TOPOLOGICAL CLASSIFICATIONS OF HYPERQUADRICS

Fundamental results in affine and projective geometry imply that every real or complex hyperquadric in n-space is affinely or projectively equivalent to an example from a finite and reasonably short list of examples. Detailed proofs are given in the following documents:

`http://math.ucr.edu/~res/progeom/pgnotes07.pdf`

`http://math.ucr.edu/~res/progeom/quadrics1.pdf`

One natural and closely related question involves the classification of such hyperquadrics up to homeomorphism or some closely related notion of equivalence. As indicated at the end of the document `quadrics1.pdf`, it is not difficult to find (affine) quadric surfaces Σ_1 and Σ_2 in $\mathbb{R}^3$ such that Σ_1 and Σ_2 are not affinely equivalent but there is a diffeomorphism φ from $\mathbb{R}^3$ to itself which maps Σ_1 onto Σ_2. Our purpose here is to give systematic classifications for hyperquadrics in the four basic cases:

Affine hyperquadrics in $\mathbb{R}^n$.

Projective hyperquadrics in $\mathbb{RP}^n$.

Affine hyperquadrics in $\mathbb{C}^n$.

Projective hyperquadrics in $\mathbb{CP}^n$.

We shall freely use background material on affine and projective geometry from the files in the directory

`http://math.ucr.edu/~res/progeom`

and we shall also freely use basic material in algebraic topology that can be found in nearly every standard, comprehensive algebraic topology textbook. We shall also used the concept of diffeomorphism, which can be found in most standard graduate (or advanced undergraduate) level textbooks on smooth manifolds or real analysis.

1. Affine hyperquadrics in $\mathbb{R}^n$

It will be convenient to define a systematic indexing for the quadratic equations listed in `quadrics1.pdf`.

$$(\mathbf{I.}p\,.r)\quad x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2 = 0 \quad (1 \le p \le r \le n, r \ge 1, p \ge \tfrac{1}{2}r)$$

$$(\mathbf{II.}p\,.r)\quad x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2 + 1 = 0 \quad (0 \le p \le r \le n, r \ge 1)$$

$$(\mathbf{III.}p\,.r)\quad x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2 + x_{r+1} = 0 \quad (1 \le p \le r < n, r \ge 1, p \ge \tfrac{1}{2}r)$$

The Roman numeral in the first position will be called the **type** of the defining equation.

Given $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, for the first two cases types shall let $\mathbf{u} = (u_1, \cdots, x_p)$ denote the first p coordinates, we shall also let $\mathbf{v} = (v_{p+1}, \cdots, v_r)$ denote the next $r-p$ coordinates, and finally we shall let $\mathbf{w} = (w_{r+1}, \cdots, w_n)$ denote the last $n-r$ coordinates. For the third type, we modify the notation slightly by taking $t = x_{r+1}$ and letting $\mathbf{w}$ denote the last $n-r-1$ coordinates. In each case the number of coordinates of a given type may be zero (provided this is consistent with the conditions on indices given above).

The topological classification results are easiest for Type III and most difficult for Type I, so we shall study the types in reverse order.

Type III. In this case, the quadratic hypersurface is the graph of a second degree polynomial in $n-1$ variables; specifically,

$$t \;=\; |\mathbf{u}|^2 \;-\; |\mathbf{v}|^2\ .$$

Therefore the quadric hypersurfaces of these types are all homeomorphic — in fact, diffeomorphic — to $\mathbb{R}^{n-1}$.

Type II. In this case we may write the defining equation as

$$|\mathbf{u}|^2 \;-\; |\mathbf{v}|^2 \;+\; 1 \;=\; 0$$

or equivalently $|\mathbf{u}|^2 - |\mathbf{v}|^2 \;=\; 1$. There are two cases, depending upon whether $p > 0$ or $p = 0$ (we exclude the case $p = r$ because the hyperquadric defined for this choice is empty). If $p = 0$ then the equation becomes $|\mathbf{v}|^2 = 1$ and the hyperquadric is given by $S^{r-1} \times \mathbb{R}^{n-r}$. On the other hand, if $p > 0$ then there is a diffeomorphism h from the hyperquadric Σ to $S^{r-p-1} \times \mathbb{R}^{n-r+p} \cong S^{r-p-1} \times \mathbb{R}^p \times \mathbb{R}^{n-r}$ given by

$$h(\mathbf{u},\mathbf{v},\mathbf{w}) \;=\; \left(\frac{1}{\sqrt{1+|\mathbf{u}|^2}},\, \mathbf{v},\, \mathbf{u},\, \mathbf{w}\right)\ .$$

It follows that the hyperquadrics defined by equations $(\mathbf{II.}p\,.r)$ and $(\mathbf{II.}p'\,.r')$ are homeomorphic if and only if $p - r = p' - r'$, in which case they are diffeomorphic.

Digression — cohomology of a space at a point

The affine hyperquadrics of Types II and III are topological $(n-1)$-manifolds, and in fact they are smooth hypersurfaces in $\mathbb{R}^n$. In contrast, the hyperquadrics of Type I are not topological manifolds, and our analysis of their homeomorphism types depends upon having a way of recognizing that certain spaces are **NOT** topological manifolds. We shall do this using local cohomology groups of a space at a point. The basic idea goes back at least to the classic textbook of Seifert and Threlfall (see pages 123–124 in the English translation, *Textbook of Topology*).

Definition. Let X be a topological space, and let $x \in X$. Then the *(local) cohomology of X at x* is given by

$$H^*(X, X - \{x\})$$

where we generally take singular cohomology with coefficients in the integers (the default choice) or some field.

