Axiomatic characterizations of products and sums

We have noted that there are at least two ways of constructing a product of three topological spaces using the binary product construction; namely, $(A \times B) \times C$ and $A \times (B \times C)$. It is important to have some means of saying that alternative constructions yield objects that are essentially the same, and one way of doing so is to give axiomatic characterizations of the construction(s).

Definition. Let $\{X_\alpha\}$ be and indexed family of spaces with indexing set α. An *abstract categorical product* of this indexed family is a pair $(P, \{p_\alpha\})$ consisting of a topological space P and a family of continuous mappings $p_\alpha : P \to X_\alpha$ (called *abstract projections*) with the following **universal mapping property:**

> *If $\{f_\alpha : Y \to X_\alpha\}$ is an indexed family of continuous mappings, then there is a* UNIQUE *continuous function $f : Y \to P$ such that $p_\alpha \circ f = f_\alpha$ for all α.*

By construction, the ordinary product space $\prod_\alpha X_\alpha$ together with the coordinate functions π_α form an abstract categorical product. The next result shows that the universal mapping property gives an essentially complete characterization of a product space.

THEOREM. *Let $(P, \{p_\alpha\})$ and $(Q, \{q_\alpha\})$ be abstract categorical products of the indexed family $\{X_\alpha\}$. Then there are unique homeomorphisms $f : P \to Q$ and $g : Q \to P$ such that $q_\alpha \circ f = p_\alpha$ and $p_\alpha \circ g = q_\alpha$ for all α.*

Proof. By the defining conditions for a categorical product, if $h : P \to P$ is a continuous mapping such that $h \circ p_\alpha = p_\alpha$ for all α, then h must be the identity on P (since the latter map obviously has the same property). A similar statement holds if we replace P and p_α by Q and q_α.

The existence of the mapppings f and g follow from the Universal Mapping Property. Furthermore the self-maps $g \circ f : X \to X$ and $f \circ g : Y \to Y$ satisfy

$$p_\alpha \circ f \circ g \;=\; q_\alpha \circ g \;=\; p_\alpha \;, \qquad q_\alpha \circ g \circ f \;=\; p_\alpha \circ f \;=\; q_\alpha$$

and therefore by the reasoning of the first paragraph we have $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. This means that f and g are homeomorphisms which are inverse to each other (why?).■

There is a similar (formally, a **dual**) characterization of disjoint unions (or coproducts or sums); this characterization is the reason for using the symbol $\amalg$ for disjoint unions, for this symbol is merely $\prod$ turned upside down.

Definition. Let $\{X_\alpha\}$ be and indexed family of spaces with indexing set α. An *abstract categorical coproduct* or *sum* of this indexed family is a pair $(S, \{i_\alpha\})$ consisting of a topological space S and a family of continuous mappings $i_\alpha : X_\alpha \to S$ (called *abstract injections*) with the following **(co)universal mapping property:**

> *If $\{f_\alpha : X_\alpha \to Y\}$ is an indexed family of continuous mappings, then there is a* UNIQUE *continuous function $f : S \to Y$ such that $f \circ i_\alpha = f_\alpha$ for all α.*

By construction, the disjoint union $\amalg_\alpha X_\alpha$ together with the standard inclusions i_α form an abstract categorical sum. The next result shows that the universal mapping property gives an essentially complete characterization of the disjoint union.

THEOREM. *Let $(S, \{i_\alpha\})$ and $(T, \{j_\alpha\})$ be abstract categorical sums of the indexed family $\{X_\alpha\}$. Then there are unique homeomorphisms $f : S \to T$ and $g : T \to S$ such that $f \circ i_\alpha = j_\alpha$ and $g \circ j_\alpha = i_\alpha$ for all α.*

The proof of this result is formally parallel to the previous one, the main difference being that the directions of all morphisms have to be reversed.

Proof. By the defining conditions for a categorical sum, if $h : T \to T$ is a continuous mapping such that $h \circ j_\alpha = j_\alpha$ for all α, then h must be the identity on T (since the latter map obviously has the same property). A similar statement holds if we replace T and j_α by S and i_α.

The existence of the mapppings f and g follow from the Universal Mapping Property. Furthermore the self-maps $g \circ f : S \to S$ and $f \circ g : T \to T$ satisfy

$$f \circ g \circ j_\alpha \quad = \quad f \circ i_\alpha \quad = \quad j_\alpha \ , \qquad g \circ f \circ i_\alpha \quad = \quad g \circ j_\alpha \quad = \quad i_\alpha$$

and therefore by the reasoning of the first paragraph we have $g \circ f = \mathrm{id}_S$ and $f \circ g = \mathrm{id}_T$. This means that f and g are homeomorphisms which are inverse to each other (why?).$\blacksquare$