# Free Groups Are Free Products

Here is the formal statement:

THEOREM. Let $X = \{x_\alpha \mid \alpha \in A\}$ be a set such that $x_\alpha$ is a generator of a cyclic group $C_\alpha$. Then the free product $\coprod_\alpha C_\alpha$ is a free group on the set $X' = \{i_\alpha(x_\alpha) \mid \alpha \in A\}$.

Here $i_\alpha : C_\alpha \longrightarrow \coprod_{\beta \in A} C_\beta$ is the map which is part of the free product data.

PROOF: Recall that the maps $\mathbb{Z} \longrightarrow C_\alpha$ sending $n$ to $x_\alpha^n$ are group isomorphisms.

To prove the result, we need to verify that $X' \subseteq \coprod_\alpha C_\alpha$ has the appropriate Universal Mapping Property. Suppose that we are given a set-theoretic map $X' \xrightarrow{f} H$, where $H$ is some group. If $i_\alpha(x_\alpha) \in X'$, define a homomorphism $\varphi_\alpha : C_\alpha \longrightarrow H$ as follows: Given $y \in C_\alpha$, there is a unique $n \in \mathbb{Z}$ such that $y = x_\alpha^n$.

Set $\varphi_\alpha(y)$ equal to $f(i_\alpha(x_\alpha))^n$; this is well defined because $n$ is uniquely determined by $y$ and $x_\alpha$, and it is a homomorphism by the Laws of Exponents. By the Universal Mapping Property for free products, there is a unique homomorphism $\Phi : \coprod_\alpha C_\alpha \longrightarrow H$ such that $\Phi \circ i_\alpha = \varphi_\alpha$ for all $\alpha$.

To complete the proof, it will suffice to show that $\Phi | X' = f$ (if this is true, then $\Phi$ must be unique because its values are specified on a set of generators!). This follows immediately from the chain of eqns.

$$\Phi(i_\alpha(x_\alpha)) = \Phi \circ i_\alpha(x_\alpha) = \varphi_\alpha(x_\alpha) = f(i_\alpha(x_\alpha))$$

To summarize, we have shown that the free product $\coprod_\alpha C_\alpha$ is a free group on the set $X'$, which is in 1-1 correspondence with $X$. ∎