar. For if Q were a plane containing all four points, then $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in P$ would imply $P = Q$, which contradicts our basic stipulation that $\mathbf{d} \notin P$.∎

THEOREM II.3. *The intersection of two distinct lines in S is either a point or the empty set.*

Proof.   Suppose that $\mathbf{x} \neq \mathbf{y}$ but both belong to $L \cap M$ for some lines L and M. By property **(I-2)** we must have $L = M$. Thus the intersection of distinct lines must consist of at most one point.∎

THEOREM II.4. *The intersection of two distinct planes in S is either a line or the empty set.*

Proof.   Suppose that P and Q are distinct planes in S with a nonempty intersection, and let $\mathbf{x} \in P \cap Q$. By **(I-5)** there is a second point $\mathbf{y} \in P \cap Q$. If L is the line $\mathbf{xy}$, then $L \subset P$ and $L \subset Q$ by two applications of **(I-4)**; hence we have $L \subset P \cap Q$. If there is a point $\mathbf{z} \in P \cap Q$ with $\mathbf{z} \notin L$, then the points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are noncollinear but contained in both of the planes P and Q. By **(I-3)** we must have $P = Q$. On the other hand, by assumption we know $P \neq Q$, so we have reached a contradiction. The source of this contradiction is our hypothesis that $P \cap Q$ strictly contains L, and therefore it follows that $P \cap Q = L$.∎

THEOREM II.5. *Let L and M be distinct lines, and assume that $L \cap M \neq \varnothing$. Then there is a unique plane P such that $L \subset P$ and $M \subset P$.*

In less formal terms, given two intersecting lines there is a unique plane containing them.

Proof.   Let $\mathbf{x} \in L \cap M$ be the unique common point (it is unique by Theorem 3). By **(I-2)** there exist points $\mathbf{y} \in L$ and $\mathbf{z} \in M$, each of which is distinct from $\mathbf{x}$. The points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are noncollinear because $L = \mathbf{xy}$ and $\mathbf{z} \in M - \{\mathbf{x}\} = M - L$. By **(I-3)** there is a unique plane P such that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in P$, and by **(I-4)** we know that $L \subset P$ and $M \subset P$. This proves the existence of a plane containing both L and M. To see this plane is unique, observe that every plane Q containing both lines must contain $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$. By **(I-3)** there is a unique such plane, and therefore we must have $Q = P$.∎

THEOREM II.6. *Given a line L and a point $\mathbf{z}$ not on L, there is a unique plane P such that $L \subset P$ and $\mathbf{z} \in P$.*

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be distinct points of L, so that $L = \mathbf{xy}$. We then know that the set $\{\,\mathbf{x},\ \mathbf{y},\ \mathbf{z}\,\}$ is noncollinear, and hence there is a unique plane P containing them. By **(I-4)** we know that $L \subset P$ and $\mathbf{z} \in P$. Conversely, if Q is an arbitrary plane containing L and $\mathbf{z}$, then Q contains the three noncollinear points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, and hence by **(I-3)** we know that $Q = P$.∎

Notation. We shall denote the unique plane in the preceding result by $L\mathbf{z}$.

Of course, all the theorems above are quite simple; their conclusions are probably very clear intuitively, and their proofs are fairly straightforward arguments. One must add Hilbert's Axioms of Order or the Euclidean Parallelism Axiom to obtain something more substantial. Since our aim is to introduce the parallel postulate at an early point, we might as well do so now (a thorough treatment of geometric theorems derivable from the Axioms of Incidence and Order appears in Chapter 12 of Coxeter, *Introduction to Geometry*; we shall discuss the Axioms of Order in Section VI.6 of these notes).

Definition. Two lines in a three-dimensional incidence space S are *parallel* if they are disjoint and coplanar (note particularly the second condition). If L and L' are parallel, we shall write $L||L'$ and denote their common plane by LL'. — Note that if $L||M$ then $M||L$ because the conditions in the definition of parallelism are symmetric in the two lines.

Affine three-dimensional incidence spaces

Definition. A three-dimensional incidence space $(S, \mathcal{L}, \mathcal{P})$ is an *affine three-space* if the following holds:

(EPP) For each line L in S and each point $\mathbf{x} \not\in L$ there is a unique line $L' \subset L\mathbf{x}$ such that $\mathbf{x} \in L$ and $L \cap L' = \varnothing$ (in other words, there is a unique line L' which contains $\mathbf{x}$ and is parallel to L).

This property is often called *the Euclidean Parallelism Property*, the *Euclidean Parallel Postulate* or *Playfair's Postulate*.[1] Actually, Euclid's *Elements* employed a logically equivalent statement which also requires the concepts of betweenness and congruence, and the advantages of using the purely incidence-theoretical statement **(EPP)** were noted explicitly by Proclus Diadochus (412–485).[2]

A discussion of the origin of the term "affine" appears in Section II.5 of the following online site:

$$\texttt{http://math.ucr.edu/{\sim}res/math133/geometrynotes2b.pdf}$$

Many nontrivial results in Euclidean geometry can be proved for arbitrary affine three-spaces. We shall limit ourselves to two examples here and leave others as exercises. In Theorems 7 and 8 below, the triple $(S, \mathcal{L}, \mathcal{P})$ will denote an arbitrary affine three-dimensional incidence space.

THEOREM II.7. *Two lines which are parallel to a third line are parallel.*

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[1]JOHN PLAYFAIR (1748–1819) was a Scottish scientist who is also known for an influential book on the philosophy of science; in his geometrical writings, he acknowledged that others had previously considered **EPP** as an axiom for geometry.

[2]The writings of Proclus provide valuable information on important, but apparently lost, works of earlier Greek mathematicians.

Proof.   There are two cases, depending on whether or not all three lines lie in a single plane; to see that the three lines need not be coplanar in ordinary 3-dimensional coordinate geometry, consider the three lines in $\mathbb{R}^3$ given by the z-axis and the lines joining $(1,0,0)$ and $(0,1,0)$ to $(1,0,1)$ and $(0,1,1)$ respectively.

THE COPLANAR CASE.   Suppose that we have three distinct lines L, M, N in a plane P such that $L||N$ and $M||N$; we want to show that $L||N$.

If L is not parallel to N, then there is some point $\mathbf{x} \in L \cap N$, and it follows that L and N are distinct lines through $\mathbf{x}$, each of which is parallel to M. However, this contradicts the Euclidean Parallel Postulate. Therefore the lines L and N cannot have any points in common.

THE NONCOPLANAR CASE.   Let α be the plane containing L and M, and let β be the plane containing M and N. By the basic assumption in this case we have $\alpha \neq \beta$. We need to show that $L \cap N = \varnothing$ but L and N are coplanar.

The lines L and N are disjoint.   Assume that the L and N have a common point that we shall call $\mathbf{x}$. Let γ be the plane determined by $\mathbf{x}$ and N (since $L||M$ and $\mathbf{x} \in L$, clearly $\mathbf{x} \notin M$). Since $\mathbf{x} \in L \subset \alpha$ and $M \subset \alpha$, Theorem 6 implies that $\alpha = \gamma$. A similar argument shows that $\beta = \gamma$ and hence $\alpha = \beta$; the latter contradicts our basic stipulation that $\alpha \neq \beta$, and therefore it follows that L and N cannot have any points in common.
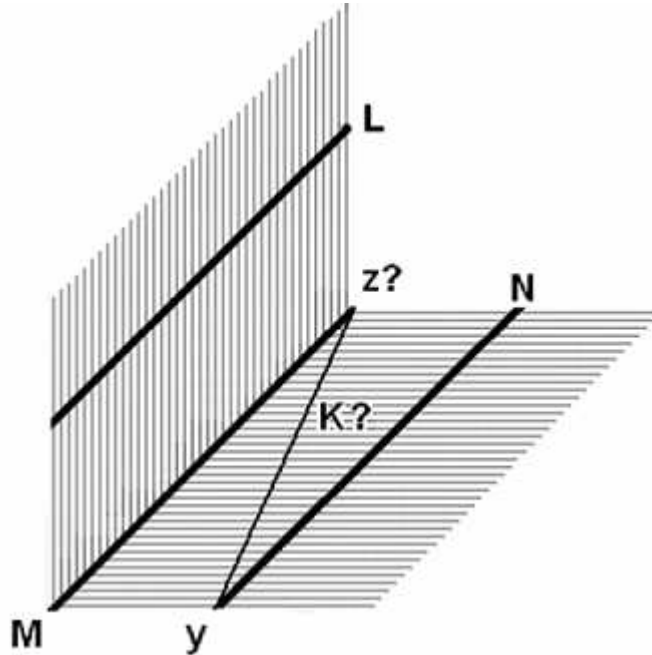

Figure II.1

The lines L and N are coplanar.   Let $\mathbf{y} \in N$, and consider the plane $L\mathbf{y}$. Now L cannot be contained in β because $\beta \neq \alpha = LM$ and $M \subset \beta$. By construction the planes $L\mathbf{y}$ and β have the point $\mathbf{y}$ in common, and therefore we know that $L\mathbf{y}$ meets β in some line K. Since L and K are coplanar, it will suffice to show that $N = K$. Since N and K both contain $\mathbf{y}$ and all three lines M, N and K are contained in β, it will suffice to show that $K||M$.

Suppose the lines are not parallel, and let $\mathbf{z} \in K \cap M$. Since $L||M$ it follows that $\mathbf{z} \notin L$. Furthermore, $L \cup K \subset L\mathbf{y}$ implies that $\mathbf{z} \in L\mathbf{y}$, and hence $\mathbf{y} = L\mathbf{z}$. Since $\mathbf{z} \in M$ and L and M are coplanar, it follows that $M \subset L\mathbf{z}$. Thus M is contained in $L\mathbf{y} \cap \beta$, and since the latter is the

line K, this shows that $M = K$. On the other hand, by construction we know that $M \cap N = \varnothing$ and $K \cap N \neq \varnothing$, so that M and K are obviously distinct. This contradiction implies that $K || M$ must hold.∎

The next result is an analog of the Parallel Postulate for parallel planes.

THEOREM II.8. *If P is a plane and $\mathbf{x} \notin P$, then there is a unique plane Q such that $\mathbf{x} \in Q$ and $P \cap Q = \varnothing$.*

Proof. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in P$ be the noncollinear points, and consider the lines A', B' through $\mathbf{x}$ which are parallel to $A = \mathbf{bc}$ and $B = \mathbf{ac}$. Let Q be the plane determined by A' and B', so that $\mathbf{x} \in Q$ by hypothesis. We claim that $P \cap Q = \varnothing$.

Assume the contrary; since $\mathbf{x} \in QA$ and $\mathbf{x} \notin P$, the intersection $P \cap Q$ is a line we shall call L.
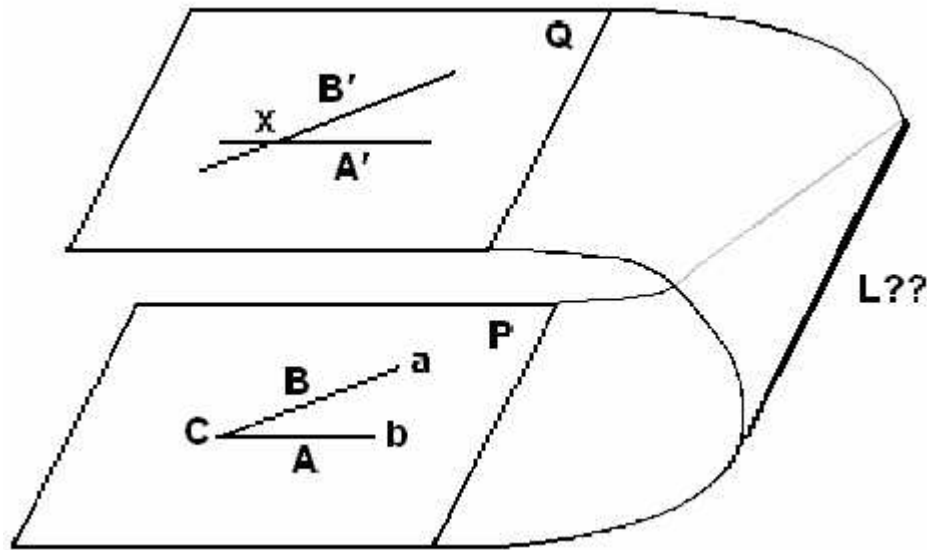

Figure II.2

STEP 1. We shall show that $L \neq A$, B. The proof that $L \neq A$ and $L \neq B$ are similar, so we shall only show $L \neq A$ and leave the other part as an exercise. — If $L = A$, then $L \subset Q$. Since A' is the unique line in Q which is parallel to A, there must be a point $\mathbf{u} \in B' \cap A$. Consider the plane $B'\mathbf{c}$. Since $\mathbf{c} \in A$, it follows that $A \subset B'\mathbf{c}$. Hence $B'\mathbf{c}$ is a plane containing A and B. The only plane which satisfies these conditions is P, and hence $B' \subset P$. But $\mathbf{x} \in B'$ and $\mathbf{x} \notin P$, so we have a contradiction. Therefore we must have $L \neq A$.

STEP 2. We claim that either $A' \cap L$ and $A \cap L$ are both nonempty or else $B' \cap L$ and $B \cap L$ are both nonempty. — We shall only show that if either $A' \cap L$ is $A \cap L$ is empty then both $B' \cap L$ and $B \cap L$ are both nonempty, since the other case follows by reversing the roles of A and B. Since L and A both lie in the plane P, the condition $A \cap L = \varnothing$ implies $A || L$. Since $A || A'$, by Theorem 7 and Theorem 1 we know that $A' || L$. Since $B \neq A$ is a line through the point $\mathbf{c} \in A$, either $B = L$ or $B \cap L \neq \varnothing$ holds by the Parallel Postulate (in fact, $B \neq L$ by Step

1). Likewise, B and B' are lines through $\mathbf{x}$ in the plane Q and $L \subset Q$, so that the $A'||L$ and the Parallel Postulate imply $B' \cap L \neq \varnothing$.

STEP 3.   There are two cases, depending upon whether $A' \cap L$ and $A \cap L$ are both nonempty or $B' \cap L$ and $B \cap L$ are both nonempty. Only the latter will be considered, since the former follows by a similar argument. Let $\mathbf{y} \in B \cap L$ and $\mathbf{z} \in B' \cap L$; since $B \cap B' = \varnothing$, if follows that $\mathbf{y} \neq \mathbf{z}$ and hence $L = \mathbf{yz}$. Let β be the plane BB'. Then $L \subset \beta$ since $\mathbf{z}$, $\mathbf{y} \in \beta$. Since $L \neq B$, the plane β is the one determined by L and B. But L, $B \subset P$ by assumption, and hence $\beta = P$. In other words, B' is contained in P. But $\mathbf{x} \in B'$ and $\mathbf{x} \notin P$, a contradiction which shows that the line L cannot exist.■

Following standard terminology, we shall say that the plane Q is *parallel* to P or that it is the plane parallel to P which passes through $\mathbf{x}$.

Corresponding definitions for incidence planes and affine planes exist, and analogs of Theorems 1, 2, 3 and 7 hold for these objects. However, incidence planes have far fewer interesting properties than their three-dimensional counterparts, and affine planes are best studied using the methods of projective geometry that are developed in later sections of these notes.

EXERCISES

Definition.   A line and a plane in a three-dimensional incidence space are *parallel* if they are disjoint.

Exercises **1**–**4** *are to be proved for arbitrary 3-dimensional incidence spaces.*

1.   Suppose that each of two intersecting lines is parallel to a third line. Prove that the three lines are coplanar.

2.   Suppose that the lines L and L' are coplanar, and there is a line M not in this plane such that $L||M$ and $L'||M$. Prove that $L||L'$.

3.   Let P and Q be planes, and assume that each line in P is parallel to a line in Q. Prove that P is parallel to Q.

4.   Suppose that the line L is contained in the plane P, and suppose that $L||L'$. Prove that either L' is parallel to P or else $L \subset P$.

In exercises **5**–**6**, *assume the incidence space is affine.*

5.   Let P and Q be parallel planes, and let L be any line which contains a point of Q and is parallel to a line in P. Prove that L is contained in Q. [*Hint:*   Let M be the line in P, and let $\mathbf{x} \in L \cap Q$. Prove that $L = M\mathbf{x} \cap Q$.]

