APPENDIX B

THE JOIN IN AFFINE GEOMETRY

In Section II.5 we defined a notion of **join** for geometrical incidence spaces; specifically, if P and Q are geometrical subspaces of an incidence space S, then the join $P \star Q$ is the unique smallest geometrical subspace which contains them both. From an intuitive viewpoint, the name "join" is meant to suggest that $P \star Q$ consists of all points on lines of the form $\mathbf{xy}$, where $\mathbf{x} \in P$ and $\mathbf{y} \in Q$. If S is a projective n-space over some appropriate scalars $\mathbb{F}$, this is shown in Exercise 16 for Section III.4, and the purpose of this Appendix is to prove a similar result for an affine n-space over some $\mathbb{F}$.

Formally, we begin with a generalization of the idea described above.

Definition. Let (S, Π, d) be an abstract geometrical incidence n-space, and let $X \subset S$. Define $\mathbf{J}(X)$ to be the set

$$X \;\cup\; \{\; \mathbf{y} \in S \mid \mathbf{y} \in \mathbf{uv} \text{ for some } \mathbf{u},\; \mathbf{v} \in X \;\} \;.$$

Thus $\mathbf{J}(X)$ is X together with all points on lines joining two points of X. Note that the construction of $\mathbf{J}(X)$ from X can be iterated to yield a chain of subsets $X \subset \mathbf{J}(X) \subset \mathbf{J}\big(\mathbf{J}(X)\big) \;\cdots$. If X is a geometrical subspace of S, then the axioms for a geometrical incidence space imply that $\mathbf{J}(X) = X$, and by Theorem II.16 and Exercise II.2.1, a subset X of $\mathbb{F}^n$ satisfies $\mathbf{J}(X) = X$ if and only if X is an affine subspace V of $\mathbb{F}^n$, provided $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_2$.

The preceding discussion and definition lead naturally to the following:

QUESTION. *If S is a geometrical incidence n-space and P and Q are geometrical subspaces of S, what is the relationship between $P \star Q$ and $\mathbf{J}(P \cup Q)$? In particular, are they equal, at least if S satisfies some standard additional conditions?*

The exercise from Section III.4 shows that the two sets are equal if S is a standard projective n-space. In general, the next result implies that the two subsets need not be equal. but one is always contained in the other.

Theorem B.1. *In the setting above, we have $\mathbf{J}(P \cup Q) \subset P \star Q$. However, for each $n \geq 2$ there is an example of a* regular *geometrical incidence n-space such that, for some choices of P and Q, the set $\mathbf{J}(P \cup Q)$ is strictly contained in $P \star Q$.*

Proof. The inclusion relationship follows from **(G-2)** and the fact that $P \star Q$ is a geometrical subspace of S. On the other hand, if we take the affine incidence space structure associated to $\mathbb{Z}_2^n$ for $n \geq 2$, then for every subset $X \subset \mathbb{Z}^n$ we automatically have $\mathbf{J}(X) = X$ because every line consists of exactly two points. Thus if W and U are vector subspaces of $\mathbb{Z}_2^n$ such that neither contains the other, then $\mathbf{J}(W \cup U)$ is not a vector subspace. Since $\mathbf{0} \in W \cap U$, we know that $W \star U$ is the vector subspace $W + U$ by Theorem II.36, and it follows in this case that $\mathbf{J}(W \cup U)$ is strictly contained in $W \star U$.■

