Corestrictions of functions

This is a highly nonstandard concept which turns out to be useful in some contexts.

FUNCTIONS. In many books a function $f: A \longrightarrow B$ is defined entirely by its graph, a subset of $A \times B$ (written Γ_f) such that for each $a \in A$ there is a unique $b \in B$ such that $(a, b) \in \Gamma_f$. Our definition of function also specifies the source (domain) and target (codomain). For example, the inclusion function $\{0\} \longrightarrow \{0,1\}$ and identity function $\{0\} \longrightarrow \{0\}$ have the same graphs, but they are different functions.

DISJOINT UNIONS. Given a family of sets $\{X_\alpha\}_{\alpha \in A}$, this yields a new set Y which is a union of pairwise disjoint subsets Y_α

where Y_α is a copy of X_α.

If each X_α has a topology, one can define a topology on Y as follows: The open sets have the form $\cup V_\alpha$ where $V_\alpha \subseteq Y_\alpha$ corresponds to an open subset of $U_\alpha \subseteq X_\alpha$.

IMAGES AND INVERSE IMAGES.

If $f: X \to Y$ is a function with $A \subseteq X, B \subseteq Y$, then the image of A under f will usually (ideally, always) be denoted by $f[A]$ and the inverse image of B by $f^{-1}[B]$. This avoids potential ambiguities.

<u>Example</u> Let $f: \{\phi\} \longrightarrow \mathbb{R}$ be the function with $f(\phi) = 0$. Then $f[\phi] = \phi$.

The following simple observation confirms that the terminology for images and inverse images does not lead to logical ambiguities:

<u>PROPOSITION.</u> Let $f: A \to B$ be 1-1 onto and let $g: B \to A$ be its inverse (so $a = g(b) \Longleftrightarrow b = f(a)$). If $C \subseteq B$, then

$$g[B] = f^{-1}[B].$$

viewed as an image of a function | viewed as an inverse image of a function.

The proof is a straightforward exercise.

CORESTRICTIONS OF FUNCTIONS.

(Highly nonstandard) If $f: X \to Y$ is a function and $A \subseteq X$, then the <u>restriction</u> of f to A, written $f|A: A \to Y$ is the function whose graph is given by

$$\Gamma_{f|A} = \Gamma_f \cap (A \times Y).$$

If $f: X \to Y$ and $B \subseteq Y$ such that $f[X] \subseteq B$, then $\Gamma_f \subseteq X \times B$ and the <u>corestriction</u> $B|f: X \to B$ is the function whose graph is Γ_f.

(This may simplify some things later on).

For continuous mappings, we have the following rewording of a basic result:

<u>PROPOSITION</u>. If $f: X \to Y$ is continuous and $f[X] \subseteq B$, then $B|f$ is also continuous (where B has the subspace topology!).

Here are a few elementary properties of restrictions that can be verified fairly easily:

1. If $f[X] \subseteq B$ and $A \subseteq X$, then
$$B|(f|A) = (B|f)|A.$$

2. If $f: X \to Y$, $g: Y \to Z$ with $f[X] \subseteq B$ and $g[Y] \subseteq C$, then
$$C|g \circ f = (C|g|B) \circ (B|f).$$

3. If f is 1-1 onto, then
$$\big(f[A]\,|\,(f|A)\big)^{-1} = A\,|\,\big(f^{-1}\,|\,f[A]\big).$$

Exercise: Verify each one!

And here is one more identity:

4. If $f[X] \subseteq B$ and $i_B : B \to Y$ is the inclusion map, then

$$f = i_B \circ (B|f).$$