6.   Two lines are said to be **skew lines** if they are not coplanar. Suppose that L and M are skew lines. Prove that there is a unique plane P such that $L \subset P$ and P is parallel to M. [*Hint:* Let $\mathbf{x} \in L$, let M' be a line parallel to M which contains $\mathbf{x}$, and consider the plane LM'.]

2. Affine subspaces of vector spaces

Let $\mathbb{F}$ be a field, and let V be a vector space over $\mathbb{F}$ (in fact, everything in this section goes through if we take $\mathbb{F}$ to be a **skew-field** as described in Section A.1 in Appendix A). Motivated by Section I.3, we define *lines* and *planes* in V to be translates of 1- and 2-dimensional vector subspaces of V. Denote these families of lines and planes by $\mathcal{L}_V$ and $\mathcal{P}_V$ respectively. If $\dim V \geq 3$ we shall prove that $(V, \mathcal{L}_V, \mathcal{P}_V)$ satisfies all the conditions in the definition of an affine incidence 3-space except perhaps for the incidence axiom $\mathbf{I} - \mathbf{5}$, and we shall show that the latter also holds if $\dim V = 3$.

THEOREM II.9. *If V, etc. are as above and $\dim V \geq 3$, then $\mathcal{L}_V$ and $\mathcal{P}_V$ are nonempty disjoint families of proper subsets of V.*

Proof. Since $\dim V \geq 3$ there are 1- and 2-dimensional vector subspaces of V, and therefore the families $\mathcal{L}_V$ and $\mathcal{P}_V$ are both nonempty. If we have $Y \in \mathcal{L}_V \cap \mathcal{P}_V$, then we may write

$$Y \;=\; \mathbf{x} \;+\; W_1 \;=\; \mathbf{y} \;+\; W_2$$

where $\dim W_i = i$. By Theorem I.4 we know that $\mathbf{y} \in \mathbf{x}+W_1$ implies the identity $\mathbf{x}+W_1 = \mathbf{y}+W_1$, and therefore Theorem I.3 implies

$$W_2 \;=\; -\mathbf{y} \;+\; (\mathbf{y} \;+\; W_1) \;=\; -\mathbf{y} \;+\; (\mathbf{y} \;+\; W_2) \;=\; W_2 \;.$$

Since $\dim W_1 \neq \dim W_2$ this is impossible. Therefore the families $\mathcal{L}_V$ and $\mathcal{P}_V$ must be disjoint. To see that an element of either family is a proper subset of V, suppose to the contrary that $\mathbf{x} + W = V$, where $\dim W = 1$ or 2. Since $\dim W < 3 \leq \dim V$, it follows that W is a proper subset of V; let $\mathbf{v} \in V$ be such that $\mathbf{v} \notin W$. By our hypothesis, we must have $\mathbf{x} + \mathbf{v} \in \mathbf{x} + W$, and thus we also have

$$\mathbf{v} \;=\; -\mathbf{x} \;+\; (\mathbf{x} \;+\; \mathbf{v}) \;\in\; -\mathbf{x} \;+\; (\mathbf{x} \;+\; W) \;=\; W$$

which contradicts our fundamental condition on $\mathbf{x}$. The contradiction arises from our assumption that $\mathbf{x}+W = V$, and therefore this must be false; therefore the sets in $\mathcal{L}_V$ and $\mathcal{P}_V$ are all proper subsets of V.∎

THEOREM II.10. *Every line in V contains at least two points, and every plane contains at least three points.*

Proof. Let $\mathbf{x}+W$ be a line or plane in V, and let $\{\, \mathbf{w}_1 \,\}$ or $\{\, \mathbf{w}_1, \mathbf{w}_2 \,\}$ be a basis for W depending upon whether $\dim W$ equals 1 or 2. Take the subsets $\{\, \mathbf{v}, \mathbf{v} + \mathbf{w}_1, \,\}$ or $\{\, \mathbf{v}, \mathbf{v} + \mathbf{w}_1, \mathbf{v} + \mathbf{w}_2 \,\}$ in these respective cases.∎

THEOREM II.11. *Given two distinct points in V, there is a unique line containing them.*

Proof. Let $\mathbf{x} \neq \mathbf{y}$ be distinct points in V, and let L_0 be the 1-dimensional vector subspace spanned by the nonzero vector $\mathbf{y} - \mathbf{x}$. Then $L = \mathbf{x} + L_0$ is a line containing $\mathbf{x}$ and $\mathbf{y}$. Suppose now that M is an arbitrary line containing $\mathbf{x}$ and $\mathbf{y}$. Write $M = \mathbf{z}+W$ where $\dim W = 1$. Then Theorem I.4 and $\mathbf{x} \in M = \mathbf{z} + W$ imply that $M = \mathbf{x} + W$. Furthermore, $\mathbf{y} \in M = \mathbf{x} + W$ then implies that $\mathbf{y} - \mathbf{x} \in W$, and since the latter vector spans L_0 it follows that $L_0 \subset W$. However, $\dim L_0 = \dim W$, and therefore $L_0 = W$ (see Theorem A.8). Thus the line $M = \mathbf{z} + W$ must be equal to $\mathbf{x} + L_0 = L$.∎

THEOREM II.12. *Given three points in V that are not collinear, there is a unique plane containing them.*

Proof. Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be the noncollinear points. If $\mathbf{y} - \mathbf{x}$ and $\mathbf{z} - \mathbf{x}$ were linearly dependent, then there would be a 1-dimensional vector subspace W containing them and hence the original three points would all lie on the line $\mathbf{x} + W$. Therefore we know that $\mathbf{y} - \mathbf{x}$ and $\mathbf{z} - \mathbf{x}$ are **linearly independent**, and thus the vector subspace W they span is 2-dimensional. If $P = \mathbf{x} + W$, then it follows immediately that P is a plane containing $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$. To prove uniqueness, suppose that $\mathbf{v} + U$ is an arbitrary plane containing all three points. As before, we must have $\mathbf{v} + U = \mathbf{x} + U$ since $bfx \in \mathbf{v} + U$, and since we also have $\mathbf{y}$, $\mathbf{z} \in \mathbf{v} + U = \mathbf{x} + U$ it also follows as in earlier arguments that $\mathbf{y} - \mathbf{x}$ and $\mathbf{z} - \mathbf{x}$ lie in U. Once again, since the two vectors in question span the subspace W, it follows that $W \subset U$, and since the dimensions are equal it follows that $W = U$. Thus we have $\mathbf{v} + U = \mathbf{x} + W$, and hence there is only one plane containing the original three points.∎

THEOREM II.13. *If P is a plane in V and $\mathbf{x}$, $\mathbf{y} \in P$, then the unique line containing $\mathbf{x}$ and $\mathbf{y}$ is a subset of P.*

Proof. As before we may write $P = \mathbf{x} + W$, where W is a 2-dimensional subspace; we also know that the unique line joining $\mathbf{x}$ and $\mathbf{y}$ has the form $L = \mathbf{x} + L_0$, where L_0 is spanned by $\mathbf{y} - \mathbf{x}$. The condition $\mathbf{y} \in P$ implies that $\mathbf{y} - \mathbf{x} \in W$, and since W is a vector subspace it follows that $L_0 \subset W$. But this immediately implies that $L = \mathbf{x} + L_0 \subset \mathbf{x} + W = P$, which is what we wanted to prove.∎

THEOREM II.14. (Euclidean Parallelism Property) *Let L be a line in V, and let $\mathbf{y} \notin L$. Then there is a unique line M such that (i) $\mathbf{y} \in M$, (ii) $L \cap M = \varnothing$, (iii) L and M are coplanar.*

Proof. Write $L = \mathbf{x} + L_0$ where $\dim L_0 = 1$, and consider the line $M = \mathbf{y} + L_0$. Then M clearly satisfies the first condition. To see it satisfies the second condition, suppose to the contrary that there is some common point $\mathbf{z} \in L \cap M$. Then the identities

$$\mathbf{z} \in L = \mathbf{x} + L_0 \qquad \mathbf{z} \in M = \mathbf{y} + L_0$$

imply that $L = \mathbf{x} + L_0 = \mathbf{z} + L_0 = \mathbf{y} + L_0 = M$, which contradicts the basic conditions that $\mathbf{y} \in M$ but $\mathbf{y} \notin L$. Therefore $L \cap M = \varnothing$. Finally, to see that M also satisfies the third condition, let W be the subspace spanned by L_0 and $\mathbf{y} - \mathbf{x}$; the latter vector does not belong to L_0 because $\mathbf{y} \notin L$, and therefore W must be a 2-dimensional vector subspace of V. If we now take $P = \mathbf{x} + W$, it follows immediately that $L \subset P$ and also

$$M = \mathbf{y} + L_0 \, \mathbf{x} + (\mathbf{y} - \mathbf{x}) + L_0 \subset \mathbf{x} + W = P$$

so that L and M are coplanar. Therefore M satisfies all of the three conditions in the theorem.

To complete the proof, we need to show that *there is only one line which satisfies all three conditions in the theorem.* — In any case, we know there is only one plane which contains L and $\mathbf{y}$, and hence it must be the plane $P = \mathbf{x} + W$ from the preceding paragraph. Thus if N is a line satisfying all three conditions in the theorem, it must be contained in P. Suppose then that $N = \mathbf{y} + L_1$ is an arbitrary line in P with the required properties. Since $\mathbf{y} + L_1 \in \mathbf{x} + W = \mathbf{y} + W$, it follows that $L_1 \subset W$. Therefore, if $\mathbf{0} \neq \mathbf{u} \in L_1$ we can write $\mathbf{u} = s(\mathbf{y} - \mathbf{x}) + t\mathbf{v}$, where $\mathbf{v}$ is a

nonzero vector in L_0 and s, $t \in \mathbb{F}$. The assumption that $L \cap N = \varnothing$ implies there are no scalars p and q such that

$$\mathbf{x} \ + \ p\mathbf{v} \ = \ \mathbf{y} \ + \ q\mathbf{u}$$

holds. Substituting for $\mathbf{u}$ in this equation, we may rewrite it in the form

$$p\mathbf{v} \ = \ (\mathbf{y} - \mathbf{x}) \ + \ q\mathbf{u} \ = \ (1 + sq)(\mathbf{y} - \mathbf{x}) \ + \ qt\mathbf{v}$$

and hence by equating coefficients we cannot find p and q such that $p = qt$ and $1 + sq = 0$. Now if $s \neq 0$, then these two equations have the solution $q = -s^{-1}$ and $p = -s^{-1}t$. Therefore, if there is no solution then we must have $s = 0$. The latter in turn implies that $\mathbf{u} = t\mathbf{v}$ and hence $L_1 = L_0$, so that $N = \mathbf{y} + L_1 = \mathbf{y} + L_0 = M$.∎

THEOREM II.15. *If* $\dim V = 3$ *and two planes in* V *have a nonempty intersection, then their intersection contains a line.*

Proof. Let P and Q be planes, and let $\mathbf{x} \in P \cap Q$. Write $P = \mathbf{x} + W$ and $Q = \mathbf{x} + U$, where $\dim W = \dim U = 2$. Then

$$P \cap Q \ = \ (\mathbf{x} + W) \ \cap \ (\mathbf{x} + U)$$

clearly contains $\mathbf{x} + (W \cap U)$, so it suffices to show that the vector subspace $W \cap U$ contains a 1-dimensional vector subspace. However, we have

$$\dim(W \cap U) \ = \ \dim W \ - \ \dim U \ - \ \dim(W + U) \ = \ 4 \ - \ \dim(W + U)$$

and since $\dim(W + U) \leq \dim V = 3$, the displayed equation immediately implies $\dim(W \cap U) \geq 4 - 3 = 1$. Hence the intersection of the vector subspaces is nonzero, and as such it contains a nonzero vector $\mathbf{z}$ as well as the 1-dimensional subspace spanned by $\mathbf{z}$.∎

The preceding results imply that Theorems 7 and 8 from Section 1, and the conclusions of Exercises 5 and 6 from that section, are all true for the models $(V, \mathcal{L}_V, \mathcal{P}_V)$ described above provided $\dim V = 3$. In some contexts it is useful to interpret the conclusions of the theorems or exercises in temrs of the vector space structure on V. For example, in Theorem 8 if P is the plane $\mathbf{x} + W$, then the parallel plane Q will be $\mathbf{y} + W$. Another example is discussed in Exercise 5 below.

Generalizing incidence to higher dimensions

The characterization of lines and planes as translates of 1- and 2-dimensional subspaces suggests a simple method for generalizing incidence structures to dimensions greater than three.[3] Namely, define a *k-plane* in a vector space V to be a translate of a *k*-dimensional vector subspace.

The following quotation from Winger, *Introduction to Projective Geometry*,[4] may help the reader understand the reasons for formulating the concepts of affine geometry in arbitrary dimensions.[5]

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[3]The explicit mathematical study of higher dimensional geometry began around the middle of the 19th century, particularly in the work of LUDWIG SCHLÄFLI (1814–1895), who also made contributions to other areas of mathematics. Many ideas in his work on higher dimensional geometry were independently discovered by others with the development of linear algebra during the second half of that century.

[4]See page 15 of that book.

[5]Actually, spaces of higher dimensions play an important role in theoretical physics. Einstein's use of a four-dimensional space-time is of course well-known, but the use of spaces with dimensions ≥ 4 in physics was at least implicit during much of the 19th century. In particular, 6-dimensional configuration spaces were implicit in work on celestial mechanics, and spaces of various dimensions were widely used in classical dynamics.

> The timid may console themselves with the reflection that the geometry of four and higher dimensions is, if not a necessity, certainly a convenience of language — a translation of the algebra — and let the philosophers ponder the metaphysical questions involved in the idea of a point set of higher dimensions.

We shall conclude this section with a characterization of k-planes in V, where V is finite-dimensional and $1 + 1 \neq 0$ in $\mathbb{F}$; in particular, the result below applies to the $V = \mathbb{R}^n$. An extension of this characterization to all fields except the field $\mathbb{Z}_2$ with two elements is given in Exercise 1 below.

Definition. Let V be a vector space over the field $\mathbb{F}$, and let $P \subset V$. We shall say P is a *flat subset* of V if for each pair of distinct points $\mathbf{x}$, $\mathbf{y} \in \mathbb{F}$ the line $\mathbf{xy}$ is contained in $\mathbb{F}$.

THEOREM II.16. *Let V be a finite-dimensional vector space over a field $\mathbb{F}$ in which $1 + 1 \neq 0$. Then a nonempty set $P \subset V$ is a flat subset if and only if it is a k-plane for some integer k satisfying $0 \leq k \leq \dim V$.*

Definition. A subset $S \subset V$ is said to be an *affine subspace* if can be written as $\mathbf{x} + W$, where $\mathbf{x} \in V$ and W is a vector subspace of V. With this terminology, we can restate the theorem to say that if $1 + 1 \neq 0$ in $\mathbb{F}$, then a nonempty subset of V is an affine subspace if and only if it is a flat subset.

Proof. We split the proof into the "if" and "only if" parts.

Every k-plane is a flat subset. Suppose that W is a k-dimensional vector subspace and $\mathbf{x} \in V$. Let $\mathbf{y}$, $\mathbf{z} \in \mathbf{x} + V$. Then we may write $\mathbf{y} = \mathbf{x} + \mathbf{u}$ and $\mathbf{z} = \mathbf{x} + \mathbf{v}$ for some distinct vectors $\mathbf{u}$, $\mathbf{v} \in W$. A typical point on the line $\mathbf{yz}$ has the form $\mathbf{y} + t(\mathbf{z} - \mathbf{y})$ for some scalar t, and we have

$$\mathbf{y} \; + \; t(\mathbf{z} - \mathbf{y}) \; = \; \mathbf{x} \; + \; \mathbf{u} \; + \; t(\mathbf{v} - \mathbf{u}) \; \in \; \mathbf{x} \; + \; W$$

which shows that $\mathbf{x} + W$ is a flat subset. Note that this implication does not require any assumption about the nontriviality of $1 + 1$.

Every flat subset has the form $\mathbf{x} + W$ for some vector subspace W. If we know this, we also know that $k = \dim W$ is less than or equal to $\dim V$. — Suppose that $\mathbf{x} \in P$, and let $W = (-\mathbf{x}) + P$; we need to show that W is a vector subspace. To see that W is closed under scalar multiplication, note first that $\mathbf{w} \in W$ implies $\mathbf{x} + \mathbf{w} \in P$, so that flatness implies every point on the line $\mathbf{x}(\mathbf{x} + \mathbf{w})$ is contained in P. For each scalar t we know that $\mathbf{x} + t\mathbf{w}$ lies on this line, and thus each point of this type lies in $P = \mathbf{x} + W$. If we subtract $\mathbf{x}$ we see that $t\mathbf{w} \in W$ and hence W is closed under scalar multiplication.