One advantage of using singular cohomology is the following localization result:

LEMMA 1. *Suppose we are using singular cohomology and that X is a Hausdorff space with an open subset U such that x lies in U. Then the restriction map from $H^*(X, X - \{x\})$ to $H^*(U, U - \{x\})$ is an isomorphism.*

The proof applies the excision property of singular theory the decomposition $X = (X - \{x\}) \cup U$.■

We also have the following topological invariance property:

LEMMA 2. *Let X and Y be topological spaces, and suppose that $h : X \to Y$ is a homeomorphism such that $h(x) = y$. Then f defines an isomorphism from $H^*(Y, Y - \{y\})$ to $H^*(X, X - \{x\})$.*■

Given a compact metric space X, the *open cone* $\mathbf{C}^{\,\mathrm{open}}(X)$ on X is defined to be the quotient of $X \times [0, \infty)$ obtained by collapsing the closed subspace $X \times \{0\}$ to a point. It is straightforward exercises to prove that $\mathbf{C}^{\,\mathrm{open}}(X)$ is homeomorphic to a separable metric space (we can isometrically embed X in a normed vector space $\mathcal{W}$ and use this to embed $\mathbf{C}^{\,\mathrm{open}}(X)$ into the product space $\mathcal{W} \times \mathbb{R}$). The point of $\mathbf{C}^{\,\mathrm{open}}(X)$ corresponding to the subset $X \times \{0\}$ will frequently be called the *vertex point* of the cone and denoted by $\mathbf{e}$.

We shall need the following basic fact:

THEOREM. *Suppose that X is a compact metric space, let $\{\mathbf{e}\}$ be the vertex point of $\mathbf{C}^{\,\mathrm{open}}(X)$, and let $\mathbf{w} \in \mathbb{R}^q$ be arbitrary. Then $H^*(\mathbf{C}^{\,\mathrm{open}}(X), \mathbf{C}^{\,\mathrm{open}}(X) - \{\mathbf{e}\})$ is isomorphic to $\widetilde{H^{*-1}}(X)$ and*

$$H^*(\mathbf{C}^{\,\mathrm{open}}(X) \times \mathbb{R}^{n-r},\ \mathbf{C}^{\,\mathrm{open}}(X) \times \mathbb{R}^{n-r} - \{(\mathbf{e}, \mathbf{w})\}) \quad \cong \quad \widetilde{H^{*-q}}(X)\ .$$

COROLLARY. *In the setting of the preceding result, suppose that M is a topological $(m-1)$-manifold such that $H^*(M)$ is not isomorphic to $H^*(S^{m-1})$. Let U be an open neighborhood of $(\mathbf{e}, \mathbf{w})$ in $\mathbf{C}^{\,\mathrm{open}}(X) \times \mathbb{R}^q$, where $\mathbf{e}$ is the vertex of the cone. Then U is not a topological $(m+q)$-manifold.*

COROLLARY. *In the situation of the preceding corollary, the subset $(\,\mathbf{C}^{\,\mathrm{open}}(X) - \{(\mathbf{e}\}\,) \times \mathbb{R}^q$ is an open dense subset, and it contains every open subset V which is a topological $(m+q)$-manifold.*

The crucial observation behind these corollaries is that if N is a topological d-manifold then we have an isomorphism $H^*(N, N - \{p\}) \cong \widetilde{H^*}(S^d)$.

Application to Type **I** *hyperquadrics*

We begin by showing that the hyperquadrics of this type are open cones on manifolds. Let Σ be a hyperquadric defined by an equation of type $(\mathbf{I}.p\,.r)$; by definition we have $p \geq r - p$, and we shall exclude the case where $r = p$ because $\Sigma = \{\mathbf{0}\}$ when this happens. In the more substantial cases where $r > p$ and $p > 0$, there is a homeomorphism

$$h : \mathbf{C}^{\,\mathrm{open}}\left(S^{p-1} \times S^{r-p-1}\right) \times \mathbb{R}^{n-r} \longrightarrow \Sigma$$

sending $\big([t; \mathbf{x}, \mathbf{y}], \mathbf{w}\big)$ to $(t\,\mathbf{x},\ t\,\mathbf{y},\ t\,\mathbf{w})$ in Σ, where the later is viewed as a subset of $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{r-p} \times \mathbb{R}^{n-r}$. Furthermore, if $\mathbf{e}$ denotes the cone vertex as before, then h maps the complement of $\{\mathbf{e}\} \times \mathbb{R}^{n-r}$ diffeomorphically to the set of all points $(\mathbf{u},\ \mathbf{v},\ \mathbf{w})$ in Σ such that $\mathbf{u}$ and $\mathbf{v}$ are both nonzero.

All spaces of the form $\mathbf{C}^{\,\mathrm{open}}(X)$ are contractible; specifically, a contracting homotopy to the vertex point $\mathbf{e}$ is given by

$$D\big(\,s, [t, x]\,\big) \quad = \quad [(1-s)t, x]\ .$$

It follows that all hyperquadrics of Type I are contractible and hence these spaces cannot be distinguished topologically by their cohomology groups. However, we can do this using the local cohomology groups which have been defined.

THEOREM. *Let Σ be the hyperquadric defined by the equation $(\mathbf{I}.p\,.r)$, and let $\mathcal{N}(\Sigma)$ be the unique maximal open dense subset which is a topological $(n-1)$-manifold. Then $\mathcal{N}(\Sigma)$ is homeomorphic to $S^{p-1} \times S^{p-r-1} \times \mathbb{R}^{n-r+1}$.*

This is true because the complement of $\mathcal{N}(\Sigma)$ is corresponds to $\{\mathbf{e}\} \times \mathbb{R}^{n-r}$ under the homeomorphism between Σ to $\mathbf{C}^{\,\mathrm{open}}(S^{p-1} \times S^{r-p-1}) \times \mathbb{R}^{n-r}$.

