COMPARING PARABOLAS AND

BRANCHES OF HYPERBOLAS

Clearly a hyperbola and parabola cannot be congruent because a parabola has only one branch and a hyperbola has two branches (in the language of topological spaces, these branches correspond to connected components). One natural question is whether a parabola can be congruent or similar to a branch of a hyperbola. If we look at drawings of these two curves, it may seem apparent that these two curves cannot be congruent or similar, but of course we want a mathematical proof. We shall do this two ways. One proof will be a mixture of techniques from projective geometry and point set topology. In particular, we shall use two facts which are common to these two subjects; namely, the real projective plane $\mathbb{RP}^2$ has a canonical topological structure such that the set of ordinary points is topologically equivalent to $\mathbb{R}^2$, and every projective collineation of the latter defines a homeomorphism from $\mathbb{RP}^2$ to itself. — The second proof will be more elementary but longer, partly because of the need to develop some necessary concepts.

For the first proof, we shall also need the following topological result:

PROPOSITION. *The circle S^1 is not homeomorphic to a proper subset of itself.*

Proof. Suppose that C is a proper subset of S^1 which is homeomorphic to S^1, let $g : C \to S^1$ be the inclusion. Given two points of S^1 there is a rotation sending one point to another, and if we compose g with such a rotation we can arrange things so that $1 \notin C$. Since $S^1 - \{1\}$ is homeomorphic to $(0, 2\pi)$, the inclusion map g factors through a 1–1 continuous mapping $g_0 : C \to (0, 2\pi)$. Let $\varphi : S^1 \to C$ be a homeomorphism, and let $h : [0, 1] \to S^1$ be the map $h(t) = \exp(2\pi\, i\, t)$. Then the restrictions of $f = g_0 \circ \varphi \circ h$ to the half-open intervals $[0, 1)$ and $[0, 1)$ are 1–1, and the values of the composite function at 0 and 1 are equal. Now a continuous 1–1 function from an interval into the reals is either strictly increasing or strictly decreasing, and since both functions are defined on the open interval $(0, 1)$ it follows that f itself is either strictly increasing or strictly decreasing and hence $f(0) \neq f(1)$. On the other hand, we have $f(0) = f(1)$ by construction, which yields a contradiction. The source of this contradiction is the assumption that S^1 has a proper subset which is homeomorphic to itself, and therefore no such subset can exist.∎

The main result

Since congruence implies similarity and similarity implies affine equivalence, the proof that a parabola and a branch of a hyperbola are neither congruent nor similar will be a consequence of the following stronger result:

THEOREM. *A parabola in $\mathbb{R}^2$ is not affinely equivalent to a branch of a hyperbola in $\mathbb{R}^2$.*

Example. There is a nonlinear changes of variables which takes the parabola defined by $y = x^2$ into the branch of the hyperbola defined by $y = \sqrt{1 + x^2}$. Specifically, let $G(x, y) = (x, y - x^2 + \sqrt{1 + x^2})$.

Proof. The results in `quadrics2.pdf` and `quadrics 3.pdf` imply that every parabola in $\mathbb{R}^2$ is affinely equivalent to the standard example Γ defined by $y = x^2$ and every hyperbola is affinely equivalent to the standard example H defined by $x^2 - y^2 = 1$, it will suffice to show that the standard parabola is not affinely equivalent to a branch of the standard hyperbola.

Suppose that there is an affine transformation $T(\mathbf{x}) = P\mathbf{x} = \mathbf{q}$ which maps P onto a branch H_0 of H. Let T^* be the projective collineation of $\mathbb{RP}^2$ extending T. The closures Γ^* and H^* of Γ and H are the nonsingular projective conics defined by the homogeneous equations $u_1^2 - u_2 u_3 = 0$ and $u_1^2 - u_2^2 - u_3^2 = 0$, and by continuity the unique point at infinity in Γ^*, which has homogeneous coordinates $(0, 1, 0)$, must be a point at infinity in H^*. It follows that T^* maps Γ^* onto a proper subset of H^*.

By the projective classification of nonsingular conics in $\mathbb{R}^2$ in Chapter 7 of

and the continuity of projective collineations, all such subsets are homeomorphic to the standard example, which is the unit circle in $\mathbb{R}^2$. Therfore the preceding paragraph implies that $S^1 \cong H^*$ contains a proper subset (corresponding to Γ^*) which is homeomorphic to S^1. By the first proposition this cannot happen, and therefore our original hypothesis — namely, Γ is affinely equivalent to a branch of H — must be false, proving the theorem.$\blacksquare$

An alternate approach

Not surprisingly, there are several ways to prove that a parabola and a branch of a hyperbola are not similar. We shall describe a proof which does not (explicitly) use projective geometry and is based upon vector parametrizations of curves as described in first and second year calculus courses (and presented in more depth in differential geometry courses). The main idea behind the approach taken here is to justify a basic intuitive way of distinguishing between a hyperbola and a parabola; namely, hyperbolas have asymptotic lines in the Euclidean plane but parabolas do not. Much of the work involves defining suitable notions of asymptotic tangent vectors for parametrized curves and deriving some of their fundamental properties.

