Solutions for Exercises 01

Notation $f : X \to Y$ continuous, V open in Y. Say V is evenly covered (by f) if there is a homeomorphism $h : V \times D \to f^{-1}[V]$ where D is some discrete space and

$$f \circ h(v, \alpha) = v \quad \text{all } \begin{cases} v \in V \\ \alpha \in D \end{cases}.$$

1. Let $y \in Y$, and choose an open neighborhood V of $h^{-1}(y)$ which is evenly covered by p. Then $h[V]$ is an open neighborhood of y which is evenly covered by $h \circ p$; specifically, the inverse image is homeo to $h[V] \times D$ etc.

   For the second part, if $H \subseteq \pi_1(Y)$ is the image of p_*, then $h_*[H]$ is the image of $(h \circ p)_* = h_* \circ p_*$. ∎

2. Let $\Psi^k : S^1 \to S^1$ be the k-th power map, which is a covering such that $\operatorname{Im} \Psi^k_* = k\mathbb{Z}$ in $\mathbb{Z} \cong \pi_1(S^1)$.

Since a product of coverings is a covering we have the covering $\Psi^a \times \Psi^b : S^1 \times S^1 \to S^1 \times S^1$ and $\mathrm{Image}\,(\Psi^a \times \Psi^b)_*$ corresponds to $\mathrm{Im}(\Psi^a)_* \times \mathrm{Im}(\Psi^b_*)$ under the natural isomorphism $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$. For our example, this image is $a\mathbb{Z} \times b\mathbb{Z}$, which is the subgroup in the exercise. By the classification of coverings, the covering space X is homeomorphic to $S^1 \times S^1$ since both coverings determine the same subgroup of $\pi_1(T^2)$. $\blacksquare$

3. (i) By hypothesis, $p\,F(x_0 a) = f p(x_0 a) = f p(x_0) = f(y_0) = y_0$. Therefore we know that $F(x_0 \cdot a) = x_0 \cdot g$ for some $g \in \pi_1(Y)$ and since X is simply connected g is unique (recall $p^{-1}[\{y_0\}] \cong \pi_1(Y)/\mathrm{Image}\,\pi_1(X)$ always holds). Set $\varphi(a) = g$.

(ii) Use the definition of the action. We get $x_0 \cdot a$ by starting with a closed curve γ in Y which represents a, then lifting to X so that $\tilde{\gamma}(0) = x_0$, and setting $x_0 \cdot a$ equal to $\tilde{\gamma}(1)$. — We can define $x_0 \cdot f_*(a)$ by starting with $f \circ \gamma$, which represents $f_*(a)$, finding some lifting starting at x_0, and taking the end point of that lifting. The key point is that we can choose the lifting of $f \circ \gamma$ to be $F \circ \tilde{\gamma}$, for $pF\tilde{\gamma} = fp\tilde{\gamma} = f\gamma$. Hence the end point of $F \circ \tilde{\gamma}$, which was called $x_0 \cdot \varphi(a)$ in part (i) is also $x_0 \cdot f_*(a)$, and hence $\varphi(a) = f_*(a)$. ∎

(iii) Follow the hint, and note that we can view $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$; more precisely, the universal covering map $\Phi : \mathbb{R}^n \longrightarrow T^n$ factors through the quotient space $\mathbb{R}^n/\mathbb{Z}^n$, usually we just say "induced"

we can check algebraically that the (co)induced map $\mathbb{R}^n/\mathbb{Z}^n \xrightarrow{q} T^n$ is 1-1 (think about the case $n=1$), and the quotient is compact because $\mathbb{R}^n/\mathbb{Z}_n$ is the image of the compact set $[0,1]^n$. Hence q is a homeomorphism.

So if we have A, an $n \times n$ integral matrix, then we have

assoc. lin transf.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T_A"] \arrow[d, "\Phi"'] & \mathbb{R}^n \arrow[d, "\Phi"] \\ \mathbb{R}^n/\mathbb{Z}^n \arrow[r, dashed, "?"] & \mathbb{R}^n/\mathbb{Z}^n \end{tikzcd}$$

and we want to fill in the diagram with a map at the bottom. This can be done if $\Phi(u) = \Phi(v) \Longrightarrow \Phi(Au) = \Phi(Av)$.

