The Polish Circle and the Seifert-van Kampen Theorem

The objective is to give an example of a subset of the plane X and arcwise connected closed subsets A, B such that

(1) X is homeomorphic to $S^1 \times [0,1]$,

(2) $A \cap B$ is nonempty, and both $A \cap B$ and $X = A \cup B$ are arcwise connected,

(3) the fundamental group of X is not a pushout of the diagram $\pi_1(A) \leftarrow \pi_1(A \cap B) \rightarrow \pi_1(B)$.

As in some of our other examples which do not behave well algebraically, our example here will involve the Polish circle, and we shall refer to the two online documents `polishcircle.pdf` and `polishcircleA.pdf` as needed.

Recall that the Polish circle $P \subset \mathbb{R}^2$ is the union of the graph of $\sin(1/x)$ for $0 < x \leq 1$ and the three closed line segments joining $(0,1)$ to $(0,-2)$, $(0,-2)$ to $(1,-2)$, and $(1,-2)$ to $(1, \sin 1)$; there is a rough sketch of P in `polishcircleA.pdf`. By Proposition 2 and Corollary 3 in `polishcircle.pdf` we know that P is simply connected.

The drawing on the first page of `polishcircleA.pdf` suggests that P is the boundary of the closed bounded region B consisting of points (x,y) in $\mathbb{R}^2$ satisfying

$$0 \quad \leq \quad x \quad \leq 1 \quad \text{and } \textbf{either}$$

$$-2 \quad \leq \quad y \quad \leq \quad \sin(1/x) \quad \text{if} \quad x \neq 0 \quad \textbf{or} \quad -2 \quad \leq \quad y \quad \leq \quad 1 \quad \text{if} \quad x = 0\ .$$

It follows immediately that $B = \text{Interior}(B) \cup P$, where the two subsets on the right hand side are disjoint, and that B is the closure of $\text{Interior}(B)$. In particular, the point $z = \left(\frac{1}{2}, -\frac{3}{2}\right)$ lies in the interior of B, and one can easily verify that the closed disk D of radius $\frac{1}{4}$ centered at z is contained in $\text{Interior}(B)$. As in `polishcircleA.pdf`, let $A = S^2 - \text{Interior}(B)$; then one can check directly that $A \cap B = P$.

Since B is a bounded region, there is some $M > 0$ such that B is contained in in the open disk $N_M(z)$; in fact, since B is contained in the solid square $[0,1] \times [-2,1]$ by definition and the solid square is contained in $N_M(z)$, we can take $M \geq 3$. Define Y to be the closed disk of radius 3 centered at z, and let $X = Y - \text{Interior}(D)$. By construction we have $P \subset X$, and therefore if we set E and F equal to $A \cap X$ and $B \cap X$ respectively, then $X = E \cup F$ and $P = E \cap F$. Choose a basepoint $x \in P$. We shall show that E and F are arcwise connected (even though this statement might seem obvious), and more importantly that *the commutative diagram of fundamental groups*

$$\begin{array}{ccc} \pi_1(P,x) & \xrightarrow{i_{1*}} & \pi_1(E,x) \\ \downarrow i_{2*} & & \downarrow j_{1*} \\ \pi_1(F,x) & \xrightarrow{j_{2*}} & \pi_1(X,x) \end{array}$$

is **NOT** *a pushout diagram.* — This example shows that one cannot prove a general version of the Seifert-van Kampen theorem in which open subsets are replaced by closed subsets.

Verification of arcwise connectedness

We shall describe arcwise connected subsets $E_0 \subset E$ and $F_0 \subset F$ such that (i) E_0 and F_0 are arcwise connected, (ii) every point in E can be joined to a point in E_0 by a continuous curve, (iii) every point in F can be joined to a point in F_0 by a continuous curve. The argument requires the right choices of the subsets, and it breaks into cases corresponding to suitable decompositions of E and F into finite unions of more tractable closed subsets.

We begin by defining closed subsets of E. The circle of radius 3 centered at z is contained in E, and it will be denoted by C_+. We shall split E into four subsets by cutting it along the vertical lines L_0 and L_1 defined by $x = 0$ and $x = 1$ respectively. There is a drawing depicting this splitting on the next page. In terms of equations and inequalities, the four subsets are defined as follows:

> E_1 is the set of all points in E such that either $0 < x \leq 1$ and $y \geq \sin(1/x)$ or else $x = 0$ and $y \geq -1$.
>
> E_2 is the set of all points in E such that $x \geq 1$.
>
> E_3 is the set of all points in E such that $0 \leq x \leq 1$ and $y \leq -2$.
>
> E_4 is the set of all points in E such that $x \leq 0$.