We can now finish the classification of affine hyperquadrics.

Type I. Suppose that the hyperquadrics Σ and Σ' defined by $(\mathbf{I}.p\,.r)$ and $(\mathbf{I}.p'\,.r')$ are homeomorphic. Then $\mathcal{N}(\Sigma)$ and $\mathcal{N}(\Sigma')$ are also homeomorphic, which implies an isomorphism between the homology groups of $S^{p-1} \times S^{r-p-1}$ and $S^{p'-1} \times S^{r'-p'-1}$. The highest dimensions with nontrivial cohomology for these spaces are $r-1$ and $r'-1$ respectively, so we must have $r = r'$. Similarly, the lowest dimensions with nontrivial cohomology are $r-p-1$ and $r'-p'-1$ respectively, so these numbers must also be equal. Combining this with $r = r'$, we see that $p = p'$. Therefore different equations of Type I yield nonhomeomorphic hyperquadrics.

Now that we have classified hyperquadrics defined by equations of the same type, we need to finish things by considering cases where the equations have different types. The following summarizing result does this and restates our previous conclusions.

THE COMPLETE CLASSIFICATION. (0) *Different types of equations yield nonhomeomorphic hyperquadrics.*

(1) *The hyperquadrics defined by* $(\mathbf{I}.p\,.r)$ *and* $(\mathbf{I}.p'\,.r')$ *are homeomorphic if and only if* $p = p'$ *and* $r = r'$.

(2) *The hyperquadrics defined by* $(\mathbf{II}.p\,.r)$ *and* $(\mathbf{II}.p'\,.r')$ *are homeomorphic if and only if* $r - p = r' - p'$.

(3) *The hyperquadrics defined by* $(\mathbf{III}.p\,.r)$ *and* $(\mathbf{III}.p'\,.r')$ *are all homeomorphic to each other.*

We have verified everything except assertion (0). To see that no hyperquadric defined by an equation of Type I is not homeomorphic to a hyperquadric defined by an equation of another type, it suffices to note that the hyperquadrics of the other types are topological $(n-1)$-manifolds while a hyperquadric defined by a Type I equation is not a topological $(n-1)$-manifold. To see that the homeomorphism types from Type II and III equations are mutually exclusive, note that a hyperquadric defined by an equation of Type III is homeomorphic to $\mathbb{R}^{n-1}$ while a hyperquadric defined by an equation of Type II is homotopy equivalent to a sphere, so hyperquadrics defined by equations of the two types have nonisomorphic cohomology groups. This completes the proof that different types of equations define nonhomeomorphic hyperquadrics.∎

In fact, one has a slightly stronger conclusion.

COMPLEMENT. *If two hyperquadrics* Σ *and* Σ' *in* $\mathbb{R}^n$ *are homeomorphic, then there is a diffeomorphism* φ *of* $\mathbb{R}^n$ *such that* $\varphi[\Sigma] = \Sigma'$.

Proof. We begin with a general observation: If $\mathbf{T}$ is an orthogonal transformation (hence diffeomorphism) which permutes the coordinates of $\mathbb{R}^n$ and Σ_0 is a hyperquadric, then Σ_0 and $\mathbf{T}[\Sigma_0]$ are in fact affinely equivalent. Therefore it will suffice to prove the assertion in the Complement for hyperquadrics obtained from Σ and Σ' by permuting coordinates. Also, we might as well assume that both Σ and Σ' are defined by equations in our standard lists of examples.

As usual, we need to consider the different equation types separately. If Σ and Σ' are both defined by Type III equations, then we may reorder the coordinates so that Σ and Σ' are respectively given by graphs $x_n = f(x_1, \cdots, x_{n-1})$ and $x_n = g(x_1, \cdots, x_{n-1})$, where f and g are homogeneous quadratic polynomials. We can now define a diffeomorphism φ on $\mathbb{R}^n \cong \mathbb{R}^{n-1} \times \mathbb{R}$ by the formula

$$\varphi(\mathbf{y}, t) \quad = \quad \big(\,\mathbf{y},\, t + g(\mathbf{y}) - f(\mathbf{y})\,\big)\ .$$

It follows immediately that φ maps Σ onto Σ'.

Now assume we are given two hyperquadrics defined by Type II equations. This means that, up to permutation of coordinates, there is a splitting of $\mathbb{R}^n$ into four pieces, with a corresponding decomposition of vectors in $\mathbb{R}^n$ into ordered quadruples $(\mathbf{u}, \mathbf{v}, \mathbf{y}, \mathbf{z})$ such that Σ is defined by an equation of the form $|\mathbf{v}|^2 - |\mathbf{u}|^2 = 1$ and Σ' is defined by an equation of the form $|\mathbf{v}|^2 - \big(|\mathbf{u}|^2 + |\mathbf{y}|^2\big) = 1$. In this case the diffeomorphism given by

$$\varphi(\mathbf{u}, \mathbf{v}, \mathbf{y}, \mathbf{z}) \quad = \quad \left(\mathbf{u},\ \frac{\sqrt{1 + |\mathbf{u}|^2 + |\mathbf{y}|^2}}{\sqrt{1 + |\mathbf{u}|^2}} \cdot \mathbf{v},\ \mathbf{y},\ \mathbf{z} \right)$$

will send Σ to Σ'; note that this is a generalization of the diffeomorphism appearing at the end of `quadrics1.pdf`.$\blacksquare$

2. Projective hyperquadrics in $\mathbb{RP}^n$

We shall find it extremely useful to look at the inverse image of a hyperquadric in $\mathbb{RP}^n$ or $\mathbb{CP}^n$ with respet to the quotient map π from S^n to $\mathbb{RP}^n$ or from S^{2n+1} to $\mathbb{CP}^n$, and the followoing result will play an important role in setting up unified approaches.