But $\Phi(u) = \Phi(v) = \ v = u + k$ where $k \in \mathbb{Z}^n$, so that $Av = Au + Ak$ with $Ak \in \mathbb{Z}^n$ (A has integer entries) and $\Phi(Av) = \Phi(Au)$. This yields the map $f : T^n \longrightarrow T^n$.

To conclude, note that the fiber of the base point is just $\mathbb{Z}^n$, and if $a \in \mathbb{Z}^n$ and $x_0 = 0 \in \mathbb{R}^n$, then $A(a) = F(x_0 \cdot a)$ in the notation of ~~Exercise~~ part (i). Hence $A(a) = f_*(a)$ by the second part of this exercise. ∎

4. (i) Covering spaces correspond to subgroups of $\pi_1(B)$ [$E \xrightarrow{p} B$ covering] and the sheets are in 1-1 correspondence with $\pi_1(B)/p_*[\pi_1(E)]$. Since $\pi_1(B) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ in this case, the number of sheets = order of some set of cosets of π_1, and hence it is 1 (if the map is a homeo), 2 or 4. ∎

(ii) We can list the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ as follows, with things written additively:

$\{(0,0)\}$, $\{(0,0),(1,0)\}$ $\{(0,0),(0,1)\}$ $\{(0,0),(1,1)\}$ $\mathbb{Z}_2 \times \mathbb{Z}_2$

So we have FIVE subgroups, and hence the same number of equivalence classes of connected coverings. ∎

5. Assume the group-theoretic result.

~~Take~~ Let $G = \pi_1(Y)$, $H = \text{Image } p_*$, so that H has finite index in G. Since the hypotheses imply there is a simply connected covering $\theta : \tilde{Y} \to Y$, the classification thm. implies that $X = \tilde{Y}/H$.

Now let $K \triangleleft G$ (normal subgp) as in the group-theoretic result, and let $W = \tilde{Y}/K$, so $W \xrightarrow{q} Y$ is a regular covering space with $|G/K|$ sheets (this is finite since $|G/K|$ is).

Generalities about quotient spaces then yield

$$\begin{tikzcd}
\tilde{Y} \arrow[r] & W = \tilde{Y}/K \arrow[r] \arrow[rr, bend left, "q"] \arrow[dr, "q'"'] & \tilde{Y}/H \arrow[r] \arrow[d, phantom, "\cong"] & \tilde{Y}/G \cong Y. \\
 & & X \arrow[ur, "p"'] &
\end{tikzcd}$$

so it is only necessary to check that q' is a regular covering with finitely many sheets. Since $|H/K| \leq |G/K|$ and $K \triangleleft G \Rightarrow K \triangleleft H$, all that is needed is to check that q' is a covering. — Let $x \in X$, and choose an open neighborhood V of $p(x)$ such that V is evenly covered for $\pi : \tilde{Y} \to Y$. Then over V we have (up to equivalence)

$$\begin{array}{ccccccc} G \times V & \longrightarrow & G/K \times V & \longrightarrow & G/H \times V & \longrightarrow & V \\ \text{in } \tilde{Y} & & \text{in } W & & \text{in } X & & \text{in } Y \end{array}$$

If $x \in \{gH\} \times V$ in X, then its inverse image of $\uparrow$ is $\bigcup_{h_\alpha} \{g h_\alpha K\} \times V$, where h_α runs through a set of elements in H which contains exactly one element from each coset in H/K. Therefor $\{gH\} \times V$ is evenly covered by q'. ∎

6. (ii) FOLLOW THE HINT(S) so $S^2 \in \Gamma_0$

We have $S^2(z) = z + 1$, ~~so~~ Note that $S^{-1}(z) = \bar{z} - \frac{1}{2}$ (solve $w = \bar{z} + \frac{1}{2}$).

Now let $T(z) = z + c$, where $c = a + bi$ with $a, b \in \mathbb{Z}$. Then $STS(z) =$

$ST(\bar{z} + \frac{1}{2}) = S(\bar{z} + \frac{1}{2} + c) = z + \bar{c} + 1$, so $STS \in \Gamma_0$.

Likewise $STS^{-1}(z) = ST(\bar{z} - \frac{1}{2}) =$

$S(\bar{z} - \frac{1}{2} + c) = z + \bar{c}$, so $STS^{-1} \in \Gamma_0$.