We shall begin with a standard observation about tangent lines to nice curves and their independence of the choice of parametrization. This will be a consequence of the following result:

PROPOSITION 0. *Let γ be a regular smooth curve in $\mathbb{R}^n$ (so that it has a continuous derivative everywhere). Assume that γ is $1 - 1$ and $\gamma((t_0) = \mathbf{p}$. If h is a regular reparametrization of γ and $h(u_0) = t_0$, then the unit tangent vectors $\mathbf{v}_\gamma(t_0)$ and $\mathbf{v}_{\gamma\, h}(u_0)$ are equal up to sign.*

A reparametrization $t = h(u)$ is regular if it has a continuous nonzero derivative everywhere (by continuity it must be either positive everywhere or negative everywhere). Note that if we take the "time reversal" reparametrization $u = -t$, then $\mathbf{v}_\gamma(t_0) = -\mathbf{v}_{\gamma\, h}(u_0)$.

Proof. By the Chain Rule we have

$$\gamma'(t) \;\; = \;\; \frac{1}{h'(u)} \cdot (\gamma \circ h)'(u)$$

and since h' is always nonzero the unit vectors in the directions of the left and right hand sides, which are $\mathbf{v}_\gamma(t_0)$ and $\mathbf{v}_{\gamma\, h}(u_0)$ respectively, are either equal (if $h' > 0$) or point in diametrically opposite directions (if $h' < 0$).$\blacksquare$

From this point on, we shall be considering regular curves which are parametrized over the entire real line such that

$$\lim_{t \to +\infty} |\gamma(t)| \;\; = \;\; \lim_{t \to -\infty} |\gamma(t)| \;\; = \;\; \infty \; .$$

Note that if h is a regular reparametrization and $\varepsilon_h = \pm 1$ depending upon the sign of h', then

$$\lim_{t \to +\infty} h(u) = \varepsilon_h \infty, \qquad \lim_{t \to -\infty} h(u) = -\varepsilon_h \infty$$

and hence the limit conditions remain true if γ is replaced by a reparametrization of itself.

Geometric intuition suggests that if γ satisfies the limit conditions in the preceding paragraph, then the latter also hold for a curve which is congruent or similar to γ. Our next result confirms this.

THEOREM 1. *Let γ be a regular smooth curve parametrized over $\mathbb{R}$ such that $|\gamma(t)| \to \infty$ as $t \to +\infty$ and as $t \to -\infty$, and let $T(\mathbf{x}) = c\, P\, \mathbf{x} + \mathbf{q}$ be a similarity transformation of $\mathbb{R}^n$, where P is an orthogonal matrix, $\mathbf{q} \in \mathbb{R}^n$ and $c > 0$. Then the following hold:*

(i) We have $|T \circ \gamma(t)| \to \infty$ as $t \to +\infty$ and as $t \to -\infty$.

(ii) If in addition the unit tangent vector $\mathbf{v}_\gamma(t)$ has limits $\mathbf{v}_\gamma(+\infty)$ and $\mathbf{v}_\gamma(-\infty)$ as $t \to +\infty$ and as $t \to -\infty$ (respectively), then the unit tangent vector $\mathbf{v}_{T\gamma}(t)$ also has such limits and

$$\lim_{t \to +\infty} \mathbf{v}_{T\gamma}(t) = P\,\mathbf{v}_\gamma(+\infty), \qquad \lim_{t \to -\infty} \mathbf{v}_{T\gamma}(t) = P\,\mathbf{v}_\gamma(-\infty).$$

Note that since $|\mathbf{v}_\gamma(t)| = 1$ for all t and the length of a vector is a continuous function, then if the limits in (ii) exist we have $|\mathbf{v}_\gamma(+\infty)| = |\mathbf{v}_\gamma(-\infty)| = 1$.

Before proving this result, we shall verify that the hypotheses in both parts are satisfied if γ is a standard example of a parametrized parabola or a parametrized branch of a hyperbola, and in the latter case the vectors $\mathbf{v}_\gamma(+\infty)$ and $\mathbf{v}_\gamma(-\infty)$ are direction vectors for the asymptotes.

Example 1. *Hyperbolas.* Suppose that γ is the branch of the hyperbola defined by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (x > 0)$$

with parametrization $\gamma(t) = (a \cosh t, b \sinh t)$ such that $A, b > 0$. For this example we have $\gamma'(t) = (a \sinh t, b \cosh t)$, and standard calculations with hyperbolic functions imply that $|\gamma(t)| \to \infty$ as $t \to +\infty$ and $t \to -\infty$ and also

$$\mathbf{v}_\gamma(+\infty) = \left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right), \quad \mathbf{v}_\gamma(-\infty) = \left(\frac{-a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right).$$

Since the asymptotes to the hyperbola are defined by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

the unit vectors $\mathbf{v}_\gamma(+\infty)$ and $\mathbf{v}_\gamma(-\infty)$ are direction vectors for the asymptotes to the hyperbola. Note that these vectors are linearly independent because $a, b > 0$.