THEOREM 1. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let $d = \dim_{\mathbb{R}}(\mathbb{F})$, and let $\pi : S^{dn+d-1} \to \mathbb{FP}^n$ be the quotient projection. Suppose we are given a nonconstant homogeneous polynomial $f(x_0, \cdots, x_m)$ of some degree $g > 0$ where $m < n$, and let $V_m \subset \mathbb{FP}^m$ and $V_n \subset \mathbb{FP}^n$ be the sets of points whose homogeneous coordinates satisfy the homogeneous equation(s) $h = 0$. If V'_m and V'_n denote the inverse images of V_m and V_n under the corresponding projections, then V'_n is homeomorphic to the join $V'_m * S^{n-m-1}$.*

Sketch of proof. Split $\mathbb{F}^{n+1}$ into $\mathbb{F}^{m+1} \times \mathbb{F}^{n-m}$. Then a nonzero ordered pair $(\mathbf{u}, \mathbf{v})$ in the latter is a set of homogeneous coordinates for a point in V_n if and only if $\mathbf{u}$ is a set of homogeneous coordinates for a point in V_m or $\mathbf{u} = \mathbf{0}$ (in which case $\mathbf{v} \neq \mathbf{0}$). If we now restrict attention to nonzero points on the unit sphere in $\mathbb{F}^{n+1} \cong \mathbb{F}^{m+1} \times \mathbb{F}^{n-m}$, which is homeomorphic to the join $S^{dm+d-1} * S^{d(n-m)-1}$, we see that the homogeneous coordinates for points of V_n correspond to classes $[\mathbf{a}, t, \mathbf{b}]$ where $\mathbf{a}$ lies in V'_m.$\blacksquare$

As shown in the previously cited document `pgnotes07.pdf`, every nontrivial hyperquadric in $\mathbb{RP}^n$ is projectively equivalent to a unique example defined by an equation from the following list:

$$(\mathbb{P}.p\,.r) \quad x_0^2 + \ \cdots \ + x_p^2 - x_{p+1}^2 - \ \cdots \ - x_r^2 \ = \ 0 \quad (0 \leq p \leq r \leq n, r \geq 1, p+1 \geq r-p)$$

Our goal is to show the following:

THEOREM 2. *Two nontrivial real projective hyperquadrics in $\mathbb{RP}^n$ are homeomorphic if and only if they are projectively equivalent.*

The nonsingular cases

There is an extensive discussion of singular and nonsingular points for hyperquadrics in the file `pgnotes07.pdf`, but for our purposes in this document the working definition of a nonsingular hyperquadric will be one definable by an equation in which $r = n$. Let $\mathbf{Q}_0(p, n)$ be the nonsingular hyperquadric in $\mathbb{RP}^n$ defined by equation $(\mathbb{P}.p\,.r)$, and let $\mathbf{Q}'_0(p, n)$ be its inverse image in S^n under the usual projection mapping π.

LEMMA. *The restriction of π to $\mathbf{Q}_0'(p,n)$ is a 2-sheeted regular covering space projection, and $\mathbf{Q}_0'(p,n)$ is homeomorphic (in fact, diffeomorphic) to the manifold $S^p \times S^{n-p-1}$. Furthermore, the covering space involution of $\mathbf{Q}_0'(p,n)$ is given by multiplication by -1 on each factor.*

Sketch of proof. Write a vector in $\mathbb{R}^{n+1} \cong \mathbb{R}^{p+1} \times \mathbb{R}^{n-p}$ as $(\mathbf{y},\mathbf{z})$. Then the defining equation for $\mathbf{Q}_0'(p,n)$ is $|\mathbf{y}|^2 - |\mathbf{z}|^2 = 0$, and if we combine this with the unit sphere equation $|\mathbf{y}|^2 + |\mathbf{z}|^2 = 1$ we see that $|\mathbf{y}| = |\mathbf{z}| = 1/\sqrt{2}$, from which the identification in the Lemma follows.■

We can use the preceding to compute the cohomology of $\mathbf{Q}_0(p,n)$ with $\mathbb{Z}_2$ coefficients:

THEOREM 3. *There is an isomorphism*

$$H^i\big(\,\mathbf{Q}_0(p,n);\mathbb{Z}_2\,\big) \;\;\cong\;\; H^i(\mathbb{RP}^n;\mathbb{Z}_2) \;\oplus H^{i-p}(\mathbb{RP}^n;\mathbb{Z}_2)$$

for all i. In particular, the nonsingular hyperquadrics $\mathbf{Q}_0(p,n)$ and $\mathbf{Q}_0(p',n')$ are homotopy equivalent *if and only if $n = n'$ and $p = p'$.*

Sketch of proof. This uses a significant amount of algebraic topology; it is also possible to give arguments using less algebraic-topological input, but (not surprisingly) they tend to be longer and more complicated.

Projection onto the last $n-p-1$ homogeneous coordinates yields a continuous mapping $\mathbf{Q}_0(p,n) \to \mathbb{RP}^{n-p-1}$ which is a fiber bundle whose fiber is S^p. Since $p \geq n-p-1$ (this is important!), we can use the Gysin sequence to show that the cohomology is given as in the statement of the theorem.