The subset E_0 is defined to be the union of the Polish Circle $P \subset E$ with the vertical line segments $L_0 \cap E$ and $L_1 \cap E$. Since each of these three pieces is arcwise connected and each vertical segment has a nonempty intersection with P, it follows that E_0 is arcwise connected.

There are fewer subsets of F to describe, but their definitions are more complicated. By construction the circle C_- with center z and radius $\frac{1}{2}$ is part of the boundary of F; note that the points of this circle are defined by the equations

$$y \quad = \quad -\frac{3}{2} \;\pm\; \sqrt{\frac{1}{16} - \left(x - \frac{1}{2}\right)^2} \qquad \text{where} \quad \frac{1}{4} \;\leq\; x \;\leq\; \frac{3}{4}\;.$$

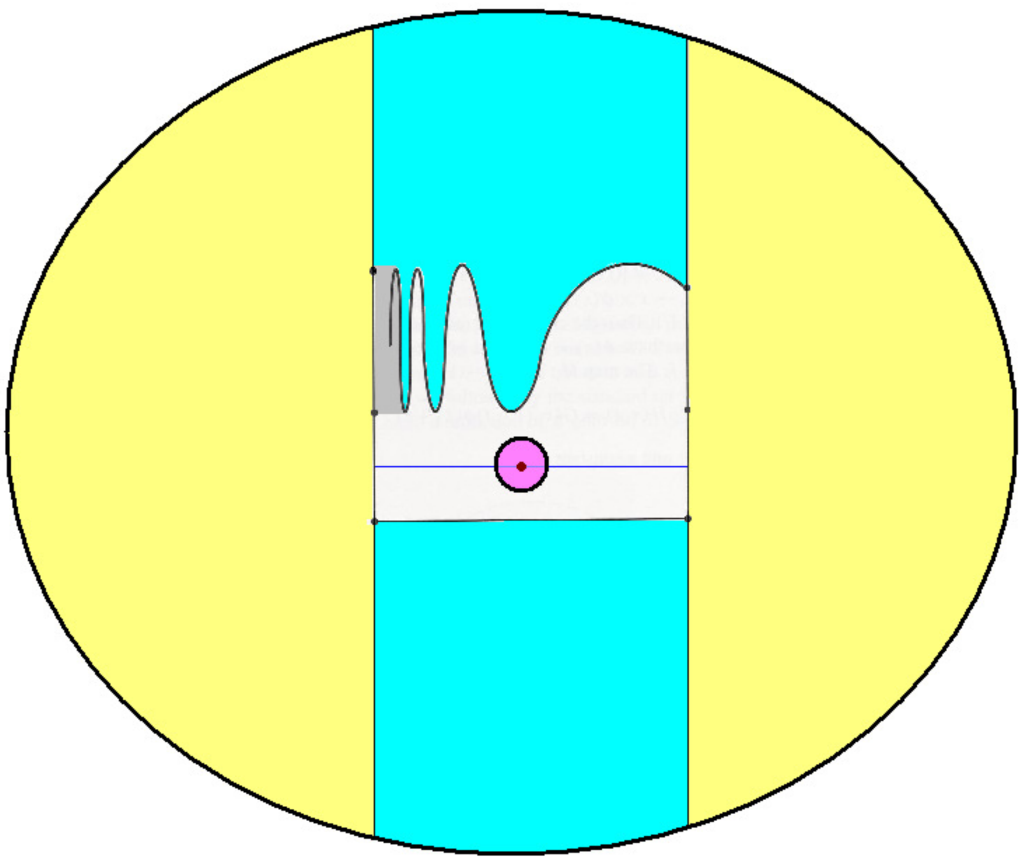
Define functions $\alpha(x)$ and $\beta(x)$ such that each function is equal to $-\frac{3}{2}$ for $x \leq \frac{1}{4}$ or $x \geq \frac{3}{4}$, and over the interval $\frac{1}{4} \leq x \leq \frac{3}{4}$ the functions $\alpha(x)$ and $\beta(x)$ are given by the displayed formula(s), with a positive sign for α and a negative sign for β. Using the preceding definitions, we shall split F into two pieces as follows:

> F_1 is the set of all points in F such that $0 \leq x \leq 1$ and $y \geq \alpha(x)$.
>
> F_2 is the set of all points in F such that $y \leq \beta(x)$.

The subset F_0 is defined to be all points in X which satisfy $0 \leq x \leq 1$ and either $y = \alpha(x)$ or $y = \beta(x)$. These sets are graphs of continuous functions defined on the interval, and hence each is arcwise connected. Since $\alpha(0) = \beta(0)$ the intersection of their graphs is nonempty and hence the union F_0 is also arcwise connected.