Example 2. *Parabolas.* Suppose that γ is the parabola defined by $y = c\,x^2$, where $c > 0$. If we take the standard graph parametrization $\gamma(t) = (t, c\,t^2)$, and standard techniques from first year calculus imply that $|\gamma(t)| \to \infty$ as $t \to +\infty$ and $t \to -\infty$ and also

$$\mathbf{v}_\gamma(+\infty) = (0,1), \qquad \mathbf{v}_\gamma(-\infty) = (0,-1).$$

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In this case the limit vectors are linearly dependent.

Proof of Theorem 1. (i) By the Chain Rule and the formula for T we have

$$\frac{d}{dt} \quad = \quad c\,P\,\gamma'(t)\ .$$

Since $\mathbf{v}(t)$ is a unit vector pointing in the same direction as $\gamma'(t)$ and P preserves lengths it follows that $\mathbf{v}_{T\gamma}(t) = P\,\mathbf{v}_\gamma(t)$ for all t. The continuity of the map sending $\mathbf{y}$ to $P\mathbf{y}$ then implies that

$$\mathbf{v}_{T\gamma}(\pm\infty) \quad = \quad \lim_{t\to\pm\infty}\mathbf{v}_{T\gamma}(t) \quad = \quad \lim_{t\to\pm\infty}P\,\mathbf{v}_\gamma(t) \quad = \quad P\,\mathbf{v}_\gamma(\pm\infty)\ .\blacksquare$$

ASYMPTOTIC ANGLES. If both $\mathbf{v}_\gamma(+\infty)$ and $\mathbf{v}_\gamma(-\infty)$ exist, then we can define an asymptotic angle for γ by the formula

$$\cos\alpha(\gamma) \quad = \quad \langle\mathbf{v}_\gamma(+\infty),\mathbf{v}_\gamma(-\infty)\rangle$$

(recall that the vectors in this expression are unit vectors, so the inner product is the cosine of the angle between them). By Theorem 1, standard limit formulas, and the bilinearity of the inner product, if h is a regular reparametrization then we have $\cos\alpha(\gamma) = \cos\alpha(\gamma\circ h)$, so that $\alpha(\gamma) = \alpha(\gamma\circ h).\blacksquare$

The next result implies that if a curve has an asymptotic angle, then this property and the measure of the angle do not change if we apply a similarity transformation to γ.

THEOREM 2. *let γ be as above, assume that $\mathbf{v}_\gamma(+\infty)$ and $\mathbf{v}_\gamma(-\infty)$ exist, and let T be a similarity transformation with the same notation as in Theorem 1. Then $\alpha(T\circ\gamma) = \alpha(\gamma)$.*

Proof. Since $|\mathbf{v}_{T\gamma}(\pm\infty)| = |\mathbf{v}_\gamma(\pm\infty)| = 1$ we have

$$\alpha(T\circ\gamma) \quad = \quad \langle P\,\mathbf{v}_\gamma(+\infty), P\,\mathbf{v}_\gamma(-\infty)\rangle \quad = \quad \langle\mathbf{v}_\gamma(+\infty),\mathbf{v}_\gamma(-\infty)\rangle \quad = \quad \alpha(\gamma)$$

where the next to last equation holds because the orthogonal matrix P preserves inner products.$\blacksquare$

We can now prove the main result.

THEOREM 3. *A parabola is not similar (and hence is also not congruent) to a branch of a hyperbola.*

Proof. By the results on congruence classification in `quadrics3.pdf` every parabola is congruent to one which is defined by an equation of the form $y = c\,x^2$ for some $c > 0$, and every hyperbola is equivalent to one defined by an equation of the form

$$\frac{x^2}{a^2} \ - \ \frac{y^2}{b^2} \quad = \quad 1 \qquad (a,b \ > \ 0)\ .$$

Furthermore, the hyperbola is congruent to itself by the isometry sending (x,y) to $(-x,y)$ and that each branch is sent to the other. Therefore it suffices to show that the given parabola is not congruent to the branch of the hyperbola defined by the displayed equation and $x > 0$; as before, this branch can be parametrized as $\gamma(t) = (a\cosh t, b\sinh t)$, so it is enough to show that a hyperbola of this type is not congruent to a parabola with parametric equations $\delta(t) = (t, c\,t^2)$. In the examples we showed that

$$\alpha(\gamma) \quad = \quad \frac{b^2 - a^2}{a^2 + b^2}\ , \qquad \alpha(\delta) \quad = \quad -1$$

so by Theorem 2 the proof of Theorem 3 reduces to observing that

$$\frac{b^2 - a^2}{a^2 + b^2} \quad = \quad -1 \ + \ \frac{2\,b^2}{a^2 + b^2} \quad > \quad -1$$

where the inequality on the right holds because $b > 0.\blacksquare$

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