If two hyperquadrics are homotopy equivalent, then their mod 2 cohomology groups are isomorphic. However, one can check directly that the mod 2 cohomology groups of $\mathbf{Q}_0(p,n)$ and $\mathbf{Q}_0(p',n')$ are isomorphic if and only if $n = n'$ and $p = p'$.■

The singular cases

We shall combine the preceding result with the first theorem of this section to study examples for which $r < n$. Let $\mathbf{Q}(p,r;n)$ denote the hyperquadric in $\mathbb{RP}^n$ defined by the equation $(\mathbb{P}.p\,.r)$, and let $\mathbf{Q}'(p,r;n)$ denote its inverse image in $\mathbb{RP}^n$. By definition we have $\mathbf{Q}(p,n;n) = \mathbf{Q}_0(p,n)$ and $\mathbf{Q}'(p,n;n) = \mathbf{Q}_0'(p,n)$. One major idea in our analysis will be to compare the singular $\mathbf{Q}(p,r;n)$ and $\mathbf{Q}'(p,r;n)$ with the nonsingular hyperquadrics $\mathbf{Q}_0(p,r)$ and $\mathbf{Q}_0'(p,r)$.

Theorem 1 implies that if $r < n$ then the inverse image of $\mathbf{Q}'(p,r,n)$ of $\mathbf{Q}(p,r,n)$ in S^n is homeomorphic to

$$\big(\,S^p \times S^{r-p-1}\,\big) \;*\; S^{n-r-1}$$

and we also know that the latter is a double covering space of the original hyperquadric. We shall begin by showing that the displayed join is not a topological $(n-1)$-manifold; it will follow that $\mathbf{Q}(p,r,n)$ also cannot be a topological $(n-1)$-manifold if $r < n$.

LEMMA. *If $r < n$ then the complement of S^{n-r-1} in*

$$\mathbf{Q}'(p,r,n) \;\;\cong\;\; \big(\,S^p \times S^{r-p-1}\,\big) \;*\; S^{n-r-1}$$

is the unique maximal open subset which is a topological $(n-1)$-manifold, and the local cohomology of this space at a point of S^{n-r-1} is isomorphic to

$$\widetilde{H}^{n-r}\,\big(\,S^p \times S^{r-p-1}\,\big)\;\;.$$

In particular, neither $\mathbf{Q}'(p,r,n)$ nor $\mathbf{Q}(p,r,n)$ is a topological $(n-1)$-manifold if $r<n$.

It follows that $\mathbf{Q}(p,r,n)$ and $\mathbf{Q}(p',r',n)$ cannot be homeomorphic if $r=n$ and $r'<n$.

Sketch of proof. This follows because the complement of S^{n-r-1} is homeomorphic to $S^p \times S^{r-p-1} \times \mathbb{R}^{n-r}$ and every point of S^{n-r-1} has an open neighborhood which is homeomorphic to the product of $\mathbb{R}^{n-r-1}$ with the open cone on $S^p \times S^{r-p-1}$.■

COROLLARY. *If as before we let $\mathcal{N}\big(\,\mathbf{Q}(p,r,n)\,\big)$ be the unique maximal open subset which is an $(n-1)$-manifold, then $\mathcal{N}\big(\,\mathbf{Q}(p,r,n)\,\big)$ is homeomorphic to the total space of an $(n-r)$-dimensional vector bundle over the nonsingular hyperquadric $\mathbf{Q}_0(p,r,n)$ in $\mathbb{RP}^r$. In particular, $\mathcal{N}\big(\,\mathbf{Q}(p,r,n)\,\big)$ is homotopy equivalent to the nonsingular hyperquadric $\mathbf{Q}_0(p,r,n)$.*

This follows because the covering space involution on the subspace $S^p \times S^{r-p-1} \times \mathbb{R}^{n-r}$ is multiplication by -1 on each of the three factors.■

PROOF OF THE CLASSIFICATION THEOREM. Suppose that the hyperquadrics associated to the equations $(\mathbb{P}.p\,.r)$ and $(\mathbb{P}.p'\,.r')$ are homeomorphic; our previous observations imply that we need only consider cases where $r<n$. We then know that the sets $\mathcal{N}\big(\,\mathbf{Q}(p,r,n)\,\big)$ and $\mathcal{N}\big(\,\mathbf{Q}(p',r',n)\,\big)$ must also be homotopy equivalent, so that the nonsingular hyperquadrics $\mathbf{Q}_0(p,r)$ and $\mathbf{Q}_0(p',r')$ must be homotopy equivalent. However, we have already seen that the latter happens if and only if $r=r'$ and $p=p'$.■

3. Projective hyperquadrics in $\mathbb{CP}^n$

This is similar to the real case, but the discussion is simpler because the classification of complex hyperquadrics involves much shorter lists of examples. Specifically, the nontrivial, projectively inequivalent examples in the complex case are given by the equations

$$z_0^2 + \ \cdots \ + z_r^2 \quad = \quad 0 \qquad (1 \le r \le n)\ .$$

Our goal in this section is to show the following:

THEOREM 1. *Two nontrivial complex projective hyperquadrics in $\mathbb{CP}^n$ are homeomorphic if and only if they are projectively equivalent.*

The nonsingular cases

As in the real case, for our purposes in this document the working definition of a nonsingular hyperquadric will be one definable by an equation in which $r=n$. Let $\mathbf{Q}_0(n)$ be the nonsingular hyperquadric in $\mathbb{CP}^n$ defined by homogeneous quadratic equation

$$\sum_{k=0}^{n} z_k^2 \quad = \quad 0$$

and let $\mathbf{Q}_0'(n)$ be its inverse image in $S^{2n+1} \subset \mathbb{C}^{n+1}$ under the usual projection mapping π.