To see that W is closed under vector addition, suppose that $\mathbf{w}_1$ and $\mathbf{w}_2$ are in W. By the previous paragraph we know that $2\mathbf{w}_1$ and $2\mathbf{w}_2$ also belong to W, so that $\mathbf{u}_1 = \mathbf{x} + 2\mathbf{w}_1$ and $\mathbf{u}_2 = \mathbf{x} + 2\mathbf{w}_2$ are in P. Our hypothesis on $\mathbb{F}$ implies the latter contains an element $\frac{1}{2} = (1+1)^{-1}$, so by flatness we also know that

$$\tfrac{1}{2}\left(\mathbf{u}_1 + \mathbf{u}_2\right) \; + \; \tfrac{1}{2}\left(\mathbf{u}_2 - \mathbf{u}_1\right) \; \in \; P \; .$$

If we now expand and simplify the displayed vector, we see that it is equal to $\mathbf{v} + \mathbf{w}_1 + \mathbf{w}_2$. Therefore it follows that $\mathbf{w}_1 + \mathbf{w}_2 \in W$, and hence W is a vector subspace of V.■

Not surprisingly, if S is an affine subspace of the finite-dimensional vector space V, then we define its *dimension* by $\dim S = \dim W$, where W is a vector subspace of V such that $S = \mathbf{x} + W$.

— *This number is well defined* because $S = \mathbf{x} + W = \mathbf{y} + U$ implies $\mathbf{y} \in \mathbf{x} + W$, so that $\mathbf{y} + W = \mathbf{x} + W = \mathbf{y} + U$ and hence

$$W = -\mathbf{y} + (\mathbf{y} + W) = -\mathbf{y} + (\mathbf{y} + U) = U.\blacksquare$$

<div align="center">Hyperplanes</div>

One particularly important family of affine subspaces in a finite-dimensional vector space V is the set of all **hyperplanes** in V. We shall conclude this section by defining such objects and proving a crucial fact about them.

Definition. Let n be a positive integer, and let V be an n-dimensional vector space over the field $\mathbb{F}$. A subset $H \subset V$ is called a *hyperplane* in V if H is the translate of an $(n-1)$-dimensional subspace. — In particular, if $\dim V = 3$ then a hyperplane is just a plane and if $\dim V = 2$ then a hyperplane is just a line.

One reason for the importance of hyperplanes is that if $k < \dim V$ then every k-plane is an intersection of finitely many hyperplanes (see Exercise II.5.4).

We have seen that planes in $\mathbb{R}^3$ and lines in $\mathbb{R}^2$ are describable as the sets of all points $\mathbf{x}$ which satisfy a nontrivial first degree equation in the coordinates (x_1, x_2, x_3) or (x_1, x_2) respectively (see Theorems I.1, I.5 and I.7). The final result of this section is a generalization of these facts to arbitrary hyperplanes in $\mathbb{F}^n$, where $\mathbb{F}$ is an arbitrary field.

THEOREM II.17. *In the notation of the previous paragraph, let H be a nonempty subset of $\mathbb{F}^n$. Then H is a hyperplane if and only if there exist scalars $c_1, \cdots, c_n$ which are not all zero such that H is the set of all $\mathbf{x} \in \mathbb{F}^n$ whose coordinates $x_1, \cdots, x_n$ satisfy the equation*

$$\sum_{i=0}^{n} c_i x_i = b$$

for some $b \in \mathbb{F}$.

Proof. Suppose that H is defined by the equation above. If we choose j such that $a_j \neq 0$ and let $\mathbf{e}_j$ be the unit vector whose j^{th} coordinate is 1 and whose other coordinates are zero, then we have $-a_j^{-1} b \, \mathbf{e}_j \in H$, and hence the latter is nonempty. Set W equal to $(-\mathbf{z}) + H$, where $\mathbf{z} \in H$ is fixed. As in Chapter I, it follows that $\mathbf{y} \in W$ if and only if its coordinates $y_1, \cdots, y_n$ satisfy the equation $\sum_i a_i y_i = 0$. Since the coefficients a_i are not all zero, it follows from Theorem A.10 that W is an $(n-1)$-dimensional vector subspace of $\mathbb{F}^n$, and therefore $H = \mathbf{x} + W$ is a hyperplane.$\square$

Conversely, suppose H is a hyperplane and write $\mathbf{x} + W$ for a suitable vector $\mathbf{x}$ and $(n-1)$-dimensional subspace W. Let $\{\mathbf{w}_1, \cdots, \mathbf{w}_{n-1}\}$ be a basis for W, and write these vectors out in coordinate form:

$$\mathbf{w}_i = (w_{i,1}, \cdots, w_{i,n})$$

If B is the matrix whose rows are the vectors $\mathbf{w}_i$, then the rank of B is equal to $(n-1)$ by construction. Therefore, by Theorem A.10 the set Y of all $\mathbf{y} = (y_1, \cdots, y_n)$ which solve the system

$$\sum_j y_j w_{i,j} = 0 \qquad (1 \leq i \leq (n-1))$$

is a 1-dimensional vector subspace of $\mathbb{F}^n$. Let $\mathbf{a}$ be a nonzero (hence spanning) vector in Y.

We claim that $\mathbf{z} \in W$ if and only if $\sum_i a_i z_i = 0$. By construction, W is contained in the subspace S of vectors whose coordinates satisfy this equation. By Theorem A.10 we know that $\dim S = (n-1)$, which is equal to $\dim W$ by our choice of the latter; therefore Theorem A.9 implies that $W = S$, and it follows immediately that H is the set of all $\mathbf{z} \in \mathbb{F}^n$ whose coordinates $z_1, \cdots, z_n$ satisfy the nontrivial linear equation $\sum_i a_i z_i = \sum_i a_i x_i$ (where $\mathbf{x} = (x_1, \cdots, x_n) \in W$ is the fixed vector chosen as in the preceding paragraph).$\blacksquare$

EXERCISES

1. Prove that Theorem 16 remains true for every field **except** $\mathbb{Z}_2$. Give an example of a flat subspace of $(\mathbb{Z}_2)^3$ which is not a k-plane for some k.

2. Let $\mathbb{F}$ be a field. Prove that the lines in $\mathbb{F}^2$ defined by the equations $ax + by + cz = 0$ and $a'x + by + c'z = 0$ (compare Theorem I.7 and Theorem 17 above) are parallel or identical if and only if $ab' - ba' = 0$.

3. Find the equation of the hyperplane in $\mathbb{R}^4$ passing through the (noncoplanar) points $(1,0,1,0)$, $(0,1,0,1)$, $(0,1,1,0)$, and $(1,0,0,1)$.

4. Suppose that $\mathbf{x}_1 + W_1$ and $\mathbf{x}_2 + W_2$ are k_1- and k_2-planes in a vector space V such that $\mathbf{x}_1 + W_1 \cap \mathbf{x}_2 + W_2 \neq \varnothing$. Let $\mathbf{z}$ be a common point of these subsets. Prove that their intersection is equal to $\mathbf{z} + W_1 \cap \mathbf{z} + W_2$, and generalize this result to arbitrary finite intersections.

5. Let V be a 3-dimensional vector space over $\mathbb{F}$, and suppose we are given the configuration of Exercise II.1.6 (L and M are skew lines, and P is a plane containing L but parallel to M). Suppose that the skew lines are given by $\mathbf{x} + W$ and $\mathbf{y} + U$. Prove that the plane P is equal to $\mathbf{x} + (U + W)$ [*Hint:* Show that the latter contains L and is disjoint from M.].

6. Suppose that $\dim V = n$ and $H = \mathbf{x} + W$ is a hyperplane in V. Suppose that $\mathbf{y} \in V$ but $\mathbf{y} \notin H$. Prove that $H' = \mathbf{y} + W$ is the unique hyperplane K such that $\mathbf{y} \in K$ and $H \cap K = \varnothing$. [*Hints:* If $\mathbf{z} \in H \cap H'$ then $N = \mathbf{x} + W = \mathbf{z} + W = \mathbf{y} + W = H'$. If $K = \mathbf{y} + U$ where U is some $(n-1)$-dimensional vector subspace different from W, explain why $\dim W \cap U = n - 2$. Choose a basis A of $(n-2)$ vectors for this intersection, and let $\mathbf{u}_0 \in U$, $\mathbf{w}_0 \in W$ such that $\mathbf{u}_0, \mathbf{w}_0 \notin U \cap W$. Show that $A \cup \{\mathbf{u}_0, \mathbf{w}_0\}$ is a basis for V, write $\mathbf{y} - \mathbf{x}$ in terms of this basis, and use this equation to find a vector which lies in $H \cap K = (\mathbf{x} + W) \cap (\mathbf{y} + U)$.]

7. Given an example of two planes P_1, P_2 in $\mathbb{R}^4$ whose intersection consists of a single point. [*Hint:* There are many examples for which the P_i are vector subspaces.]

3. Affine bases

We shall need analogs of linear independence and spanning that apply to arbitrary k-planes and not just k-dimensional vector subspaces. The starting points are two basic observations.

THEOREM II.18. *Suppose that S is a k-plane in a vector space V over the field $\mathbb{F}$. Given $\mathbf{a}_1, \cdots, \mathbf{a}_r \in S$, let $t_1, \cdots, t_r \in S$ be such that $\sum_j t_j = 1$. Then $\sum_j t_j \mathbf{a}_j \in S$.*

Proof. Write $S = \mathbf{x} + W$, where $\mathbf{x} \in S$ and W is a k-dimensional vector subspace, and for each i write $\mathbf{a}_i = \mathbf{x} + \mathbf{w}_i$ where $\mathbf{w}_i \in \mathbb{F}$. Then

$$\sum_j t_j \mathbf{a}_j = \sum_j t_j(\mathbf{x} + \mathbf{w}_j) = \sum_j t_j \mathbf{x} + \sum_j t_j \mathbf{w}_j = \mathbf{x} + \sum_j t_j \mathbf{w}_j$$

where the latter holds because $\sum_j t_j = 1$. Since W is a vector subspace, we know that $\sum_j t_j \mathbf{w}_j \in W$, and therefore it follows that $\sum_j t_j \mathbf{a}_j \in W$.∎

THEOREM II.19. *If V is as above and $T \subset V$ is an arbitrary subset, define the **affine hull** of T by*

$$\mathcal{H}(T) = \{\, \mathbf{x} \in V \mid \mathbf{x} = \sum_j t_j \mathbf{w}_j, \text{ where } \mathbf{v}_i \in T \text{ for all } t \text{ and } \sum_j t_j = 1 \}$$

(note that the sum is finite, but T need not be finite). Then $\mathcal{H}(T)$ is an affine subspace of V.

Sometimes we shall also say that $\mathcal{H}(T)$ is the *affine span* of T, and we shall say that T *affinely spans* an affine subspace S if $S = \mathcal{H}(T)$.

Proof. Suppose that $\mathbf{x}, \mathbf{y} \in \mathcal{H}(T)$, and write

$$\mathbf{x} = \sum_i s_i \mathbf{u}_i = \sum_j t_j \mathbf{v}_j$$

where $\mathbf{u}_i \in T$ and $\mathbf{v}_j \in T$ for all i and j, and the coefficients satisfy $\sum_i s_i = \sum_j t_j = 1$. We need to show that $\mathbf{x} + c(\mathbf{y} - \mathbf{x}) \in \mathcal{H}(T)$. But

$$\mathbf{x} + c(\mathbf{y} - \mathbf{x}) = \sum_i s_i \mathbf{u}_i + c \cdot \left(\sum_j t_j \mathbf{v}_j - \sum_i s_i \mathbf{u}_i \right) =$$

$$(1 - c) \cdot \sum_i s_i \mathbf{u}_i + c \cdot \sum_j t_j \mathbf{v}_j = \sum_i s_i (1 - c) \mathbf{u}_i + \sum_j t_j c \mathbf{v}_j .$$

We have no a *priori* way of knowing whether any of the vectors $\mathbf{u}_i$ are equal to any of the vectors $\mathbf{v}_j$, but in any case we can combine like terms to rewrite the last expression as $\sum_q r_q \mathbf{w}_q$, where the vectors $\mathbf{w}_q$ run through all the vectors $\mathbf{u}_i$ and $\mathbf{v}_j$, and the coefficients r_q are given accordingly; by construction, we then have

$$\sum_q r_q = \sum_i s_i (1 - c) + \sum_j t_j c = (1 - c) \cdot \sum_i s_i + c \cdot \sum_j t_j = (1 - c) \cdot 1 + c \cdot 1 = 1$$

and therefore it follows that the point $\mathbf{x} + c(\mathbf{y} - \mathbf{x})$ belongs to $\mathcal{H}(T)$.∎

Thus *a linear combination of points in an affine subspace will also lie in the subspace* **provided** *the coefficients add up to* 1, and by Theorem 19 this is the most general type of linear combination one can expect to lie in S.

Definition.   A vector $\mathbf{v}$ is an *affine combination* of the vectors $\mathbf{x}_0, \cdots, \mathbf{x}_n$ if we have $\mathbf{x} = \sum_j t_j \mathbf{x}_j$, where $\sum_j t_j = 1$. Thus the affine hull $\mathcal{H}(T)$ of a set T is the set of all (finite) affine combinations of vectors in T.

AFFINE VERSUS LINEAR COMBINATIONS. If a vector $\mathbf{y}$ is a linear combination of the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$, then it is automatically an affine combination of $\mathbf{0}, \mathbf{v}_1, \cdots, \mathbf{v}_n$, for if $\mathbf{y} = \sum_i t_i \mathbf{x}_i$ then

$$\mathbf{y} \;=\; \left(1 \;-\; \sum_j t_j \right) \cdot \mathbf{0} \;+\; \sum_i t_i \mathbf{x}_i \; .$$

Definition.   Let V be a vector space over a field $\mathbb{F}$. and let $X \subset V$ be the set $\{\mathbf{x}_0, \cdots, \mathbf{x}_n\}$. We shall say that X is *affinely dependent* if one element of X is expressible as an affine combination of the others and *affinely independent* otherwise. By convention, one point subsets are affinely independent.

The next result gives the fundamental relationship between affine dependence and linear dependence (and, by taking negations, it also gives the fundamental relationship between affine independence and linear independence).

THEOREM II.20. *In the setting above, the finite set $X = \{\mathbf{x}_0, \cdots, \mathbf{x}_n\} \subset V$ is affinely dependent if and only if the set $X' = \{\mathbf{x}_1 - \mathbf{x}_0, \cdots, \mathbf{x}_n - \mathbf{x}_0\}$ is linearly dependent. Likewise, the finite set $X = \{\mathbf{x}_0, \cdots, \mathbf{x}_n\} \subset V$ is affinely independent if and only if the set $X' = \{\mathbf{x}_1 - \mathbf{x}_0, \cdots, \mathbf{x}_n - \mathbf{x}_0\}$ is linearly independent.*

Proof.   Since affine dependence and affine independence are the negations of each other and similarly for linear dependence and linear independence, it will suffice to prove the first conclusion in the theorem.

Proof that X is affinely dependent if X' is linearly dependent.   By linear dependence there is some $k > 0$ such that

$$\mathbf{x}_k - \mathbf{x}_0 \;=\; \sum_{i \neq 0, k} c_i \left(\mathbf{x}_i - \mathbf{x}_0 \right)$$

and therefore we also have

$$\mathbf{x}_k \;=\; \mathbf{x}_0 \;+\; \sum_{i \neq 0, k} c_i \, \mathbf{x}_i \;-\; \sum_{i \neq 0, k} c_i \, \mathbf{x}_0 \;=\; \left(1 \;-\; \sum_{i \neq 0, k} c_i \right) \mathbf{x}_0 \;+\; \sum_{i \neq 0, k} c_i \, \mathbf{x}_i \; .$$

Therefore $\mathbf{x}_k$ is also an affine combination of all the $\mathbf{x}_j$ such that $j \neq k$.