The latter implies that $\underbrace{S \cdot \Gamma_0}_{\text{left coset}} = \underbrace{\Gamma_0 \cdot S}_{\text{right coset}}$

<u>CLAIM</u> $\Gamma_0 \cup \overset{= \Gamma_0 \cdot S}{S \cdot \Gamma_0}$ is a subgroup.

<u>Closed under composition/multiplication</u>

<u>$g_1, g_2 \in \Gamma_0$</u>: no problem, Γ_0 is a subgp.

<u>$g_1 \in \Gamma_0 \; g_2 \in S \cdot \Gamma_0$</u>: Since $S \cdot \Gamma_0 = \Gamma_0 S$,

$g_2 = S g_2'$ (some $g_2' \in \Gamma_0$) and $g_1 \cdot S g_2' = (g_1 S) g_2' =$

$S g_1' g_2$, some $g_1' \in \Gamma_0$.

<u>$g_1 \in S \cdot \Gamma_0$, $g_2 \in \Gamma_0$</u>: Like the first case, no problem.

<u>$g_1 \in S \cdot \Gamma_0 \quad g_2 \in S \cdot \Gamma_0$</u> Write $g_1 = S g_1'$, $g_2 = S g_2'$ (some g_1', g_2' in Γ_0)

Then $S g_1' S g_2' =$

$S^2 g_1'' g_2$ (some g_1'') $= g_0 g_1'' g_2$ where $g_0 \in \Gamma_0$.

So the set is closed under mult.

In particular, this means that $\Gamma_0 \subseteq \Gamma$ has at most two cosets, and since $S \notin \Gamma_0$ there must be exactly two. We know that $|\Gamma/\Gamma_0| = 2 \Rightarrow \Gamma_0 \triangleleft \Gamma$ in general, so in our case we also have $\Gamma_0 \triangleleft \Gamma$ as claimed. $\blacksquare$

(iii) More generally, if $X \xrightarrow{p} Y$ is a covering and $Y \xrightarrow{q} Z$ is a <u>finite</u> covering, then $X \to Z$ is a covering (this fails if $Y \to Z$ is not finite). Let $z \in Z$ and let $z \in U$ open such that U is evenly covered; denote the sheets over U by $U_1, \ldots, U_n$, with $z_i \in U_i$ mapping to z. Inside U_i, take a nbhd. V_i of z_i which is evenly covered by $X \to Y$, and let $W = \cap\, p[V_i]$, which is an open nbhd of z in U. One can then check that V is evenly covered by $q \circ p$. Hence in our case $\mathbb{R}^2 \longrightarrow T^2 \longrightarrow K$ is a covering.

let $W = \bigcap_i p[V_i]$, an open set containing z.
It follows that W is evenly covered by $q \circ p$.
In particular $\mathbb{R}^2 \to T^2 \to K$ is a covering, and Γ acts on $\mathbb{R}^2$ by covering transformations. In fact we have

$$\mathbb{R}^2/\Gamma = (\mathbb{R}^2/\Gamma_0)/(\Gamma/\Gamma_0) = T^2/\mathbb{Z}_2 = K.$$

Since $\mathbb{R}^2$ is simply connected, Γ is transitive on the inverse image of a point, and hence $\Gamma \cong \pi_1(K)^{op}$. Now Γ is not abelian (if $T(z) = z+i$, then $STS^{-1}(z) = z-i$) so $\pi_1(K)$ is not abelian. ∎

(iii) No element of Γ_0 except the identity has finite order since $\Gamma_0 \cong \mathbb{Z} \oplus \mathbb{Z}$. We need to show the same for $S \cdot \Gamma_0$. In general, if $g^2 \in G$ has ∞ order, then so does g; therefore it suffices to show that if $ST \in S\Gamma_0$

then $(ST)^2 \neq I$.

Let $T(z) = z + c$ as usual, so that
$STS(z) = z + \bar{c} + 1$ and hence
$STST(z) = STS(z+c) = z + (c+\bar{c}) + 1$.
If we write $c = a + bi$ with $a, b \in \mathbb{Z}$, this becomes $z + (2a+1)$, which is not z because $(2a+1)$ is an odd integer. Hence $(ST)^2 \neq I$, and as before this implies that ST has ∞ order in Γ. ∎