We shall begin by showing that $\mathbf{Q}_0'(n)$ can be identified with an object which arises frequently in topology and geometry:

THEOREM 2. *The restriction of π to $\mathbf{Q}_0'(n)$ is a principal S^1-bundle projection, and $\mathbf{Q}_0'(n)$ is homeomorphic (in fact, diffeomorphic) to the* real *Stiefel manifold $\mathbb{V}_{n+1,2}$ of orthonormal 2-frames in $\mathbb{R}^{n+1}$. In particular, the cohomology groups $H^k\big(\mathbf{Q}_0'(n);\mathbb{Z}\big)$ are given as follows:*

(i) If n is odd, then H^k is trivial unless $k = 0, n-1, n$ or $2n-1$, while $H^k \cong \mathbb{Z}$ in these cases.

(ii) If n is even, then H^k is trivial unless $k = 0, n$ or $2n-1$, while $H^k \cong \mathbb{Z}$ if $k = 0$ or $2n-1$, and $H^n \cong \mathbb{Z}_2$.

Formally, the Stiefel manifold $\mathbb{V}_{n+1,2}$ consists of all ordered pairs of unit vectors $(\mathbf{x}, \mathbf{y}) \in S^n \times S^n$ such that $\mathbf{x}$ and $\mathbf{y}$ are perpendicular.

Proof. Given a typical vector $\mathbf{z} = (z_0, \cdots, z_n) \in \mathbb{C}^{n+1}$, express it in the form $\mathbf{x} + i\,\mathbf{y}$ where $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^{n+1}$. Then the equation $\Sigma\, z_k^2 = 0$ can be rewritten in the form

$$0 \quad = \quad \big(\,|\mathbf{x}|^2 - |\mathbf{y}|^2\,\big) \ + \ 2\,i\,\langle \mathbf{x},\, \mathbf{y}\rangle$$

and if we equate the real and imaginary parts of the two sides of this equation we see that $|\mathbf{x}|^2 = |\mathbf{y}|^2$ and $\langle \mathbf{x},\, \mathbf{y}\rangle = 0$.

A vector $\mathbf{z}$ lies on $\mathbf{Q}_0'(n)$ if and only if it lies on the unit sphere and satisfies the equations in the preceding paragraph. Since the defining equation for the unit sphere can be written in the form

$$1 \quad = \quad |\mathbf{z}|^2 \quad = \quad |\mathbf{x}|^2 \ + \ |\mathbf{y}|^2$$

and if we combine this with the previously defined equations we see that the defining equations for $\mathbf{Q}_0'(n)$ are $|\mathbf{x}| = |\mathbf{y}| = 1/\sqrt{2}$ and $\langle \mathbf{x},\, \mathbf{y}\rangle = 0$. The required diffeomorphism from $\mathbf{Q}_0'(n)$ to $\mathbb{V}_{n+1,2}$ is given by sending $\mathbf{z} = (\mathbf{x},\, \mathbf{y})$ to $(\sqrt{2}\,\mathbf{x},\, \sqrt{2}\,\mathbf{y})$.

The formulas for the cohomology groups follow by applying the Gysin sequence to the fiber bundle

$$S^{n-1} \ \longrightarrow \ \mathbb{V}_{n+1,2} \ \longrightarrow \ S^n \ .$$

We should note that this sphere bundle is equal to the unit sphere bundle for the tangent bundle of S^n, and in fact this observation can be used to compute the cohomology of $\mathbb{V}_{n+1,2}$.■

We can use the preceding and the Gysin sequence for the bundle

$$S^1 \ \longrightarrow \ \mathbf{Q}_0'(n) \ \longrightarrow \ \mathbf{Q}(n)$$

to compute the cohomology of $\mathbf{Q}_0(n)$ with $\mathbb{Z}$ coefficients:

THEOREM 3. *Given $n \geq 2$, let $m = \frac{1}{2}(n-2)$ if n is even and $\frac{1}{2}(n-1)$ if n is odd. Then for all i there is an isomorphism*

$$H^i\big(\,\mathbf{Q}_0(n);\mathbb{Z}\,\big) \quad \cong \quad H^i(\mathbb{CP}^m;\mathbb{Z}) \ \oplus H^{i-p}(\mathbb{CP}^m;\mathbb{Z})$$

where $p = 2n - 2m - 2$. In particular, the nonsingular hyperquadrics $\mathbf{Q}_0(n)$ and $\mathbf{Q}_0(n')$ are homotopy equivalent if and only if $n = n'$.■

More broadly based computations of these cohomology groups and further information on the topology of $\mathbf{Q}_0(n)$ are also contained in the following papers:

J. Ewing and S. Moolgavkar. *On the signature of Fermat surfaces.* Michigan Mathematical Journal **22** (1975), 257–268.

R. S. Kulkarni and J. C. Wood. *Topology of nonsingular complex hypersurfaces.* Advances in Mathematics **35** (1980), 239–263.

The singular cases

We shall combine the preceding result with the first theorem of this section to study examples for which $r < n$. Let $\mathbf{Q}(r;n)$ denote the hyperquadric in $\mathbb{CP}^n$ defined by the equation

$$\sum_{k=0}^{r} z_k^2 \quad = \quad 0$$

and let $\mathbf{Q}'(r;n)$ denote its inverse image in $\mathbb{CP}^n$. By definition we have $\mathbf{Q}(n;n) = \mathbf{Q}_0(n)$ and $\mathbf{Q}'(n;n) = \mathbf{Q}'_0(n)$.