Additional examples of regular indicence spaces for which $\mathbf{J}(P \cup Q)$ is strictly contained in $P \star Q$ may be constructed using Exercise II.5.7 in the notes. Specifically, let $D \subset \mathbb{R}^{n+1}$ be the set of all points $(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R} \cong \mathbb{R}^{n+1}$ such that $-1 < x_i < 1$ for all i, where $\mathbf{x} = (x_1, \;\cdots\;, x_n)$, so that D is the solid tower for which the base B is the open hypercube defined by the inequalities $|x_i| < 1$ and the tower is unbounded both above and below. By the previously cited exercise we know that D is a regular geometrical incidence $(n+1)$-space if we define k-planes to be all nonempty intersections $P \cap D$, where P is a k-plane in $\mathbb{R}^{n+1}$. Furthermore, if S is $B \times \{0\}$ and T is given by the point $(\mathbf{0}, 1)$, then $S \star T = D$ but $\mathbf{J}(S \cup T)$ is a proper subset. For example, if $\mathbf{x} \in B$, $0 < t < 1$ and t is greater than all of the coordinate absolute values $|x_i|$, then $(\mathbf{0}, t)$ does not lie in $\mathbf{J}(S \cup T)$; if $n = 1$ this can be seen directly (try drawing a picture for motivation!) and one can extend everything directly to higher values of n. One can also construct many other such examples, but we shall stop here.

Note that the examples constructed in the proof of Theorem 1 are in fact *affine* incidence spaces. The main objective of this appendix is to prove that $\mathbf{J}(P \cup Q) = P \star Q$ if V is a vector space of dimension ≥ 2 over a field $\mathbb{F}$ which is not (isomorphic to) $\mathbb{Z}_2$.

Theorem B.2. *Let V be a vector space of dimension ≥ 2 over a field $\mathbb{F}$ which is not (isomorphic to) $\mathbb{Z}_2$, and suppose that $P = \mathbf{a} + U$ and $Q = \mathbf{b} + W$ are geometrical subspaces of V. Then the following hold:*

(i) The join $P \star Q$ is the affine span of $P \cup Q$.

(ii) $P \star Q = \mathbf{J}(P \cup Q)$.

Proof. FIRST STATEMENT. If R is the affine span of P and Q, then R is an affine subspace containing P and Q by Theorem II.19, Theorem II.16 and Exercise 1 for Section II.2 (this is where we use the assumption that $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_2$). Therefore it follows that R also contains $P \star Q$. On the other hand, if R' is a geometrical subspace containing P and Q, then by Theorem II.18 it contains all affine combinations of points in $P \cup Q$, and hence R' must contain R. Combining these observations, we conclude that R must be equal to $P \star Q$.

SECOND STATEMENT. By the previous theorem we know that $\mathbf{J}(P \cup Q) \subset P \star Q$, so it suffices to show that we also have the converse inclusion $P \star Q \subset \mathbf{J}(P \cup Q)$.

Let $\mathbf{x} \in P \star Q$, and let $\{\mathbf{d}_0, \cdots, \mathbf{d}_p\}$ and $\{\mathbf{c}_0, \cdots, \mathbf{c}_q\}$ be affine bases for P and Q respectively. Then by the conclusion of the first part of the theorem we may write

$$\mathbf{x} \;=\; \sum_{i=0}^{p} r_i \mathbf{d}_i \;+\; \sum_{j=0}^{p} s_j \mathbf{c}_j$$

where $\sum_i r_i + \sum_j s_j = 1$. Let $t = \sum_i r_i$, so that $\sum_j s_j = 1 - t$. There are now two cases, depending upon whether either or neither of the numbers t and $1-t$ is equal to zero. If $t = 0$ or $1 - t = 0$ (hence $t = 1$), then we have $\mathbf{x} \in P \cup Q$. Suppose now that both t and $1-t$ are nonzero. If we set

$$\alpha \;=\; \sum_{i=0}^{p} \frac{r_i}{t} \cdot \mathbf{d}_i \qquad\qquad \beta \;=\; \sum_{j=0}^{q} \frac{s_j}{(1-t)} \cdot \mathbf{c}_j$$

then $\alpha \in P$, $\beta \in Q$, and $\mathbf{x} = t\,\alpha + (1-t)\,\beta$; therefore it follows that $\mathbf{x} \in \mathbf{J}(P \cup Q)$.■