Proof that X' is linearly dependent if X is affinely dependent.   By assumption there is some k such that

$$\mathbf{x}_k \;=\; \sum_{j \neq k} c_j \, \mathbf{x}_j$$

where $\sum_{j \neq k} c_j = 1$. Therefore we also have

$$\mathbf{x}_k - \mathbf{x}_0 = \left(\sum_{j \neq k} c_j \, \mathbf{x}_j \right) - \mathbf{x}_0 = \sum_{j \neq k} c_j (\mathbf{x}_j - \mathbf{x}_0) \ .$$

Note that we can take the summation on the right hand side to run over all j such that $j \neq k$, 0 because $\mathbf{x}_0 - \mathbf{x}_0 = \mathbf{0}$.

There are now two cases depending on whether $k > 0$ or $k = 0$. In the first case, we have obtained an expression for $\mathbf{x}_k - \mathbf{x}_0$ in terms of the other vectors in X', and therefore X' is linearly dependent. Suppose now that $k = 0$, so that the preceding equation reduces to

$$\mathbf{0} = \sum_{j > 0} c_j (\mathbf{x}_j - \mathbf{x}_0) \ .$$

Since $\sum_{j>0} c_j = 1$ it follows that $c_m \neq 0$ for some $m > 0$, and this now implies

$$\mathbf{x}_m - \mathbf{x}_0 = \sum_{j \neq m, 0} \frac{-c_j}{c_m} (\mathbf{x}_j - \mathbf{x}_0)$$

which shows that X' is linearly dependent.∎

One important characterization of linear independence for a set Y is that an arbitrary vector has at most one expression as a linear combination of the vectors in Y. There is a similar characterization of affine independence.

THEOREM II.21. *A (finite) set X of vectors in a given vector space V is affinely independent if and only if every vector in V has **at most one** expression as an affine combination of vectors in X.*

Proof. Suppose X is affinely independent and that

$$\mathbf{y} = \sum_j t_j \mathbf{v}_j = \sum_j s_j \mathbf{v}_j$$

where $\mathbf{v}_j$ runs through the vectors in V and

$$\sum_j t_j = \sum_j s_j = 1 \ .$$

Then we have

$$\left(\sum_j t_j \mathbf{v}_j \right) - \mathbf{v}_0 = \left(\sum_j s_j \mathbf{v}_j \right) - \mathbf{v}_0$$

which in turn implies

$$\left(\sum_j t_j \mathbf{v}_j \right) - \left(\sum_j t_j \right) \mathbf{v}_0 = \left(\sum_j s_j \mathbf{v}_j \right) - \left(\sum_j s_j \right) \mathbf{v}_0$$

so that we have

$$\sum_{j>0} t_j (\mathbf{v}_j - \mathbf{v}_0) = \sum_j s_j (\mathbf{v}_j - \mathbf{v}_0) \ .$$

Since the vectors $\mathbf{v}_j - \mathbf{v}_0$ (where $j > 0$) is linearly independent, it follows that $t_j = s_j$ for all $j > 0$. Once we know this, we can also conclude that $t_0 = 1 - \sum_{j>0} t_j$ is equal to $1 - \sum_{j>0} s_j = s_0$, and therefore all the corresponding coefficients in the two expressions are equal.$\square$

Conversely, suppose X satisfies the uniqueness condition for affine combinations, and suppose that we have

$$\sum_{j>0} c_j(\mathbf{v}_j - \mathbf{v}_0) \;=\; \mathbf{0} \ .$$

We then need to show that $c_j = 0$ for all j. But the equation above implies that

$$\mathbf{v}_0 \;=\; \left(\sum_{j>0} c_j(\mathbf{v}_j - \mathbf{v}_0) \right) \;+\; \mathbf{v}_0$$

and if we simplify the right hand side we obtain the equation

$$\mathbf{v}_0 \;=\; \left(1 \;-\; \sum_{j>0} c_j \right) \mathbf{v}_0 \;+\; \sum_{j>0} c_j \mathbf{v}_j \ .$$

The coefficients on both sides add up to 1, so by the uniqueness assumption we must have $c_j = 0$ for all $j > 0$; but this implies that the vectors $\mathbf{v}_j - \mathbf{v}_0$ (where $j > 0$) are linearly independent.$\blacksquare$

Definition.   If S is an affine subspace of the finite-dimensional vector space V and $T \subset S$ is a finite subset, then T is said to be an *affine basis* for S if T is affinely independent and affinely spans T.

There is a fundamental analog of the preceding results which relates affine bases of affine subspaces and vector space bases of vector subspaces.

THEOREM II.22. *Let V be a finite-dimensional vector space, let S be an affine subspace, and suppose that $S = \mathbf{z} + W$ for a suitable vector $\mathbf{z}$ and vector subspace V. Then the finite set $X = \{\mathbf{x}_0, \cdots, \mathbf{x}_m\} \subset S$ is an affine basis for S if and only if the set $X' = \{\mathbf{x}_1 - \mathbf{x}_0, \cdots, \mathbf{x}_n - \mathbf{x}_0\}$ is linear basis for W.*

Proof.   First of all, since $\mathbf{x}_0 \in S$ we may write $S = \mathbf{x}_0 + W$ and forget about the vector $\mathbf{z}$.

Suppose that $X = \{\mathbf{x}_0, \cdots, \mathbf{x}_m\}$ is an affine basis for S, and let $\mathbf{y} \in W$. Since $\mathbf{x}_0 + \mathbf{y} \in \mathbf{x}_0 + W = S$, there exist $s_0, \cdots, s_m \in \mathbb{F}$ such that $\sum_i s_i = 1$ and $\mathbf{x}_0 + \mathbf{y} = \sum_i s_i \mathbf{x}_i$. Subtracting $\mathbf{x}_0$ from both sides and using the equation $\sum_i s_i = 1$, we see that

$$\mathbf{y} \;=\; \sum_{i>0} s_i(\mathbf{x}_i - \mathbf{x}_0)$$

and hence X' spans W. Since X' is linearly independent by Theorem 20, it follows that X' is a basis for W.$\square$

Conversely, suppose that $X' = \{\mathbf{x}_1 - \mathbf{x}_0, \cdots, \mathbf{x}_n - \mathbf{x}_0\}$ is linear basis for W. Since X' is linearly independent, by Theorem 20 we also know that X is affinely independent. To see that X affinely spans S, let $\mathbf{u} \in S$, and write $\mathbf{u} = \mathbf{x}_0 + \mathbf{v}$, where $\mathbf{v} \in S$. Since X' spans W we know that

$$\mathbf{u} \;=\; \mathbf{x}_0 \;+\; \sum_{i>0} s_i(\mathbf{x}_i - \mathbf{x}_0)$$

for appropriate scalars s_i, and if we set $s_0 = 1 - \sum_{i>0} s_i$, then we may rewrite the right hand side of the preceding equation as $\sum_{i \geq 0} s_i \mathbf{x}_i$, where by construction we have $\sum_{i \geq 0} s_i = 1$. Therefore X affinely spans S, and it follows that X must be an affine basis for X.∎

Definition. Suppose that we are in the setting of the theorem and X is an affine basis for S, so that each $\mathbf{y} \in S$ can be uniquely written as an affine combination $\sum_i t_i \mathbf{x}_i$, where $\sum_i t_i = 1$. Then the unique coefficients t_i are called the *barycentric coordinates* of $\mathbf{y}$ with respect to X. The physical motivation for this name is simple: Suppose that $\mathbb{F}$ is the real numbers and we place weights $w_i > 0$ at the vectors $\mathbf{v}_i$ in S such that the total weight is w units. Let $t_i = w_i/w$ be the normalized weight at $\mathbf{v}_i$; then $\sum_i t_i \mathbf{x}_i$ is the center of gravity for the resulting physical system (a version of this is true even if one allows some of the coefficients t_i to be negative).

In analogy for linear bases of vector subspaces, the number of elements in an affine basis for an affine subspace S depends on S itself. However, as illustrated by the final result of this section, there is a crucial difference in the formula relating the dimension of S to the number of elements in an affine basis.

THEOREM II.23. *If V is a finite-dimensional vector space and S is an affine subspace, then S has an affine basis. Furthermore, if we write $S = \mathbf{y} + W$ for suitable $\mathbf{y}$ and W, then every affine basis for S has exactly $\dim W + 1$ elements.*

Proof. If $X = \{\mathbf{x}_0, \cdots, \mathbf{x}_m\} \subset S$ is an affine basis for $S = \mathbf{y} + W = \mathbf{x}_0 + W$, then $X' = \{\mathbf{x}_1 - \mathbf{x}_0, \cdots, \mathbf{x}_n - \mathbf{x}_0\}$ is a linear basis for W by Theorem 22, and conversely. Therefore the existence of an affine basis for S follows from Theorem 22 and the existence of a linear basis for W. Furthermore, since every linear basis for W contains exactly $\dim W$ elements, by Theorem 22 we know that every affine basis for S contains exactly $\dim W + 1$ elements.∎

EXERCISES

In the exercises below, assume that all vectors lie in a fixed finite-dimensional vector space V over a field $\mathbb{F}$.

1. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in V$ (a vector space over some field) be noncollinear and for $i = 1, 2, 3$ let $\mathbf{x}_i = t_i \mathbf{a} + u_i \mathbf{b} + v_i \mathbf{c}$, where $t_i + u_i + v_i = 1$. Prove that the points $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are collinear if and only if

$$\begin{vmatrix} t_1 & u_1 & v_1 \\ t_2 & u_2 & v_2 \\ t_3 & u_3 & v_3 \end{vmatrix} = 0$$

where the left hand side is a 3×3 determinant.

2. Prove the **Theorem of Menelaus**:[6] Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in V$ (a vector space over some field) be noncollinear, and suppose we have points $\mathbf{p} \in \mathbf{bc}$, $\mathbf{q} \in \mathbf{ac}$, and $\mathbf{r} \in \mathbf{ab}$. Write these three vectors as

$$\begin{aligned} \mathbf{p} &= \mathbf{b} + t(\mathbf{c} - \mathbf{b}) = t\mathbf{c} + (1-t)\mathbf{b} \\ \mathbf{q} &= \mathbf{a} + u(\mathbf{c} - \mathbf{a}) = u\mathbf{c} + (1-u)\mathbf{a} \\ \mathbf{r} &= \mathbf{a} + v(\mathbf{b} - \mathbf{a}) = v\mathbf{b} + (1-v)\mathbf{a} \end{aligned}$$

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[6]MENELAUS OF ALEXANDRIA (*c.* 70 A. D.–*c.* 130 A. D.) worked in geometry and astronomy, and he is particularly given credit for making numerous contributions to spherical geometry and trigonometry.

where t, u, v are appropriate scalars. Then $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are collinear if and only if

$$\frac{t}{1-t} \cdot \frac{u}{1-u} \cdot \frac{v}{1-v} = -1 \ .$$

3.   Prove the **Theorem of Ceva**:[7] In the setting of the preceding exercise, the lines $\mathbf{ap}$, $\mathbf{bq}$ and $\mathbf{cr}$ are concurrent (there is a point which lies on all three lines) if and only if

$$\frac{t}{1-t} \cdot \frac{u}{1-u} \cdot \frac{v}{1-v} = +1 \ .$$

4.   Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in V$ (as above) be noncollinear, and suppose we have points $\mathbf{y} \in \mathbf{ba}$ and $\mathbf{x} \in \mathbf{bc}$ which are distinct from $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and satisfy these three vectors as

$$\begin{aligned}
\mathbf{x} &= \mathbf{b} + t(\mathbf{a} - \mathbf{b}) = t\mathbf{a} + (1-t)\mathbf{b} \\
\mathbf{y} &= \mathbf{b} + u(\mathbf{c} - \mathbf{a}) = u\mathbf{c} + (1-u)\mathbf{b}
\end{aligned}$$

where t, u are appropriate scalars (neither of which is 0 or 1). Prove that the lines $\mathbf{ay}$ and $\mathbf{cx}$ have a point in common if and only if $ut \neq 1$. [*Hint:*   Explain why the lines have no points in common if and only if $\mathbf{y} - \mathbf{a}$ and $\mathbf{x} - \mathbf{c}$ are linearly dependent. Write both of these vectors as linear combinations of $\mathbf{a} - \mathbf{b}$ and $\mathbf{c} - \mathbf{b}$, and show that if $\mathbf{z}$ and $\mathbf{w}$ are linearly independent, then $p\mathbf{z} + q\mathbf{w}$ and $r\mathbf{z} + s\mathbf{w}$ are linearly dependent if and only if $sp = rq$. Finally, compare the two conclusions in the preceding sentence.]

5.   Let V be a finite-dimensional vector space over the field $\mathbb{F}$, let $W \subset V$ be a vector subspace, suppose that $\dim W = k$, and let $X = \{\mathbf{x}_1, \cdots , \mathbf{x}_m\}$ be a finite subset of W. Prove that W is a basis for W in the sense of linear algebra if and only if the set $X \cup \{\mathbf{0}\}$ is an affine basis for $W = \mathbf{0} + W$ if the latter is viewed as a k-plane.

6.   In the setting of Exercises 2 and 3, suppose that the lines $\mathbf{ap}$, $\mathbf{bq}$ and $\mathbf{cr}$ are concurrent with $t = \frac{1}{2}$ and $v = 1-u$. Express the common point $\mathbf{g}$ of these three lines as a linear combination of $\mathbf{a}$ and $\mathbf{d}$ with the coefficients expressed in terms of u.

7.   Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the noncollinear points in $\mathbb{R}^2$ with Cartesian coordinates $(1,0)$, $(5,0)$ and $(5,4)$ respectively. Find the barycentric coordinates of the following points with respect to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$:

(i) The point $(2,2)$.

(ii) The point $(4,4)$.

(iii) The point $(2,-1)$.

[*Hint:*   For each vector $\mathbf{x}$ as above, write $\mathbf{x} - \mathbf{a}$ as a linear combination of $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{z}$.]

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[7]Giovanni Ceva (1647–1734) is known for the result bearing his name, his rediscovery of Menelaus' Theorem, and his work on hydraulics.

4. Properties of coordinate affine spaces

In this section we shall generalize certain classical theorems of Euclidean geometry to affine planes over fields in which $1 + 1 \neq 0$. Similar material is often presented in many mathematics courses as *proofs of geometric theorems using vectors.* In fact, the uses of vectors in geometry go far beyond yielding alternate proofs for some basic results in classical Greek geometry; they are often the method of choice for studying all sorts of geometrical problems ranging from purely theoretical questions to carrying out the computations needed to create high quality computer graphics. We shall illustrate the uses of vector algebra in geometry further by proving some nonclassical theorems that figure significantly in the next two chapters.

Let $\mathbb{F}$ be a field in which $1 + 1 \neq 0$, and set $\frac{1}{2}$ equal to $(1 + 1)^{-1}$. Given a vector space V over $\mathbb{F}$ and two distinct vectors $\mathbf{a}, \ \mathbf{b} \in V$, the *midpoint* of $\mathbf{a}$ and $\mathbf{b}$ is the vector $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.

THEOREM II.24. *Let V and $\mathbb{F}$ as above, and let $\mathbf{a}, \ \mathbf{b}, \ \mathbf{c}$ be noncollinear points in V. Then the lines joining $\mathbf{x} = \mathsf{midpoint}(\mathbf{a}, \mathbf{b})$ and $\mathbf{y} = \mathsf{midpoint}(\mathbf{a}, \mathbf{c})$ is parallel to $\mathbf{bc}$.*

In ordinary Euclidean geometry one also knows that the length of the segment joining $\mathbf{x}$ to $\mathbf{y}$ is half the length of the segment joining $\mathbf{b}$ to $\mathbf{c}$ (the length of the segment is just the distance between the endpoints). We do not include such a conclusion because our setting does not include a method for defining distances (in particular, an arbitrary field has no *a priori* notion of distance).
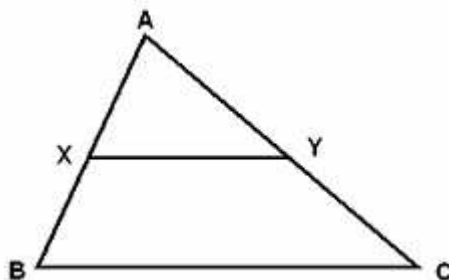

Figure II.3

Proof, Let W be the subspace spanned by $\mathbf{c} - \mathbf{b}$; it follows that $\mathbf{bc} = \mathbf{b} + W$. On the other hand, the line joining the midpoints is given by $\frac{1}{2}(\mathbf{b} + \mathbf{c}) + U$, where U is the subspace spanned by

$$\tfrac{1}{2}(\mathbf{a} + \mathbf{c}) \ - \ \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) \ = \ \tfrac{1}{2}(\mathbf{c} - \mathbf{b}) \ .$$

Since there is a vector $\mathbf{w}$ such that W is spanned by $\mathbf{w}$ and U is spanned by $\frac{1}{2}\mathbf{w}$, clearly $W = U$, and therefore the line joining the midpoints is parallel to $\mathbf{bc}$ by the construction employed to prove Theorem 14.∎

Definition. Let V and $\mathbb{F}$ as above, and let $\mathbf{a}, \ \mathbf{b}, \ \mathbf{c}$ be noncollinear points in V. The *affine triangle* $\triangle \mathbf{abc}$ is given by $\mathbf{ab} \ \cup \ \mathbf{ac} \ \cup \ \mathbf{bc}$, and the *medians* of this affine triangle are the lines joining $\mathbf{a}$ to $\mathsf{midpoint}(\mathbf{b}, \mathbf{c})$, $\mathbf{b}$ to $\mathsf{midpoint}(\mathbf{a}, \mathbf{c})$, and $\mathbf{c}$ to $\mathsf{midpoint}(\mathbf{a}, \mathbf{b})$.