All of the subsets defined above are depicted in the drawing on the next page.

The following drawing shows the decompositions of E and F into closed subspaces described on the preceding page. The turquoise and yellow regions are in E, and the light shaded regions are in F. The rectangle shaded in gray is a region where the graph of the function $y = \sin(1/x)$ oscillates too much to be drawn easily. The regions E_1, E_2, E_3 and E_4 are the upper turquoise region, the right hand yellow region, the lower turquoise region and the left hand yellow region respectively. The regions F_1 and F_2 are the regions above and below the blue horizontal line, which is defined by the equation $y = -3/2$. The dark red point is the center of the two circles which form the boundary of X, and the interior of the pink disk is not contained in X or F. For the sake of completeness, we note that the Polish circle P is the union of the graph of the function $y = \sin(1/x)$ together with the vertical and horizontal line segments whose endpoints are indicated by black dots; by construction, this subset is the intersection of E and F.



The coordinates of the dark red point are $(1/2, -3/2)$

The set E_0 is the union of the Polish Circle P with the vertical chords, and the set F_0 is the union of the two blue horizontal line segments with the circle whose radius is ¼ and whose center is the dark red point.

Motivated by the drawings, we can verify that E and F are arcwise connected as follows:

(E1) If $(x, y) \in E_1$ and $x > 0$, then the vertical straight line curve joining (x, y) to the graph of $y = \sin(1/x)$ lies entirely inside E_1. If $(x, y) \in E_1$ and $x = 0$, then (x, y) lies on L_0. In both cases, we have continuous curves in E_1 joining the given point to some point in E_0.

(E2) If $(x, y) \in E_2$, then the horizontal straight line curve joining (x, y) to $(1, y)$ lies entirely in E_2 and its endpoint lies in E_0.

(E3) If $(x, y) \in E_3$, then the vertical straight line curve joining (x, y) to the graph of $y = -2$ lies entirely inside E_3 and its endpoint lies in E_0.

(E4) If $(x, y) \in E_4$, then the horizontal straight line curve joining (x, y) to $(0, y)$ lies entirely in E_4 and its endpoint lies in E_0.

(F1) If $(x, y) \in F_1$, then the vertical straight line curve joining (x, y) to the graph of $y = \alpha(x)$ lies entirely inside F_1 and its endpoint lies in F_0.

(F2) If $(x, y) \in F_2$, then the vertical straight line curve joining (x, y) to the graph of $y = \beta(x)$ lies entirely inside F_2 and its endpoint lies in F_0.

If we combine these statements, we see that every point in E can be joined to a point in the arcwise connected set E_0 by a continuous curve, and similarly every point in F can be joined to a point in the arcwise connected set F_0 by a continuous curve. Since E_0 and F_0 are arcwise connected, it follows that the subsets E and F must be arcwise connected.

Fundamental group computations

By construction we know that $\pi_1(X) \cong \mathbb{Z}$. We claim that $\pi_1(E)$ and $\pi_1(F)$ are at least that large.

LEMMA. *The circle C_+ is a retract of E, and the circle C_- is a retract of F.*

Proof. Since X is homeomorphic to $S^1 \times [0, 1]$ such that C_- corresponds to $S^1 \times \{0\}$ and C_+ corresponds to $S^1 \times \{1\}$, the first coordinate projection and the identifications define retractions $r_\pm : X \to C_\pm$. The desired retractions on E and F are given by restricting r_- to E and r_- to F.■

In fact, one can prove the stronger conclusion that the circles are deformation retracts of E and F, but we shall not need this additional information.

COROLLARY. *The fundamental groups of E and F are infinite; in fact, each has a quotient which is isomorphic to $\mathbb{Z}$.*

Proof. This is true because the retractions $E \to C_+$ and $F \to C_-$ induce surjections of fundamental groups.■

Completion of the argument

We know that $\pi_1(X, x) \cong \mathbb{Z}$, so it is enough to show that the pushout of the diagram

$$\pi_1(E, x) \longleftarrow \pi_1(P, x) \longrightarrow \pi_1(F, x)$$

is not infinite cyclic. Since P is simply connected, the pushout in this case is a free product of $\pi_1(E, x)$ and $\pi_1(F, x)$. By the preceding discussion, neither of these groups is trivial. Since a free product of two nontrivial groups is always nonabelian (use the result on unique factorizations!), it follows that the pushout group is also nonabelian and hence is not isomorphic to $\pi_1(X, x) \cong \mathbb{Z}$.■