Exercise 01 #3

(i) ~~Let~~ Let $D = p^{-1}[\{y_0\}]$ the fiber over y_0. Then $f \circ p = p \circ F \Rightarrow$ $f \circ p[D] = p \circ F[D]$. The left side is $\{f(y_0) = y_0\}$, so $F[D] \subseteq p^{-1}[\{y_0\}] = D$.
Every point in D can be written $x_0 \cdot g$ for some unique $g \in \pi_1(Y, y_0)$. In particular, $x_0 \cdot a$ can be so written.

Define $\varphi(a) = g$.

(ii) By the construction of the π_1 action, $x_0 \cdot g$ is the end point of a lifting $\tilde{\gamma}$ of a closed curve γ representing $g \in \pi_1(Y, y_0)$*. Since $f \circ p = p \circ F$ and $F(x_0) = x_0$, it follows that $F \circ \tilde{\gamma}$ lifts $f \circ \gamma$, so $F\tilde{\gamma}(1) = x_0 \cdot f_*(g)$.
On the other hand, $\tilde{\gamma}(1) = x_0 \cdot g$, so this means that $f_*(g) = \varphi(g)$.

$\boxed{\text{* } \tilde{\gamma} \text{ starts at } x_0}$

(iii) Follow the hint and define f_A so that $F_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ on the universal covering is given by the matrix A. ~~By (iii)~~ This means that if $v \in \mathbb{Z}^n$ = inverse image of $\{1\}$ in $\mathbb{R}^n \longrightarrow (S^1)^n = T^n$, then $F_A(v) = Av$. On the other hand, $f_A \circ p = p \circ F_A \Rightarrow f_{A*}(v) = Av$ by (ii) because the action of $\mathbb{Z}^n = \pi_1(T^n)$ on D = fiber of $\{1\} = \mathbb{Z}^n$ is just by translations.

Exercise 01 #5 $\quad \underset{H}{\underset{\|}{\pi_1(X, x_0)}} \longrightarrow \underset{G}{\underset{\|}{\pi_1(Y, y_0)}}$ is 1-1.

By group theory, there is a normal subgroup $K \subseteq G$ such that G/K is finite and $K \subseteq H \subseteq G$

Let $q' : W \xrightarrow{\text{[COVERING SPACE]}} X$, correspond to $K \subseteq H$. Since $X \to Y$ is a finite covering, it follows that

$$\underbrace{W \xrightarrow{q'} X \xrightarrow{p} Y}_{q}$$

is also a covering space projection.

It is regular and the number of sheets is the order of $|G/K|$ because $K \triangleleft G$.

Exercise 01 #6

(i) Elements of Γ are monomials of the forms $ST_1ST_2\dots ST_m$,

$T_1S\dots ST_mS$, or

$T_1S\dots ST_m$ since

$$\boxed{S^2 = T_0 \quad T_0(z) = z+1}$$ and $S^{-1} = ST^*$ where $T^*(z) = z-1$.

So all we really need to show is that if T is a translation then $TS(z) =$

$Tz = z+c$

$T(\bar{z} + \tfrac{1}{2}) = \bar{z} + \tfrac{1}{2} + c =$

$S(z+\bar{c}) = ST'(z)$ where $T'(z) = z+\bar{c}$.

This shows the existence of the factorization and the fact that Γ_0 has index 2 in Γ (note that $S \notin \Gamma_0$).

$S^{-1} \notin \Gamma_0$

To prove uniqueness, suppose
$S^{\alpha}T = S^{\beta}T'$, so that
$S^{-\beta}S^{\alpha} = T'T^{-1} \in \Gamma_0$.
Since $\{\alpha,\beta\} \subseteq \{0,1\}$, this on the left can only happen if $\alpha=\beta$. But this also means
$1 = T'T^{-1}$, so that $T=T'$.

(ii) As before $\mathbb{R}^2 \to T^2$ covering $\Rightarrow$
$T^2 \to K$ double covering
$\mathbb{R}^2 \to K$ is a covering. By construction Γ is a group of covering transformations, so $\Gamma \subseteq \pi_1(K)$. To see that it is <u>all</u> of $\pi_1(K)$, note that the fiber of $\mathbb{R}^2 \to K =$ all $(x,y) \in \mathbb{R}^2$ such that $y, 2x \in \mathbb{Z}$ and check that for each point in the fiber there is some element of Γ which maps $(0,0)$ to (x,y).

Finally since $\Gamma \cong \pi_1(K)$, the latter is nonabelian $\Leftrightarrow$ Γ is. But we have shown that if $T(z) = z+c$, then $TS\cancel{\mp} = ST'$ where $T'(z) = z+\bar{c}$, and $T' \neq T$ if c is not real, so that $S^{-1}TS \neq T$ in such cases. Hence Γ cannot be abelian.

(iii) $\Gamma_0 \cong \mathbb{Z}^2 \Rightarrow \Gamma_0$ has no nontrivial elements of finite order. Suppose there is an elt of finite order not in Γ. Write it as ST where $T \in \Gamma_0$. Since $g^2 \in G$ has finite order if g does, it follows that $(ST)^2$ has finite order, and since $(ST)^2 \in \Gamma_0$ (recall $\Gamma/\Gamma_0 \cong \mathbb{Z}_2$) it follows that $(ST)^2 = Id$.

Now $ST(z) = \bar{z} + \bar{c} + \frac{1}{2}$ if $T(z) = z+c$,

~~so $(ST)^2(z) = z + c + 1$ (check this). Hence $(ST)^2 = I \Rightarrow c+1 = 0 \Rightarrow c = -1$.~~

so that

$$(ST)^2(z) = ST(\bar{z} + \bar{c} + \tfrac{1}{2}) =$$
$$z + \bar{c} + c + \tfrac{1}{2} + \tfrac{1}{2} =$$
$$z + 2\mathrm{Re}(c) + 1.$$

This implies $0 = 2\mathrm{Re}(c) + 1$ since $(ST)^2 = 1 \Rightarrow ST^2(z) = z$. Note that $c = a + bi$ with $a, b \in \mathbb{Z}$, so we have $2a + 1 = 0$. There is no choice of $a \in \mathbb{Z}$ for which this is true. Therefore every elt. of $\Gamma - \Gamma_0$ also has ∞ order. Combining this with the first sentence of the solution, we see that no nontrivial elt. of Γ has finite order.