THEOREM II.25. *Let V, $\mathbb{F}$, $\mathbf{a}, \ \mathbf{b}, \ \mathbf{c}$ be as above. Then the medians of the affine triangle $\triangle \mathbf{abc}$ are concurrent (pass through a single point) if $1 + 1 + 1 \neq 0$ and parallel in pairs if $1 + 1 + 1 = 0$.*
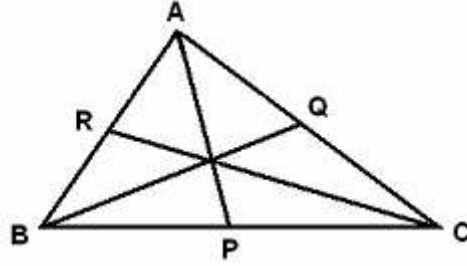
Figure II.4

Proof.   *First case.*   Suppose that $1 + 1 + 1 \neq 0$, and let $\frac{1}{3} = (1 + 1 + 1)^{-1}$. Assume that the point $\mathbf{x}$ lies on the line joining $\mathbf{a}$ to $\frac{1}{2}(\mathbf{b} + \mathbf{c})$ and also on the line joining $\mathbf{b}$ to $\frac{1}{2}(\mathbf{a} + \mathbf{c})$. Then there exist $s, t \in \mathbb{F}$ such that

$$s\mathbf{a} \; + \; (1 - s)\tfrac{1}{2}(\mathbf{b} + \mathbf{c}) \; = \; \mathbf{x} \; = \; t\mathbf{b} \; + \; (1 - t)\tfrac{1}{2}(\mathbf{a} + \mathbf{c}) \; .$$

Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are affinely independent, and in both expansions for $\mathbf{x}$ the coefficients of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ add up to 1, each expansion gives $\mathbf{x}$ as an affine combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Therefore we may equate the barycentric coordinates in the two expansions for $\mathbf{x}$. In particular, this implies $s = \frac{1}{2}(1 - t)$ and $t = \frac{1}{2}(1 - s)$. If we solve these equations, we find that $s = t = \frac{1}{3}$, so that

$$\mathbf{x} \; = \; \tfrac{1}{3}(\mathbf{a} \; + \; \mathbf{b} \; + \; \mathbf{c}) \; .$$

A routine computation shows that this point does lie on both lines.

In a similar fashion one can show that the lines joining $\mathbf{a}$ to $\frac{1}{2}(\mathbf{b} + \mathbf{c})$ and $\mathbf{c}$ to $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ also meet at the point $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$, and therefore we conclude that the latter point lies on all three medians.∎

Second case.   Suppose that $1 + 1 + 1 = 0$; in this case it follows that $\frac{1}{2} = -1$. The line joining $\mathbf{a}$ to $\frac{1}{2}(\mathbf{b} + \mathbf{c})$ is then given by $\mathbf{a} + W$, where W is spanned by

$$\tfrac{1}{2}(\mathbf{b} + \mathbf{c}) \; - \; \mathbf{a} \; = \; -(\mathbf{a} + \mathbf{b} + \mathbf{c}) \; .$$

Similarly computations show that the other two medians are given by $\mathbf{b} + W$ and $\mathbf{c} + W$. To complete the proof, we need to show that no two of these lines are equal.

However, if, say, we had $\mathbf{b} + W = \mathbf{c} + W$ then it would follow that $\mathbf{c}$, $\frac{1}{2}(\mathbf{a} + \mathbf{b})$, $\mathbf{b}$, and $\frac{1}{2}(\mathbf{a} + \mathbf{c})$ would all be collinear. Since the line joining the second and third of these points contains $\mathbf{a}$ by construction, it would then follow that $\mathbf{a} \in \mathbf{bc}$, contradicting our noncollinearity assumption. Thus $\mathbf{b} + W \neq \mathbf{c} + W$; similar considerations show that $\mathbf{a} + W \neq \mathbf{c} + W$ and $\mathbf{a} + W \neq \mathbf{b} + W$, and therefore the three medians are distinct (and by the preceding paragraph are each parallel to each other).∎

HYPOTHESIS FOR THE REMAINDER OF THIS SECTION.   *For the rest of this section, the vector space V is assumed to be* **two-dimensional.**

Definition.   Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in V$ be four ordered points, no three of which are collinear. If $\mathbf{ab}\|\mathbf{cd}$ and $\mathbf{ad}\|\mathbf{bc}$, we shall call the union

$$\mathbf{ab} \; \cup \; \mathbf{bc} \; \cup \; \mathbf{cd} \; \cup \; \mathbf{da}$$

the *affine parallelogram* determined by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, and we shall write it $\square\mathbf{abcd}$. The *diagonals* of the parallelogram are the lines $\mathbf{ac}$ and $\mathbf{bd}$.

THEOREM II.26. *Suppose that* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in V$ *as above are the vertices of an affine parallelogram. Then the diagonals* $\mathbf{ad}$ *and* $\mathbf{bc}$ *have a point in common and this point is equal to* $\mathsf{midpoint}(\mathbf{a}, \mathbf{d})$ *and* $\mathsf{midpoint}(\mathbf{b}, \mathbf{c})$.

If we take $\mathbb{F} = \mathbb{R}$ and $V = \mathbb{R}^2$, this result reduces to the classical geometric result that *the diagonals of a parallelogram bisect each other*.

Proof. The classical *parallelogram law* for vector addition states that

$$(\mathbf{c} - \mathbf{a}) \;=\; (\mathbf{d} - \mathbf{a}) \;+\; (\mathbf{b} - \mathbf{a})$$
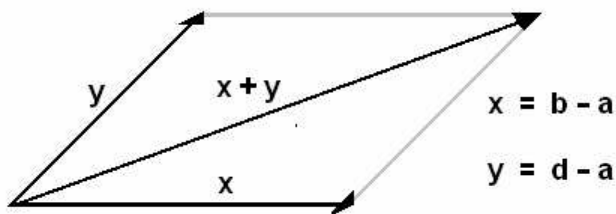
(see the accompanying figure).

Figure II.5

In fact, it is trivial to verify that $\mathbf{d} + \mathbf{b} - \mathbf{a}$ lies on both the parallel to $\mathbf{ad}$ through $\mathbf{b}$ and the parallel to $\mathbf{ab}$ and $\mathbf{c}$, and hence this point must be $\mathbf{c}$. Therefore the common point of $\mathbf{ac}$ and $\mathbf{bd}$ satisfies the equations

$$t\mathbf{b} \;+\; (1 - t)\mathbf{d} \;=\; s\mathbf{c} \;+\; (1 - s)\mathbf{a} \;=\; s(\mathbf{d} + \mathbf{b} - \mathbf{a}) \;+\; (1 - s)\mathbf{a} \;=\; s\mathbf{d} \;+\; s\mathbf{b} + (1 - 2s)\mathbf{a} \;.$$

Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$ are affinely independent, if we equate barycentric coordinates we find that $s = t = \frac{1}{2}$. But this implies that the two lines meet at a point which is equal to both $\mathsf{midpoint}(\mathbf{a}.\mathbf{c})$ and $\mathsf{midpoint}(\mathbf{b}, \mathbf{d})$.∎

We include the following two theorems because they help motivate the construction of projective space in Chapter III.

THEOREM II.27. *Let* $\mathbf{v}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *be four points, no three of which are collinear (but coplanar). Let* $\mathbf{a}' \in \mathbf{va}$, $\mathbf{b}' \in \mathbf{vb}$, *and* $\mathbf{c}' \in \mathbf{vc}$ *be distinct from* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *such that* $\mathbf{ab}||\mathbf{a}'\mathbf{b}'$ *and* $\mathbf{ac}||\mathbf{a}'\mathbf{c}'$. *Then* $\mathbf{bc}||\mathbf{b}'\mathbf{c}'$.
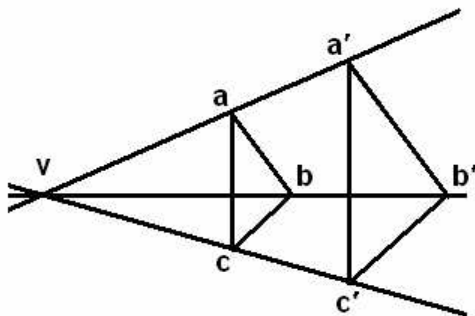

Figure II.6

Proof.   Since $\mathbf{a}' \in \mathbf{va}$ and $\mathbf{b}' \in \mathbf{vb}$, we may write $\mathbf{a}' - \mathbf{v} + s(\mathbf{a} - \mathbf{v})$ and $\mathbf{b}' - \mathbf{v} + t(\mathbf{b} - \mathbf{v})$ for suitable scalars s and t. Since $\mathbf{ab}||\mathbf{a}'\mathbf{b}'$, it follows that $\mathbf{b}' - \mathbf{a}' = k(\mathbf{b} - \mathbf{a})$ for some scalar k. But

$$k(\mathbf{b} - \mathbf{v}) \; - \; k(\mathbf{a} - \mathbf{v}) \; = \; k(\mathbf{b} - \mathbf{a}) \; = \; \mathbf{b}' - \mathbf{a}' \; = \; (\mathbf{b}' - \mathbf{v}) \; - \; (\mathbf{a}' - \mathbf{v}) \; = \; t(\mathbf{b} - \mathbf{v}) \; - \; s(\mathbf{a} - \mathbf{v}) \, .$$

Since $\mathbf{b} - \mathbf{v}$ and $\mathbf{a} - \mathbf{v}$ are linearly independent, it follows that $s = t = k$. For similar reasons we also have $\mathbf{c}' - \mathbf{v} = t(\mathbf{c} - \mathbf{v})$.

To prove $\mathbf{bc}||\mathbf{bc}'$, it suffices to note that

$$\mathbf{c}' - \mathbf{b}' \; = \; (\mathbf{c}' - \mathbf{v}) \; - \; (\mathbf{b}' - \mathbf{v}) \; = \; t(\mathbf{c} - \mathbf{v}) \; - \; t(\mathbf{b} - \mathbf{v}) \; = \; t(\mathbf{c} - \mathbf{b})$$

by the preceding paragraph, and hence $\mathbf{bc}||\mathbf{b}'\mathbf{c}'$ follows.∎

Here is a similar result with slightly different hypotheses:

THEOREM II.28. *Let $\mathbf{v}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be four points, no three of which are collinear (but coplanar). Let $\mathbf{a}' \in \mathbf{va}$, $\mathbf{b}' \in \mathbf{vb}$, and $\mathbf{c}' \in \mathbf{vc}$ be distinct from $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ such that $\mathbf{ab}||\mathbf{a}'\mathbf{b}'$ but $\mathbf{ac}$ and $\mathbf{a}'\mathbf{c}'$ meet in some point $\mathbf{x}$. Then $\mathbf{bc}$ and $\mathbf{b}'\mathbf{c}'$ also meet in some point $\mathbf{y}$ and we have $\mathbf{xy}||\mathbf{ab}$.*
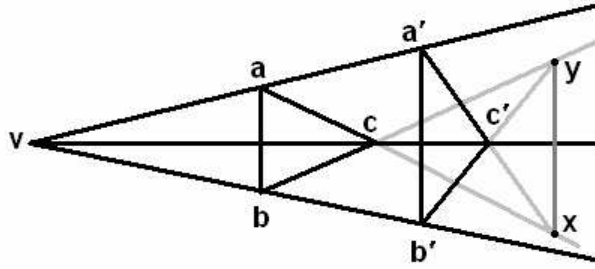

Figure II.7

Proof.   As in Theorem 27 we may write $\mathbf{a}' - \mathbf{v}' = t(\mathbf{a} - \mathbf{v})$ and $\mathbf{b}' - \mathbf{v}' = t(\mathbf{b} - \mathbf{v})$; however, $\mathbf{c}' - \mathbf{v}' = s(\mathbf{c} - \mathbf{v})$ for some $s \neq t$ (otherwise $\mathbf{ac}||\mathbf{a}'\mathbf{c}'$). Expressions for the point $\mathbf{x}$ may be computed starting with the equation

$$\mathbf{x} \; = \; r\mathbf{a} \; + \; (1 - r)\mathbf{c} \; = \; q\mathbf{a}' \; + \; (1 - q)\mathbf{c}'$$

which immediately implies that

$$\mathbf{x} - \mathbf{v} \; = \; r(\mathbf{a} - \mathbf{v}) + (1 - r)(\mathbf{c} - \mathbf{v}) \; = \; q(\mathbf{a}' - \mathbf{v}) + (1 - q)(\mathbf{c}' - \mathbf{v}) \; = \; qt(\mathbf{a} - \mathbf{v}) + (1 - q)s(\mathbf{c} - \mathbf{v}) \, .$$

Since $\mathbf{a} - \mathbf{v}$ and $\mathbf{c} - \mathbf{v}$ are linearly independent, we find that $r = qt$ and $1 - r = (1 - q)s$. These equations determine r completely as a function of s and t:

$$r(s, t) \; = \; \frac{t(1 - s)}{s - t}$$

A similar calculation shows that any common point to $\mathbf{bc}$ and $\mathbf{b}'\mathbf{c}'$ has the form

$$r(s, t)\mathbf{b} \; + \; \big(1 - r(s, t)\big)\,\mathbf{c}$$

and a reversal of the previous argument shows that this point is common to $\mathbf{bc}$ and $\mathbf{b}'\mathbf{c}'$. Therefore

$$\mathbf{y} - \mathbf{x} \; = \; r(s, t)\,(\mathbf{b} - \mathbf{a})$$

which shows that $\mathbf{xy}||\mathbf{ab}$.∎

There is a third result of a similar type; its proof is left as an exercise (we should note that all these results will be improved upon later).

THEOREM II.29. *In the notation of the theorems, assume that all three pairs of lines* $\{\,\mathbf{ab},\,\mathbf{a'b'}\,\}$, $\{\,\mathbf{ac},\,\mathbf{a'c'}\,\}$, *and* $\{\,\mathbf{bc},\,\mathbf{b'c'}\,\}$ *all have points of intersection. Then the three intersection points are collinear.*

EXERCISES

Definition.   Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in V$ be four ordered points, no three of which are collinear (with no parallelism assumptions). The union

$$\mathbf{ab} \ \cup \ \mathbf{bc} \ \cup \ \mathbf{cd} \ \cup \ \mathbf{da}$$

is called the *affine quadrilateral* determined by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, and as before we shall write it $\square\mathbf{abcd}$. The *diagonals* of the quadrilateral are the lines $\mathbf{ac}$ and $\mathbf{bd}$. The *sides* of the quadrilateral are four lines whose union forms the affine quadrilateral;

> *In the exercises below, assume that all vectors lie in the vector space* $\mathbb{F}^2$, *where* $\mathbb{F}$ *is a field in which* $1 + 1 \neq 0$.

1.   Prove that an affine quadrilateral is a parallelogram if and only if its diagonals bisect each other (in the sense of Theorem 26).