Theorem 1 in Section 2 implies that if $r < n$ then the inverse image of $\mathbf{Q}'(r,n)$ of $\mathbf{Q}(r,n)$ in S^n is homeomorphic to

$$\mathbb{V}_{r+1,2} \;*\; S^{2(n-r)-1}$$

and we also know that the latter is a principal S^1-bundle over the original hyperquadric. We shall begin by showing that the displayed join is not a topological $(2n-1)$-manifold; it will follow that $\mathbf{Q}(r,n)$ also cannot be a topological $(2n-2)$-manifold if $r < n$. Since the nonsingular hyperquadric $\mathbf{Q}_0(n)$ is a smooth $(2n-2)$-manifold (and it has a canonical complex analytic structure!), we can also conclude that $\mathbf{Q}(r,n)$ and $\mathbf{Q}(n,n)$ cannot be homeomorphic if $r < n$.

LEMMA. *If $r < n$, then the complement of $S^{2(n-r)-1}$ in*

$$\mathbf{Q}'(r,n) \quad \cong \quad \mathbb{V}_{r+1,2} \;*\; S^{2(n-r)-1}$$

is the unique maximal open subset $\mathcal{N}\big(\,\mathbf{Q}'(r,n)\big)$ which is a topological $(2n-1)$-manifold, and the local cohomology of this space at a point of $S^{2(n-r)-1}$ is isomorphic to

$$\widetilde{H}^{2(n-r)}\left(\,\mathbb{V}_{r+1,2};\mathbb{Z}\,\right) \ .$$

Furthermore, the local cohomology of $\mathbf{Q}(r,n)$ at a point in the image $\pi\left[S^{2(n-r)-1}\right]$ of $S^{2(n-r)-1}$ in $\mathbf{Q}(r,n)$ is isomorphic to

$$\widetilde{H}^{2(n-r)-1}\left(\,\mathbb{V}_{n+1,2};\mathbb{Z}\,\right) \ .$$

In particular, $\mathbf{Q}'(r,n)$ is not a topological $(2n-1)$-manifold if $r < n$, and $\mathbf{Q}(r,n)$ is not a topological $(2n-2)$-manifold if $r < n$.

Sketch of proof. The assertion regarding $\mathbf{Q}'(r,n)$ follows because the complement of $S^{2(n-r)-1}$ is homeomorphic to

$$\mathbb{V}_{r+1,2} \times \mathbb{C}^{n-r}$$

and every point of $S^{2(n-r)-1}$ has an S^1-invariant open neighborhood which is S^1-equivariantly homeomorphic to the product of $S^1 \times \mathbb{C}^{n-r-1}$ (translation action on the first coordinate, trivial action on the second) with the open cone on $\mathbb{V}_{r+1,2}$. The analagous assertion regarding $\mathbf{Q}(r,n)$ follows from consideration of the quotient of such a neighborhood with respect to the action of S^1.■

We shall also need the following result about the complement of $\pi\left[S^{2(n-r)-1}\right]$:

PROPOSITION. *If $\mathcal{N}\big(\,\mathbf{Q}(r,n)\big)$ is the unique maximal open subset which is a $(2n-2)$-manifold, then this set is the image of $\mathcal{N}\big(\,\mathbf{Q}'(r,n)\big)$ with respect to the projection map π.*

Proof. The set $\mathcal{N}\big(\mathbf{Q}'(r,n)\big)$ is in fact S^1-homeomorphic to $\mathbf{Q}'(r,n)\times\mathbb{C}^{n-r}$ in which the S^1-action is given by the smooth principal bundle action on the first coordinate and scalar multiplication by unit complex numbers on the second. It follows immediately that the quotient map is a smooth principal S^1-bundle projection and the quotient space is a smooth manifold of dimension $(2n-2)$. Therefore the image of $\mathcal{N}\big(\mathbf{Q}'(r,n)\big)$ under π is contained in $\mathcal{N}\big(\mathbf{Q}(r,n)\big)$. On the other hand, by the computations of local cohomology groups we know that no other points of $\mathbf{Q}(r,n)$ have open neighborhoods which are homeomorphic to open subsets of $\mathbb{R}^{2n-2}$, so therefore $\mathcal{N}\big(\mathbf{Q}(r,n)\big)$ must be equal to the image of $\mathcal{N}\big(\mathbf{Q}'(r,n)\big)$.$\blacksquare$

One immediate consequence of the preceding result is that $\mathcal{N}\big(\mathbf{Q}(r,n)\big)$ is homotopy equivalent to $\mathbf{Q}_0(r)$. This observation leads directly to our main objective.

PROOF OF THE CLASSIFICATION THEOREM. As in the real case, it suffices to consider cases $\mathbf{Q}(r,n)$ and $\mathbf{Q}(r',n)$, where $r,r'<n$. If the hyperquadrics $\mathbf{Q}(r,n)$ and $\mathbf{Q}(r',n)$ are homeomorphic, then we know that the sets $\mathcal{N}\big(\mathbf{Q}(r,n)\big)$ and $\mathcal{N}\big(\mathbf{Q}(r',n)\big)$ must also be homeomorphic, so by the preceding observations the nonsingular hyperquadrics $\mathbf{Q}_0(r)$ and $\mathbf{Q}_0(r')$ must be homotopy equivalent. However, we already know that the latter happens if and only if $r=r'$.$\blacksquare$

4. Affine hyperquadrics in $\mathbb{C}^n$

In this case we know that every hyperquadric is affinely equivalent to a unique example defined by a quadratic equation in the following indexed list from `quadrics1.pdf`.

$$\textbf{(I.}r\textbf{)}\ z_1^2+\ \cdots\ +z_r^2 \ = \ 0 \qquad (1\le r\le n)$$

$$\textbf{(II.}r\textbf{)}\ z_1^2+\ \cdots\ +z_r^2 \ = \ 1 \qquad (1\le r\le n)$$

$$\textbf{(III.}r\textbf{)}\ z_1^2+\ \cdots\ +z_r^2 \ = \ z_{r+1} \qquad (1\le r< n)$$

As in the real case, the Roman numeral in the first position will be called the **type** of the defining equation.