2.   Suppose we are given an affine parallelogram. Prove that a line joining the midpoints of a pair of parallel sides contains the intersection point of the diagonals.
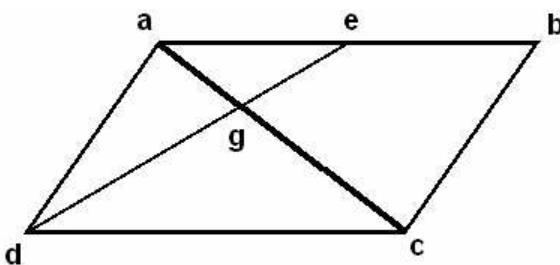

Figure II.8

3.   In the figure above, assume we are given a parallelogram $\square\mathbf{abcd}$ such that

$$\mathbf{e} \ = \ \mathsf{midpoint}(\mathbf{a}, \mathbf{b})$$

and $1 + 1 + 1 \neq 0$ in $\mathbb{F}$. Prove that

$$\mathbf{g} \ = \ \tfrac{1}{3}\mathbf{c} \ + \ \tfrac{2}{3}\mathbf{a} \ = \ \tfrac{1}{3}\mathbf{c} \ + \ \tfrac{2}{3}\mathbf{e} \ .$$

Definition.   An affine quadrilateral $\square\mathbf{abcd}$ is said to be an *affine trapezoid* if either $\mathbf{ab}||\mathbf{cd}$ or $\mathbf{bc}||\mathbf{ad}$ but not both (and generally the points are labeled so that the first is true). The two parallel sides are called the *bases*.

4.   Suppose we are given affine trapezoid □**abcd** as above with bases **ab** and **cd**. Prove that the line joining midpoint(**a**, **d**) and midpoint(**b**, **c**) is parallel to the bases.

5.   In the same setting as in the previous exercise, prove that the line joining midpoint(**a**, **c**) and midpoint(**b**, **d**) is parallel to the bases.

6.   In the same setting as in the previous exercises, prove that the line joining midpoint(**a**, **d**) and midpoint(**b**, **c**) is equal to the line joining midpoint(**a**, **c**) and midpoint(**b**, **d**).

7.   Prove Theorem 29. [*Hint:*   This can be done using the Theorem of Menelaus.]

8.   Suppose we are given an affine parallelogram **abcd**, and let **e** and **f** denote midpoint(**a**, **b**) and midpoint(**c**, **d**) respectively. Prove that **de** is parallel to **bf**.

9.   Let L_1, L_2 and L_3 be mutually parallel lines in the vector space $\mathbb{F}^2$, and let M_1 and M_2 be two lines which are not parallel to the L_i, so that each intersection $L_i \cap M_j$ consists of a single point. Define **a**, **b**, **c**, **x**, **y**, **z** so that the following hold:

$$\mathbf{a} \in M_1 \cap L_1 \qquad \mathbf{x} \in M_2 \cap L_1$$
$$\mathbf{b} \in M_1 \cap L_2 \qquad \mathbf{y} \in M_2 \cap L_2$$
$$\mathbf{c} \in M_1 \cap L_3 \qquad \mathbf{z} \in M_2 \cap L_3$$

Suppose that none of these points lie on both M_1 and M_2, and write $\mathbf{c} = bfa + t(\mathbf{b} - \mathbf{a})$ and $\mathbf{z} = bfx + t(\mathbf{y} - \mathbf{x})$ for suitable scalars t, $u \in \mathbb{F}$. Prove that $t = u$. [*Hint:*   The points **a**, **b** and **x** are noncollinear, and $L_1 = \mathbf{ax}$.]

10.   Suppose we are in the setting of the previous excercise, but now we assume that $\mathbf{a} = \mathbf{x}$ is a common point of M_1 and M_2. Prove that $t = u$ in this case too. [*Hint:*   In this case the points $\mathbf{a} = \mathbf{x}$, **b** and **y** are noncollinear, and $L_2 = \mathbf{by}$.]

> *Remark.*   The preceding exercises are abstract versions of the classical geometric result that *parallel lines determine proportional segments on two transversals.*

11.   Let **a**, **b** and **c** be noncollinear points, and define **d**, **e**, **f** by the equations $\mathbf{d} = \mathbf{b} + \mathbf{c} - \mathbf{a}$, $\mathbf{e} = \mathbf{a} + \mathbf{c} - \mathbf{b}$, and $\mathbf{f} = \mathbf{a} + \mathbf{b} - \mathbf{c}$. Prove the following:

(*i*) **a**, **b** and **c** are equal to midpoint(**d**, **e**), midpoint(**d**, **f**), and midpoint(**e**, **f**) respectively.

(*ii*) We have **bc** || **ef**, **ac** || **df**, and **ab** || **de**.

> In the next exercise, *assume that all vectors lie in a vector space V over a field $\mathbb{F}$ in which $1 + 1 = 0$; the most basic example is the field $\mathbb{Z}_2$ which has exactly two elements.*

12.   Suppose that $1 + 1 = 0$ in $\mathbb{F}$, and aside from this we are in the setting of Theorem II.26: Specifically, let **a**, **b**, **c**, $\mathbf{d} \in V$ be four noncollinear points such that the first three are noncollinear the four points are the vertices an affine parallelogram. Prove that in this case *the diagonal lines **ad** and **bc** are **parallel***. Is the converse also true? Give reasons for your answer.

5. Generalized geometrical incidence

> Our geometry is an abstract geometry. The reasoning could be followed by a disembodied
> spirit who had no understanding of a physical point, just as a man blind from birth could
> understand the Electromagnetic Theory of Light. — **H. G. Forder (1889–1981)**

Although we have not yet defined a geometrical incidence space of arbitrary dimension, it is clear
that the families of k-planes in $\mathbb{F}^n$ should define an n-dimensional incidence space structure on
$\mathbb{F}^n$. Given this, it is not difficult to guess what the correct definitions should be.

Definition.   A *geometrical incidence space* is a triple (S, Π, d) consisting of a set S, a family
of subsets Π the *geometrical subspaces*), and a function d from Π to the positive integers (the
dimension) satisfying the following conditions:

(G-1) : If $x_0 \cdots, x_n$ are distinct points of S such that no $P \in \Pi$ with $d(P) < n$ contains
them all, then there is a unique $P \in \Pi$ such that $d(P) = n$ and $x_i \in P$ for all i.

Notation.   We denote the geometrical subspace P in the preceding statement by $x_0 \cdots x_n$. The
condition on the x_i is expressed in the statement that *the set $\{ x_0 \cdots, x_n \}$ is (geometrically)*
independent.

(G-2) : If $P \in \Pi$ and $\{ x_0 \cdots, x_m \}$ is a set of geometrically independent points in P, then
the geometrical subspace $x_0 \cdots x_m$ is contained in P.

(G-3) : If $P \in \Pi$, then P contains at least $d(P) + 1$ points.

If $P \in \Pi$ and $d(P) = k$, then we shall say that P **is a** k-**plane**; the set of all k-planes is denoted
by Π_k. By convention, a 1-plane is often called a *line* and a 2-plane is often simply called a
plane. Note that the defining conditions do not give any *a priori* information about whether or
not there are any k-planes in S (however, if S contains at least two elements, it follows that Π_1
must be nonempty).

For most of the examples that we shall consider, the whole space S is one of the geometrical
subspaces. If this is the case and $\dim S = n$, then we shall say that the system (S, Π, d) is an
abstract n-dimensional geometrical incidence space. When $n = 2$ we shall also say the system is
an *abstract incidence plane*; we shall often shorten these terms by omitting the word "abstract."
— Also, if we are given a geometrical incidence space and the explicit data Π and d are either
clear from the context or are not really needed in a discussion, we shall often simply say that
"S is a geometrical incidence space."

EXAMPLE 1.   A three-dimensional incidence space (as defined above) is a geometrical incidence
space if we take $\Pi_1 = \mathcal{L}$, $\Pi_2 = \mathcal{P}$, and $\Pi_3 = \{S\}$.

EXAMPLE 2.   The **minimal** geometrical indicidence structures: Let Π be the family of all finite
subsets of a set S with at least two elements, and for each such subset Q let $d(Q) = \#(Q) - 1$,
where $\#(Q)$ is the number of elements in Q. The figure below illustrates the special case in
which S has three elements $\{\mathbf{a}, \ \mathbf{b}, \ \mathbf{c}\}$ and $d(S) = 2$.
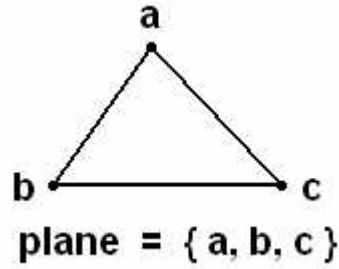
Figure II.9

EXAMPLE 3. An important class of examples mentioned at the beginning of this section: Let V be an n-dimensional vector space over a field $\mathbb{F}$ (where $n > 0$), and define the *affine incidence structure on V* such that for each positive integer k the set Π_k is the set of k-planes considered in earlier sections of this chapter. The properties **(G-1)**–**(G-3)** may be verified as follows:

Proof of **(G-1)**. If a set of vectors $\{\, \mathbf{v}_0, \;\cdots\;, \mathbf{v}_k \,\}$ is not geometrically independent, then the set is also affinely dependent, for if $\{\, \mathbf{v}_0, \;\cdots\;, \mathbf{v}_k \,\}$ is contained in a q-plane $\mathbf{y} + W$ for some $q < k$, then we have $\mathbf{y} + W = \mathbf{x}_0 + W$ and the k vectors $\mathbf{v}_1 - \mathbf{v}_0, \;\cdots\;, \mathbf{v}_k - \mathbf{v}_0 \in W$ must be linearly dependent because $\dim W < k$. Hence the original vectors are affinely dependent as claimed. Taking negations, we see that if $\{\, \mathbf{v}_0, \;\cdots\;, \mathbf{v}_k \,\}$ is geometrically independent, then the set is also affinely independent.

Let Q be the affine span of $\mathbf{v}_0, \;\cdots\;, \mathbf{v}_k$; then $Q = \mathbf{x}_0 + W$ where W is the linear span of $\mathbf{v}_1 - \mathbf{v}_0, \;\cdots\;, \mathbf{v}_k - \mathbf{v}_0$, and W is k-dimensional because these vectors are linearly independent. Therefore Q is a k-plane containing the vectors $\mathbf{v}_i$. Conversely, if Q' is an arbitrary k-plane containing the vectors $\mathbf{v}_i$, then we may write $Q' = \mathbf{v}_0 + U$ where U is a vector subspace of dimension k which contains the difference vectors $\mathbf{v}_1 - \mathbf{v}_0, \;\cdots\;, \mathbf{v}_k - \mathbf{v}_0$; it follows that U contains the k-dimensional vector subspace W described above, and since $\dim U = \dim W$ it follows that $W = U$, so that $Q = Q'$.∎

Proof of **(G-2)**. Since a k-plane is closed under forming affine combinations, if $\mathbf{v}_0, \;\cdots\;, \mathbf{v}_m$ is contained in P then the affine span of $\mathbf{v}_0, \;\cdots\;, \mathbf{v}_m$ is also contained in P.∎

Proof of **(G-3)**. Given a k-plane P, express it as $\mathbf{v}_0 + W$, where W is a k-dimensional vector subspace, and let $\mathbf{v}_1, \;\cdots\;, \mathbf{v}_k$ be a linear-algebraic basis for W. Then the set

$$\{\, \mathbf{v}_0, \; \mathbf{v}_0 + \mathbf{v}_1, \;\cdots\;, \mathbf{v}_k + \mathbf{v}_1 \,\}$$

forms an affine basis for P by Theorem 22. Hence P contains at least $(k + 1)$ points.∎

We have not yet described an analog of the axiom implying that two planes in 3-space intersect in a line if they are not disjoint. Before formulating an appropriate generalization of this,[8] we derive some consequences of **(G-1)**–**(G-3)**:

THEOREM II.30. *Let $x_0, \;\cdots\;, x_m$ be geometrically independent points in a geometrical incidence space S, and suppose $y \notin \{x_0, \;\cdots\;, x_m\}$. Then the set $\{x_0, \;\cdots\;, x_m, y\}$ is geometrically independent.*

---

[8]The definition follows the proof of Theorem 35 below.

Proof. If the points are not geometrically independent then for some $k \le m$ there is a k-plane P which contains all of them. Since $\{x_0, \cdots, x_m\}$ is geometrically independent, it follows that $d(P) \ge m$, so that $d(P) = m$ and $P = x_0 \cdots x_m$. But this contradicts the hypothesis.∎

THEOREM II.31. *Let P be a k-plane in the geometrical incidence space S. Then there is a set of independent points $\{x_0, \cdots, x_k\}$ such that $P = x_0 \cdots x_k$.*

Proof. Let $\mathcal{F}$ be the family of finite independent subsets of P. No element in this family contains more than $m + 1$ elements, for every subset with more points will be geometrically dependent. Let m be the largest integer for which some element of $\mathcal{F}$ has $m + 1$ elements; by the preceding sentence, we must have $m \le k$. We need to show that $m = k$, for in this case we know that P is the unique k-plane containing the given set of geometrically independent points and hence $P = x_0 \cdots x_k$.

Assume the contrary, so that $m < k$. Let $\{x_0, \cdots, x_m\} \in \mathcal{F}$. Then the m-plane $Q = x_0 \cdots x_m$ is contained in P by **(G-2)**, and $m < k$ implies that Q is a proper subset of P. Let x_{k+1} be a point which is in P but not in Q. Then Theorem 30 implies that $\{x_0, \cdots, x_k, y\}$ is geometrically independent. But this contradicts the defining condition for m, which is that there are no geometrically independent subsets of P with $m + 2$ elements. Therefore we must have $m = k$.∎

STANDARD CONVENTIONS. Given a geometrical incidence space (S, Π, d), we shall often say that the empty set is a geometrical subspace whose dimension is -1, each one point subset of S is a geometrical subspace whose dimension is zero, and subsets of both these types are geometrically independent. We shall say that ordinary geometrical subspaces, one point subsets and the empty set are *geometrical subspaces in the extended sense.* — Properties **(G-1)** – **(G-3)** of a geometrical incidence space remain valid if we extend the definition of geometrical subspace using these conventions, and the latter allow us to state the next result more simply.

THEOREM II.32. *Every finite subset X of the geometrical incidence space S (in the extended sense) contains a maximal independent subset $Y = \{y_0, \cdots, y_k\}$. Furthermore, X is contained in $y_0 \cdots y_k$, and the latter is the unique minimal geometrical subspace containing X.*

Henceforth, unless stated otherwise the term "geometrical subspace" will be taken to mean a geometrical subspace in the extended sense.

Proof. Let $Y \subset X$ be an independent subset with a maximum number of elements, let Q be the k-plane determined by Y, and let $w \in X$ be an arbitrary element not in Y. Since $y \notin Q$ would imply that $Y \cup \{y\}$ would be independent, it follows that $y \in Q$. Thus $X \subset Q$ as claimed. Suppose that Q' is another geometrical subspace containing X; then $Q \subset Q'$ by **(G-2)**, and hence every geometrical subspace that contains X must also contain Q.∎

When we work with vector subspaces of a vector space, it is often useful to deal with their intersections and sums. The next two results show that similar constructions hold for geometrical subspaces:

THEOREM II.33. *The intersection of a nonempty family of geometrical subspaces of S is a geometrical subspace (with the Standard Conventions for 0- and (-1)-dimensional subspaces preceding Theorem 32).*

Proof. Clearly it suffices to consider the case where the intersection contains at least two points. Let $\{\, P_\alpha \mid \alpha \in A \,\}$ be the family of subspaces, and let $\mathcal{F}$ be the set of all finite independent subsets of $\cap_\alpha P_\alpha$. As before, the number of elements in a member of $\mathcal{F}$ is at most $d(P_\alpha) + 1$ for all α. Thus there is a member of $\mathcal{F}$ with a maximal number of elements; call this subset $\{x_0, \cdots, x_k\}$. By **(G-2)** we know that $x_0 \cdots x_k$ is contained in each P_α and hence in the intersection $\cap_\alpha P_\alpha$. If it were a proper subset and y is a point in the intersection which does not belong to $x_0 \cdots x_k$, then $\{x_0 \cdots x_k, y\}$ would be an independent subset of the intersection with more elements than $\{x_0, \cdots, x_k\}$. This contradiction means that $x_0 \cdots x_k = \cap_\alpha P_\alpha$.∎

Although the union of two geometrical subspaces P and Q is usually not a geometrical subspace, the next result shows that there is always a minimal geometrical subspace containing both P and Q; this is analogous to the concept of sum for vector subspaces of a vector space.

THEOREM II.34. *If P and Q are geometrical subspaces of the geometrical incidence space S, then there is a unique minimal geometrical subspace containing them both.*

Proof. By Theorem 31 we may write $P = x_0 \cdots x_m$ and $Q = y_0 \cdots y_n$. Let $A = \{x_0, \cdots, x_m, y_0, \cdots, y_n\}$, and let T be the smallest subspace containing A (which exists by Theorem 32). Then $P, Q \subset T$, and if T' is an arbitrary geometrical subspace containing P and Q then it also contains A, so that T' must contain T as well.∎

The subspace given in the preceding result is called the **join** of P and Q, and it is denoted by $P \star Q$. Motivation for this definition is given by Exercise III.4.17 and Appendix B.

THEOREM II.35. *If P and Q are geometrical subspaces of S, then $d(P \star Q) \le d(P) + d(Q) - d(P \cap Q)$.*

It is easy to find examples of geometrical subspaces in $\mathbb{R}^n$ (even for $n = 2$ or 3) in which one has **strict** inequality. For example, suppose that L and M are parallel lines in $\mathbb{R}^3$; then the left hand side is equal to 2 but the right hand side is equal to 3 (recall that the empty set is (-1)-dimensional). Similarly, one has strict inequality in $\mathbb{R}^3$ if P is a plane and Q is a line or plane which is parallel to P.

Proof. Let $P \cap Q = x_0 \cdots x_m$. By a generalization of the argument proving Theorem 31 (see Exercise 1 below), there exist independent points $y_0, \cdots, y_p \in P$ and $z_0, \cdots, z_q \in Q$ such that

$$P \;=\; x_0 \cdots x_m y_0 \cdots y_p \,, \qquad Q \;=\; x_0 \cdots x_m z_0 \cdots z_q \,.$$

Let $X = \{x_0, \cdots, x_m, y_1, \cdots, y_p, z_1, \cdots, z_q\}$, and let T be the unique smallest geometrical subspace containing X. It is immediate that $P \subset T$ and $Q \subset T$, so that $P \star Q \subset T$. On the other hand, if a geometric subspace B contains $P \star Q$, then it automatically contains X and hence automatically contains T. Therefore we have $T = P \star Q$.

It follows from Theorem 32 that $d(P \star Q) = \dim S \le \#(X) + 1$, and therefore we have

$$\dim(P \star Q) \;\le\; m + 1 + p + q \;=\; (m+p+1) + (m+q+1) - (m+1) \;=$$
$$d(P) + d(Q) - d(P \cap Q)$$

which is what we wanted to prove.∎

The following definition contains an abstract version of the 3-dimensional incidence assumption about two planes intersecting in a line or not at all.

Definition. A geometrical incidence space is *regular* if the following holds:

> **(G-4) :** If P and Q are geometrical subspaces such that $P \cap Q \neq \varnothing$, then
> $$d(P \star Q) \;=\; d(P) \;+\; d(Q) \;-\; d(P \cap Q)\,.$$

EXAMPLE 1. Ordinary 3-dimensional incidences as defined in Section II.1 are regular. The only nontrivial case of the formula arises when P and Q are distinct planes, so that $P \star Q = S$.

EXAMPLE 2. The minimal examples at the beginning of this section are regular. The formula in this case follows immediately from the standard *inclusion-exclusion identity* for counting the elements in finite sets:
$$\#(P \cup Q) \;=\; \#(P) \;+\; \#(Q) \;-\; \#(P \cap Q)\,.$$

EXAMPLE 3. Here is an example which is **not** regular. Let $\mathbb{F}$ be a field, take the standard notions of lines and planes for $\mathbb{F}^4$, and let $d(\mathbb{F}^4) = 3$. Then the planes V and W spanned by $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{e}_3, \mathbf{e}_4\}$ have exactly one point in common.

Logical indpendence of the regularity condition. The preceding example shows that it is **not** possible to prove the assumption in the definition of regularity from the defining assumptions for a geometrical incidence space. — For if it were possible to prove the regularity condition from the definition, then it would NOT be possible to construct an example of a geometrical incidence space that did not satisfy the regularity condition.

The preceding remark illustrates the mathematical approach to concluding that one statement (say $\mathbf{Q}$) cannot be derived as a logical consequence of other statements (say $\mathbf{P}_1, \cdots, \mathbf{P}_n$): *It is only necessary to produce* **one example** *of an object which satisfies* $\mathbf{P}_1, \cdots, \mathbf{P}_n$ *but does not satisfy* $\mathbf{Q}$.

EXAMPLE 4. The incidence structure associated to a finite-dimensional vector space V over a field $\mathbb{F}$ is regular. To prove this, we first establish the following.

THEOREM II.36. *Let P and Q be affine subspaces of the incidence space structure associated to a vector space V, and assume $P \cap Q \neq \varnothing$. Write $P = \mathbf{x} + W$ and $Q = \mathbf{x} + U$ where $\mathbf{x} \in P \cap Q$. Then $P \star Q = \mathbf{x} + (W_1 + W_2)$.*

REMARK. Since $P \cap Q = \mathbf{x} + (W_1 \cap W_2)$ is readily established (see Exercise II.2.4), Theorem 36 and the dimension formula for vector subspaces (Theorem A.9) imply the regularity of V.∎

Proof. The inclusion $P \star Q \subset \mathbf{x} + (W_1 + W_2)$ is clear since the right hand side is a geometrical subspace containing P and Q. To see the reverse inclusion, first observe that $P \star Q = \mathbf{x} + U$ for some vector subspace U; since P, $Q \subset \mathbf{x} + U$ it follows that W_1, $W_2 \subset U$ and hence $W_1 + W_2 \subset U$. The latter yields $\mathbf{x} + (W_1 + W_2) \subset P \star Q$, and therefore the two subsets are equal. ∎

Finally, we introduce an assumption reflecting the Euclidean Parallel Postulate.

Definition. Two lines in a geometrical incidence space are *parallel* if they are coplanar but disjoint. A regular incidence space is said to be affine if given a line L and a point $\mathbf{x} \notin L$, then there is a unique line M such that $\mathbf{x} \in M$ and M is parallel to L. — If S is an affine incidence space and $d(S) = n$, then we say that it is an *affine n-space*. If $n = 2$, it is also called an *affine plane*.

One can use the argument proving Theorem 14 to verify that the affine incidence space associated to a vector space is affine in the sense defined above.

Equivalent mathematical structures

Our discussion of geometrical incidence fits closely with the main themes of these notes, but the formulation is definitely nonstandard. Normally such incidence structures are viewed in equivalent but more abstract terms. One approach involves specifying a sequence of *dependence relations* on finite subsets of a given set S; in this formulation, there is a class of such subsets that are called *independent* and satisfy a few suitable properties. As one might expect, a finite subset of $k+1$ points is said to be independent if and only if there is no k-dimensional subspace containing them; the resulting structure is called a **matroid**. Details of abstract dependence theory and matroids are described in the paper by H. Whitney[9] listed in the bibliography, and a more recent account of matroid theory is given in the following online reference:

http://home.gwu.edu/~jbonin/survey.ps

The matroid approach to independence leads naturally to another interpretation in terms of partially ordering relations on sets. Every matroid has a family of subsets which satisfy the conditions for a geometrical incidence space, and the associated ordered family of subsets satisfies the conditions for a partially ordered set to be a **geometric lattice**. A classic reference for the theory of lattices is the book by G. Birkhoff[10] (*Lattice Theory*) cited in the bibliography.

EXERCISES

1.   Let P be a geometrical subspace of S, and suppose that $\{x_0, \cdots, x_k\}$ is an independent subset of P. Prove that there is a (possibly empty) set of points $x_{k+1}, \cdots, x_m \in P$ such that $\{x_0, \cdots, x_m\}$ is independent and $P = x_0 \cdots x_m$. [*Hint:* Imitate the proof of Theorem 31 using the family $\mathcal{G} \subset \mathcal{F}$ of all subsets containing $\{x_0, \cdots, x_k\}$.]

2.   (*i*) Prove that a subset of an independent set of points in a geometrical incidence space is independent.

(*ii*) Suppose that P and Q are geometrical subspaces of a geometrical incidence space such that $P \subset Q$. Prove that $d(P) \leq d(Q)$, with equality if and only if $P = Q$. [*Hint:* For the first part, choose $\{x_0, \cdots, x_k\}$ such that $P = x_0 \cdots x_k$ and apply Exercise 1. For the second part, suppose that $P \subset Q$ and $d(P) = d(Q) = k$. Choose $\{x_0, \cdots, x_k\}$ such that $P = x_0 \cdots x_k$ and apply **(G-1)** to conclude that $P = Q$.]

3.   (*i*) Let (S, Π, d) be a geometrical incidence space, let $T \subset S$ be a geometrical subspace, and let Π_T be the set of all geometrical subspaces in Π which are contained in T. Prove that (S, Π_T, d_T) is a geometrical incidence space, where d_T is the restriction or T.

(*ii*) Prove that a geometrical subspace of a regular geometrical incidence space is again regular. Is the analog true for affine spaces? Give reasons for your answer.

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[9]HASSLER WHITNEY (1907–1989) did his earliest work in graph theory and related areas, but he his best known for his fundamental, innovative, and highly influential research in topology.

[10]GARRETT BIRKHOFF (1911–1996) is known for his work in abstract algebra, applied mathematics and numerical computation. Several of his books have been extremely influential.

4. Let S be a geometrical incidence space with $d(S) = n$. A *hyperplane* in S is an $(n-1)$-plane. Prove that for every $k < n$, every k-plane in S is an intersection of $(n-k)$ distinct hyperplanes. [*Hint:* If T is a k-plane and $T = x_0 \cdots x_k$, choose $y_1, \cdots, y_{n-k}$ such that $\{x_0 \cdots x_k, y_1, \cdots, y_{n-k}\}$ is independent. Let P_i be the hyperplane determined by all these points except y_j.]

5. (*i*) Prove that the join construction on geometrical subspaces of a geometrical incidence space satisfies the associativity condition $(P \star Q) \star R = P \star (Q \star R)$ for geometrical subspaces $P, Q, R \subset S$. [*Hint:* Show that both of the subspaces in the equation are equal to the smallest geometrical subspace containing $P \cup Q \cup R$.]

(*ii*) Prove that the join and intersection operations for subspaces satisfy the **absorption identities** $P \star (P \cap Q) = P = P \cap (P \star Q)$.

6. Show that the Euclidean parallelism property is not a logical consequence of the defining conditions for a regular n-dimensional geometrical incidence space. [*Hint:* Look at the so-called trivial examples from the previous section in which Π is the collection of all finite subsets and if P is finite then $d(P) = \#(P) - 1$. Why do standard counting formulas imply that every pair of coplanar lines in this system will have a common point?]

Remark. For affine planes and the examples considered in the previous section, the following modification of the Euclidean Parallelism Property is valid: *Given a line L and a point $\mathbf{x} \notin L$, then there is* **at most one** *line M such that $\mathbf{x} \in M$ and M is parallel to L.* It is also not possible to prove this from the conditions defining a regular incidence space, and the following *Beltrami-Klein incidence plane* is an example: Let P be the set of all points in $\mathbb{R}^2$ which are interior to the unit circle; this set is defined by the inequality $x^2 + y^2 < 1$ (see the figure below). The lines in P are taken to be the open chords of the unit circle with equation $x^2 + y^2 = 1$, or equivalently all **nonempty** intersections of ordinary lines in $\mathbb{R}^2$ with P. As the figure suggests, this example has the following property: *Given a line L and a point $\mathbf{x} \notin L$, then there are* **at least two** *lines M such that $\mathbf{x} \in M$ and M is parallel to L.* — In fact, there are *infinitely many* such lines.
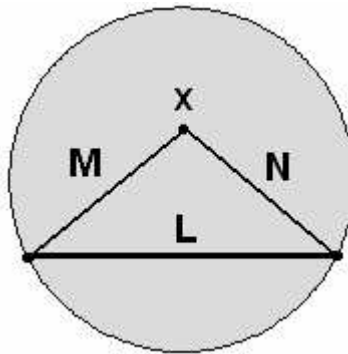

Figure II.10

The plane is the shaded region inside the circle, with the boundary excluded. The open chord L has no points in common with the open chords determined by M and N.

7. (*Further examples as above*) Define a *finitely supportable convex body* in $\mathbb{R}^n$ to be a nonempty set D defined by finitely many strict inequalities $f_i(\mathbf{x}) > 0$, where each f_i is a linear polynomial in the coordinates of $\mathbf{x}$. Let Π_D denote the set of all nonempty intersections $P \cap D$, where P is a geometrical subspace of $\mathbb{R}^n$, and set the dimension $d_D(P \cap D$ equal to $d(P)$. Prove

that this structure makes D into a regular geometrical incidence n-space. [*Hint:* Try to do this first for $n = 2$, then for $n = 3$.]

8.   Assume we are in the setting of the preceding exercise.

(i) Let D be the open square in $\mathbb{R}^2$ defined by $x > 0$, $y > 0$, $-x > -1$ and $-y > -1$ (hence the coordinates lie between 0 and 1). Prove that D has the parallelism properties that were stated but not proved for the Beltrami-Klein plane described above.

(ii) Let D be the upper half plane in $\mathbb{R}^2$ defined by $y > 0$. Prove that for some lines L and external points $\mathbf{x}$ there is a unique parallel to L through $\mathbf{x}$, but for others there are infinitely many such parallels.

9.   Let (S, Π, d) be a geometrical incidence space, and let U, V, W be geometrical subspaces such that $U \subset W$.

(i) Prove the *semimodularity property* $U \star (V \cap W) \subset (U \star V) \cap W$. [*Hint:* Show that the left hand side is contained in $U \star V$ and W.]

(ii) Describe positive dimensional geometrical subspaces U, V, W of $\mathbb{R}^3$ (with the usual 3-dimensonal incidence structure) such that $U \star (V \cap W)$ is properly contained in $(U \star V) \cap W$. [*Hint:* Take W to be a plane containing the line U, and let V be a line which is disjoing from W.]

(iii) In the setting of the problem, suppose that S is regular and $U \cap V$ is nonempty. Prove the *modularity property* $U \star (V \cap W) = (U \star V) \cap W$. [*Hint:* By ($i$) and part ($ii$) of Exercise 2, it suffices to prove that

$$d\bigl(U \star (V \cap W)\bigr) \;=\; d\bigl((U \star V) \cap W\bigr) .$$

Let Δ be equal to the difference of the right hand side and the left hand side, and use **(G-4)** repeatedly to show that $\Delta = 0$.]

6. Isomorphisms and automorphisms

Although the mathematical concepts of *isomorphism* and *automorphism* are stated abstractly, they reflect basic concepts that arise very quickly in elementary geometry. Congruent triangles are fundamental examples of isomorphic objects: The statement $\Delta\,ABC \cong \Delta\,DEF$ means that the obvious 1–1 correspondence of vertices from $\{A, B, C\}$ to $\{D, E, F\}$ preserves the basic structural measurements, so that the distances between the vertices satisfy

$$d(A, B) \;=\; d(D, E), \qquad d(B, C) \;=\; d(E, F), \qquad d(A, C) \;=\; d(D, F)$$

and the (degree or radian) measurements of the vertex angles satisfy similar equations:

$$\begin{aligned}
\text{measure}(\angle\,ABC) &\;=\; \text{measure}(\angle\,DEF) \\
\text{measure}(\angle\,ACB) &\;=\; \text{measure}(\angle\,DFE) \\
\text{measure}(\angle\,CAB) &\;=\; \text{measure}(\angle\,FDE)
\end{aligned}$$

Suppose now that we have an isosceles triangle $\Delta\,ABC$ in which $d(A, B) = d(A, C)$. In such cases one has a natural symmetry of the triangle with respect to the line joining A to the midpoint of B and C, and one aspect of this symmetry is a nontrivial congruence of the isosceles triangle with itself; specifically, $\Delta\,ABC \cong \Delta\,ACB$. In mathematics, an isomorphism from an object to itself is called an automorphism. The identity map from the object to itself is a trivial example of an automorphism, and the isosceles triangle example indicates that some structures may have other automorphisms.