The following result will lead directly to the topological classification of affine hyperquadrics in the complex case:

PROPOSITION. *Let $F_2(n)\subset\mathbb{C}^n$ denote the set of all points $\mathbf{z}=(z_1,\ \cdots\ ,z_n)$ satisfying equation* **(II.**n**)**. *Then $F_2(n)$ is diffeomorphic to the tangent space $\mathbb{T}(S^{n-1})$.*

Proof. This is similar to an argument in the preceding section. If we now take $\mathbf{z}=(z_1,\ \cdots\ ,z_n)\in\mathbb{C}^n$ and express it in the form $\mathbf{x}\ +\ i\,\mathbf{y}$ where $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$, then the equation $\Sigma\ z_k^2\ =\ 0$ can be rewritten in the form

$$1 \ = \ \big(|\mathbf{x}|^2-|\mathbf{y}|^2\big)\ +\ 2\,i\,\langle\mathbf{x},\,\mathbf{y}\rangle$$

and if we equate the real and imaginary parts of the two sides of this equation we see that $|\mathbf{x}|^2=|\mathbf{y}|^2+1$ and $\langle\mathbf{x},\,\mathbf{y}\rangle=0$.

We may view $\mathbb{T}(S^{n-1})$ as the set of all points $(\mathbf{u},\,\mathbf{v})\in S^{n-1}\times\mathbb{R}^n$ such that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, and the diffeomorphism from the hyperquadric to the tangent space is defined by sending $\mathbf{z}=(\mathbf{x},\,\mathbf{y})$ to

$$\left(\frac{1}{|\mathbf{x}|}\cdot\mathbf{x},\,\mathbf{y}\right)\ .$$

One can check directly that this sends points in the hyperquadric into $\mathbb{T}(S^{n-1})\subset S^{n-1}\times\mathbb{R}^n$.$\blacksquare$

This yields the following result on the homeomorphism types of the quadrics in our list:

PROPOSITION. *Suppose that $\Sigma \subset \mathbb{C}^n$ is defined by a quadratic equation in the short list of examples:*

(i) *If Σ is defined by equation* **(I.r)**, *then Σ is homeomorphic to $\mathbf{C}^{\,\mathrm{open}}\big(\,\mathbb{V}_{r,2}\,\big) \times \mathbb{C}^{n-r}$.*

(ii) *If Σ is defined by equation* **(II.r)**, *then Σ is diffeomorphic to $\mathbb{T}(S^{r-1}) \times \mathbb{C}^{n-r}$.*

(ii) *If Σ is defined by equation* **(III.r)**, *then Σ is diffeomorphic to $\mathbb{C}^{n-1}$.*

Proof. The first case follows because Σ is homeomorphic to the open cone on $\Sigma \cap S^{2n-1}$, and by the results of the preceding section we know the latter is homeomorphic to the indicated Stiefel manifold. The second follows immediately from the previous proposition, and the third follows because in this case Σ is merely the graph of a quadratic polynomial function.∎

As in the real case, we now have the following result:

THE COMPLETE CLASSIFICATION. (0) *Different types of equations yield nonhomeomorphic hyperquadrics.*

(1) *The hyperquadrics defined by* **(I.r)** *and* **(I.r')** *are homeomorphic if and only if $r = r'$.*

(2) *The hyperquadrics defined by* **(II.r)** *and* **(II.r')** *are homeomorphic if and only if $r = r'$.*

(3) *The hyperquadrics defined by* **(III.r)** *and* **(III.r')** *are all homeomorphic to each other.*

Sketch of proof. As usual, let $\mathcal{N}(\Sigma)$ be the unique maximal open subset which is a topological $(2n-2)$-manifold. Then $\mathcal{N}(\Sigma) = \Sigma$ for equations of Type II or III, while $\mathcal{N}(\Sigma)$ corresponds to the proper subset $(0,\infty) \times \mathbb{V}_{r,2} \times \mathbb{C}^{n-r}$ for the equation **(I.r)**. It follows immediately that a hyperquadric defined by an equation of Type I is not topologically equivalent to one which is defined by an equation of Type II or III.

To finish the proof of (0), we need to show that a pair of hyperquadrics defined by equations of Types II and III cannot be homeomorphic. Since a hyperquadric defined by an equation **(II.r)** is homotopy equivalent to S^{r-1} and a hyperquadric defined by an equation of Type III is contractible, the assertion regarding Types II and III also follows quickly.

To see that hyperquadrics Σ and Σ' defined by equations of the form **(I.r)** and **(I.r')** are not homeomorphic if $r \neq r'$, note that if they were homeomorphic then $\mathcal{N}(\Sigma)$ and $\mathcal{N}(\Sigma')$ would also be homeomorphic, and since these sets have the homotopy types of the Stiefel manifolds $\mathbb{V}_{r,2}$ and $\mathbb{V}_{r',2}$ it would follow that the latter would be homotopy equivalent. However, this only happens if $r = r'$. Similarly, to see that hyperquadrics Σ and Σ' defined by equations of the form **(II.r)** and **(II.r')** are not homeomorphic if $r \neq r'$, note that if they were homeomorphic then S^{r-1} and $S^{r'-1}$ and would be homotopy equivalent, and this only happens if $r = r'$. Finally, we already know that all hyperquadrics of Type III are diffeomorphic, so this completes the argument.∎

Our final result follows formally from the same type of argument given at the end of Section 1.

COMPLEMENT. *If two hyperquadrics Σ and Σ' in $\mathbb{C}^n$ are homeomorphic, then there is a complex analytic diffeomorphism φ of $\mathbb{C}^n$ such that $\varphi[\Sigma] = \Sigma'$.*∎