The word "isomorphic" means "identical shape" or "same form," and in mathematics it means that one has a rule for passing between two objects that preserves all the mathematical structure that is essential in a given context. In particular, *if two objects are isomorphic, a statement about the properties or structure of the first object is true if and only if the corresponding statement about the second object is true* (for example, in Euclidean geometry if we know that $\Delta ABC \cong \Delta DEF$, then ΔABC is equilateratl if and only if ΔDEF is equilateral). Such principles can be extremely useful, especially when one of the objects is relatively easy to work with and the other is less so; one example of this is given by Theorem II.39 below).

The precise mathematical definitions of isomorphisms and automorphisms vary, and the details depend upon the sort of objects being considered. Also, there may be several different notions of isomorphism or automorphism, depending upon the amount of structure that remains unchanged. For example, if we are working with triangles, it is often useful to consider triangles that might not be congruent but are *similar*; in other words, one still has a 1–1 correspondence correspondence of vertices from $\{A, B, C\}$ to $\{D, E, F\}$ as before such that the angle measure equations are satisfied, but now we only know that there is a positive number r (the *ratio of similitude*) such that

$$d(A, B) \;=\; r \cdot d(D, E), \quad d(B, C) \;=\; r \cdot d(E, F), \quad d(A, C) \;=\; r \cdot d(D, F) \,.$$

Of course, congruence is the special case of similarity for which $r = 1$.

Here is the appropriate concept of isomorphism for geometrical incidence spaces:

Definition. Let (S, Π, d) and (S', Π', d') be geometrical incidence spaces. An *isomorphism* of geometrical incidence spaces from (S, Π, d) to (S', Π', d') is a 1–1 correspondence $f : S \to S'$ such that if $P \subset S$, then $P \in \Pi$ if and only if its image $f[P]$ belongs to Π', and in this case $d(P) = d'(\,f[P]\,)$. In other words, P is a k-plane in S if and only if $f[P]$ is a k-plane in S'.[11]

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[11]As noted in the Prerequisites, we use $f[P]$ to denote the image of P under the mapping f.

It is standard to write $A \cong B$ when two mathematical systems are isomorphic, and we shall do so throughout these notes.

The first theorem of this section implies that isomorphism of geometrical incidence spaces is an equivalence relation.

THEOREM II.37. (*i*) *If* $f : S \to S'$ *is an isomorphism of geometrical incidence spaces, then so is its inverse* $f^{-1} : S' \to S$.

(*ii*) *If* $f : S \to S'$ *and* $g : S' \to S''$ *are isomorphisms of geometrical incidence spaces, then so is their composite* $g \circ f : S \to S''$.

Proof.   (*i*) If f is 1–1 and onto, then it has an inverse map f^{-1} which is also 1–1 and onto. If $Q \subset S$ is a k-plane, then the identity $Q = f\left[\, f^{-1}[Q]\,\right]$ implies that $f^{-1}[Q]$ is a k-plane in S. Similarly, if $f^{-1}[Q]$ is a k-plane in S then so is $Q = f\left[\, f^{-1}[Q]\,\right]$.

(*ii*) If f and g are 1–1 onto, then so is $g \circ f$. If $P \subset S$ is a k-plane, then so is $f[P]$ because f is an isomorphism of geometrical incidence spaces, and since g is also an isomorphism of geometrical incidence spaces then

$$g \circ f[P] \;=\; g\big[\, f[P]\,\big]$$

is also a k-plane. Conversely, if the latter is a k-plane, then so is $f[P]$ since g is an isomorphism of geometrical incidence spaces, and therefore P is too because f is an isomorphism of geometrical incidence spaces.∎

The next result illustrates earlier assertions that isomorphisms preserve the basic properties of mathematical structures like abstract geometrical incidence spaces.

THEOREM II.38. *Let* $f : S \to S'$ *is an isomorphism of geometrical incidence spaces, and let* $X = \{\, \mathbf{x}_0, \, \cdots \, \mathbf{x}_m \,\}$ *be a finite subset of* S. *Then* X *is a geometrically independent subset of* S *if and only if* $f[X]$ *is a geometrically independent subset of* X'.

Proof.   Suppose that X is independent, and assume that $f[X]$ is not. Then there is some k-plane $Q \subset S'$ such that $d(P) < q$ and $f[X] \subset Q$. Let $P = f^{-1}[Q]$. By the definition of isomorphism, it follows that P is also a k-plane, where $k < q$, and $X = f^{-1}\big[\, f[X]\,\big]$ is contained in P. Therefore X is not independent, which contradicts our choice of X. Thus it follows that $f[X]$ is geometrically independent if X is. The converse statement follows by applying the preceding argument to the inverse isomorphism f^{-1}.∎

If $\mathcal{G}$ is a family of geometrical incidence spaces, then a *classifying family* for $\mathcal{G}$ is a subfamily $\mathcal{C}$ such that each object in $\mathcal{G}$ is isomorphic to a unique space in $\mathcal{C}$. The standard coordinate affine spaces $\mathbb{F}^n$ (where $\mathbb{F}$ is some field) comes relatively closed to being a classifying family for n-dimensional affine incidence spaces for $n \geq 3$. It is only necessary to add standard coordinate affine n-spaces over *skew-fields* (see the second paragraph of Appendix A). The only difference between fields and skew-fields is that multiplication is not necessarily commutative in the latter; the standard nontrivial example is given by the **quaternions**, which are discussed in Appendix A. — If $\mathbb{F}$ is a skew-field, let $\mathbb{F}^n$ be the **right** vector space of ordered n-tuples as defined in Appendix A. As indicated there, all of the linear algebra used to prove that $\mathbb{F}^n$ is an affine n-space goes through if $\mathbb{F}$ is a skew-field. In particular, the right vector space $\mathbb{F}^n$ is an affine n-space. The classification of n-dimensional affine spaces (where $n \geq 3$) is then expressible as follows:

THEOREM II.39. *Let $(S, etc.)$ be an affine n-space, where $n \geq 3$. Then there is a skew-field $\mathbb{F}$ such that S is isomorphic to $\mathbb{F}^n$ as a geometrical incidence space. Furthermore, if $\mathbb{E}$ ind $\mathbb{F}$ are skew-fields such that S is geometrically isomorphic to $\mathbb{E}^n$ and $\mathbb{F}^n$, then $\mathbb{E}$ and $\mathbb{F}$ are algebraically isomorphic.*

The proof of this result involves several concepts we have not yet introduced and is postponed to Remark 3 following the proof of Theorem IV.19. Specifically, it will be a fairly simple application of the projective coordinatization theorem (Theorem IV.18) to the synthetic projective extension of S (which is defined in the Addendum to Chapter 3).

Automorphisms and symmetry

A standard dictionary definition of the word *symmetry* is "regularity in form and arrangement," and the significance of automorphisms in mathematics is that they often yield important patterns of regularity in a mathematical system. This is apparent in our previous example involving isosceles triangles. Of course, the amount of regularity can vary; for example, an equilateral triangle has a more regular structure than, say, an isosceles right triangle. Generally speaking, the more regular an object is, the more automorphisms it has.

Definition.   An *automorphism* of a geometrical incidence space (S, Π, d) and (S', Π', d') is a 1–1 onto map from S to itself which is an isomorphism of geometrical incidence spaces. — The identity map of a mathematical object is normally an example of an automorphism, and it is a routine exercise to check that for each geometrical incidence space (S, Π, d), the identity map of S defines an automorphism of (S, Π, d).

The following result is an immediate consequence of Theorem 38 and the preceding sentence.

THEOREM II.40. *The set of all automorphisms of a geometrical incidence space (S, Π, d) forms a group with respect to composition of mappings.*∎

Notation.   This group is called the *geometric symmetry group* of (S, Π, d) and is denoted by $\mathbf{\Sigma}(S, \Pi, d)$.

EXAMPLE 1.   Let (S, Π, d) be the affine incidence space associated to a vector space V. If T is an invertible linear self-map of V, then T also defines an incidence space automorphism. This is true because the condition $\mathrm{Kernel}(T) = \{\mathbf{0}\}$ implies that T maps each k-dimensional vector subspace W to another subspace $T[W]$ of the same dimension (see Theorem A.14.(iv)). Furthermore, if S is an arbitrary subset and $\mathbf{x} \in V$, then it is easy to check that $T[\mathbf{x} + S] = T(\mathbf{x}) + T[S]$ (this is left to the reader as an exercise). .

EXAMPLE 2.   If V is as above and $\mathbf{x} \in V$, define the mapping $T_{\mathbf{x}} : V \to V$ (*translation by* $\mathbf{x}$) to be the mapping $T_{\mathbf{x}}(\mathbf{v}) = \mathbf{x} + \mathbf{v}$. Then $T_{\mathbf{x}}$ is clearly 1–1 and onto. Furthermore, it defines a geometrical space automorphism because $T_{\mathbf{x}}[\mathbf{y} + W] = (\mathbf{x} + \mathbf{y}) + W$ shows that P is a k-plane in V if and only if $T_{\mathbf{x}}[P]$ is (again, filling in the details is left to the reader).∎

The preceding observations show that affine incidence spaces associated to vector spaces generally have many geometric symmetries. Define the *affine group* of V — denoted by $\mathrm{AFF}(V)$ — to be all symmetries expressible as a composite $T \circ S$, where T is a translation and S is an invertible linear transformation. We shall say that the elements of $\mathrm{AFF}(V)$ are *affine transformations*. The

terminology suggests that the affine transformations form a subgroup, and we shall verify this in the next result.

THEOREM II.41. *The set* AFF(V) *is a subgroup of the group of all geometric symmetries of V. It contains the groups of linear automorphisms and translations as subgroups.*

Proof. First of all, the identity map is a linear transformation and it is also translation by $\mathbf{0}$, so clearly the identity belongs to AFF(V). If A is an arbitrary affine transformation, then for each $\mathbf{x} \in V$ we have $A(\mathbf{x}) = S(\mathbf{x}) + \mathbf{y}$, where $\mathbf{y} \in V$ and S is an invertible transformation. If A' is another such transformation, write $A'(\mathbf{x}) = S'(\mathbf{x}) + \mathbf{y}'$ similarly. Then we have

$$A' \circ A(\mathbf{x}) \;=\; A'\big(S(\mathbf{x}) + \mathbf{y} \big) \;=\;$$
$$S'\big((S(\mathbf{x}) + \mathbf{y}) \big) + \mathbf{y}' \;=\; S'S(\mathbf{x}) + \big(S'(\mathbf{y}) + \mathbf{y}' \big)$$

showing that $A' \circ A$ is also affine; hence AFF(V) is closed under multiplication. To see that this family is closed under taking inverses, let A be as above, and consider the transformation

$$B(\mathbf{x}) \;=\; S^{-1}(\mathbf{x}) \;-\; S^{-1}(\mathbf{y}) \;.$$

By the above formula, $B \circ A$ and $A \circ B$ are both equal to the identity, so that A^{-1} is the affine transformation B. The second statement in the theorem follows from Examples 1 and 2 above.∎

In general, AFF(V) is **not** the full group of geometric symmetries. For example, if $\mathbb{F} = \mathbb{C}$ (the complex numbers), the cooreinatewise conjugation map

$$\chi : \mathbb{C}^n \;\longrightarrow\; \mathbb{C}^n$$

taking $(z_1, \cdots, z_n)$ to $(\overline{z_1}, \cdots, \overline{z_n})$ is a geometrical symmetry of $\mathbb{C}^n$ which does not lie in AFF$(\mathbb{C}^n)$.[12] However, if $\mathbb{F}$ is the integers mod p (where p is prime), the rational numbers, or the real numbers, then AFF(V) does turn out to be the entire group of geometric symmetries of $\mathbb{F}^n$.

Historical note. Although the concept of geometric symmetry was not formulated explicitly until the early 19$^{\text{th}}$ century, special symmetries of $\mathbb{R}^n$ (for at least $n = 2, 3$) known as *rigid motions* or *isometries* are implicit in classical Greek geometry, particularly in attempts at "proof by superposition" (a rigid motion is defined to be a 1–1 correspondence T from $\mathbb{R}^n$ to itself that preserves distances (*i.e.*, $d(\mathbf{x}, \mathbf{y}) = d\big(T(\mathbf{x}), T(\mathbf{y}) \big)$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ — such maps also preserve angles).[13] More precisely, superposition phrases of the form, "Place figure A so that points B coincide with points C," may be interpreted as saying, "Find a rigid motion of $\mathbb{R}^n$ that maps points B to points C." Indeed, it seems that the inability of classical geometry to justify the notion of superposition resulted from the lack of a precise definition for rigid motions.

EXERCISES

1. Give a detailed verification of the assertion $T[\mathbf{x} + S] = T(\mathbf{x}) + T[S]$ which appears in Example 1 above.

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[12]See Exercise V.2.5 for a full explanation.

[13]The fact that rigid motions are geometrical symmetries follows because they all lie in the group AFF(V) defined previously. A proof is given in the Addendum to Appendix A; see Theorem A.23.

2. If V and W are vector spaces over a field $\mathbb{F}$, a map $T : V \to W$ is called *affine* if it has the form $T(\mathbf{v}) = S(\mathbf{v}) + \mathbf{w}_0$, where S is linear and $\mathbf{w}_0 \in W$. Prove that

$$T\big(t\mathbf{x} + (1-t)\mathbf{y}\big) = tT(\mathbf{x}) + (1-t)T(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in V$ and $t \in \mathbb{F}$.

3. Suppose that $\{\mathbf{v}_0, \cdots, \mathbf{v}_n\}$ and $\{\mathbf{w}_0, \cdots, \mathbf{w}_n\}$ are affine bases for the vector space V. Prove that there is a unique $T \in \mathrm{AFF}(V)$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 0, \cdots, n$. [*Hint:* The sets $\{\mathbf{v}_1 - \mathbf{v}_0, \cdots, \mathbf{v}_n - \mathbf{v}_0\}$ and $\{\mathbf{w}_1 - \mathbf{w}_0, \cdots, \mathbf{w}_n - \mathbf{w}_0\}$ are bases. Apply Theorem A.13(v).]

4. Suppose $\mathbb{F} \neq \mathbb{Z}_2$ is a field, V is a vector space over $\mathbb{F}$, and $T : V \to V$ is a 1–1 onto map satisfying the condition in Exercise 2:

$$T\big(t\mathbf{x} + (1-t)\mathbf{y}\big) = tT(\mathbf{x}) + (1-t)T(\mathbf{y})$$

Prove that T is an affine transformation. [*Hint:* Set $S(\mathbf{v}) = T(\mathbf{v}) - T(\mathbf{0})$ and prove S is linear. Observe that S maps $\mathbf{0}$ to itself. — If the problem seems too hard as stated, try to prove if for fields in which $1 + 1 \neq 0$.]

5. Let $f : (S, \Pi, d) \to (S', \Pi', d')$ be an isomorphism of geometrical incidence spaces, and let P and Q be nonempty geometrical subspaces of S in the extended sense (*i.e.*, we include the possibility that either consists of one point). If "$\star$" is the join construction described in the notes, prove that $f[P \star Q] = f[P] \star f[Q]$. [*Hint:* Recall that the join of two geometrical subspaces is the unique smallest geometrical subspace containing both of them.]

6. Let $f : (S, \Pi, d) \to (S', \Pi', d')$ be an isomorphism of geometrical incidence spaces.

(*i*) Prove that (S, Π, d) is regular if and only if (S', Π', d') is.

(*ii*) Prove that (S, Π, d) is an affine n-space if and only if (S', Π', d') is.

(*iii*) Let $m > 2$ be an integer. Prove that every line in (S, Π, d) contains exactly m points if and only if the same is true for (S', Π', d'). Explain why a similar conclusion holds if we substitute "infinitely many" for "exactly m" in the statement.

7. Let V be a vector space over a field of characteristic zero (*i.e.*, no iterated finite sum $1 + \cdots + 1$ is equal to zero), and let $T \in \mathrm{AFF}(V)$ be an element such that some finite power $T^n = T \circ \cdots \circ T$ is the identity. Prove that there is some point $\mathbf{x} \in V$ such that $T(\mathbf{x}) = \mathbf{x}$ (*i.e.*, a **fixed point**). [*Hints:* Let $\mathbf{v} \in V$ be arbitrary, and consider the vector

$$\frac{1}{n} \sum_{i=0}^{n-1} T^i(\mathbf{v}) \, .$$

Note that the transformation T may have more than one fixed point; for example, if we take the reflection map T on $\mathbb{R}^2$ which sends (x, y) to $(x, -y)$, then T^2 is the identity and T sends every point on the x-axis to itself.]