Construction of Universal Covering Spaces

In addition to the usual default hypotheses, assume B is locally simply connected* and connected (hence arcwise connected).

* Every point has a neighborhood base of open simply connected sets.

<u>Theorem</u> There is a covering space projection $(E, e_0) \longrightarrow (B, b_0)$ such that E is simply connected.

<u>Construction</u> b_0 = chosen base point.

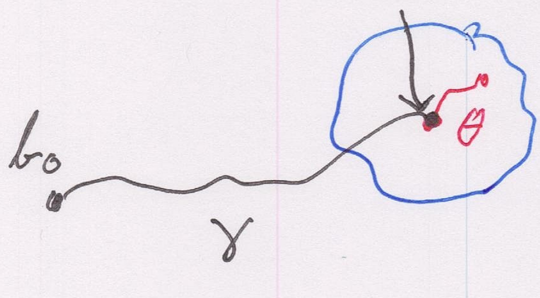
$\Pi(B, b_0)$ = endpoint preserving homotopy classes of curves $\gamma : [0,1] \to B$ such that $\gamma(0) = b_0$.

$$p : \Pi(B, b_0) \longrightarrow B \qquad p([\gamma]) = \gamma(1).$$

We need to put a topology on $\Pi(B, b_0)$ so that $\Pi(B, b_0) \to B$ is a covering space projection, with $\Pi(B, b_0)$ Hausdorff and simply connected.

For each $[\gamma]$ construct a neighborhood base as follows:

Let $b = p([\gamma])$ and let $\{U_\alpha\}$ be a neighborhood base for b consisting of simply connected open sets. Set $U_\alpha^*[\gamma]$ equal to all classes of the form $[\gamma]+[\theta] = [\gamma+\theta]$ where θ is a curve in U_α starting at b.



Since $\pi_1(U_\alpha) = \{1\}$, $[\theta] = [\theta'] \Leftrightarrow$ both have the same end point

It follows that p maps $U_\alpha^*([\gamma])$ to U_α bijectively. Note simply connected $\Rightarrow$ arcwise connected

<u>CLAIM:</u>

If $p([\gamma]) = p([\gamma'])$ but $[\gamma] \neq [\gamma']$, then $U_\alpha^*([\gamma]) \cap U_\alpha^*([\gamma']) = \phi$.

First step in showing p is a covering space projection.

Suppose not. Then $[\gamma] + [\theta] = [\gamma'] + [\theta']$ for some θ'. But the images of θ and θ' lie in U_α, and their endpoints are the same because

$$p(\underbrace{[\gamma + \theta]}_{\theta(1)}) = p(\underbrace{[\gamma' + \theta']}_{\theta'(1)}),$$

So $[\theta] = [\theta']$ since U_α is simply connected.

Hence $[\gamma+\theta]+\left(-[\gamma'+\theta']\right) \simeq$ Constant curve at b_0

But LHS $= [\gamma]+[\theta]+\left(-[\theta']\right)+\left(-[\gamma']\right)$

and $[\theta]=[\theta'] \Rightarrow$ LHS $= [\gamma]+\left(-[\gamma']\right) \Rightarrow$

$[\gamma]=[\gamma']$, contradiction. Hence

$U_\alpha^*([\gamma]) \cap U_\alpha^*([\gamma']) = \emptyset$. ∎

<u>Now</u> define a topology on $\Pi(B,b_0)$ as follows: We know that $p \mid U_\alpha^*([\gamma])$ is 1-1 onto, for if $p([\gamma]+[\theta])=p([\gamma]+[\theta'])$ then $\theta(1)=\theta'(1)$, so by simple connectivity $[\theta]=[\theta']$. Also, each $y \in U_\alpha$ is $\theta(1)$ for some θ since U_α is arcwise connected.

The topology on $\Pi(B, b_0)$ is generated by all sets of the form $p^{-1}[V] \cap U_\alpha^*[\gamma]$ where $V \subseteq U_\alpha$ is open and $p([\gamma]) = b$.

CLAIM p is continuous.

Suppose $p([\gamma]) = b$, and let U_α be a simply connected open neighborhood of b. Then p maps $U_\alpha^*([\gamma])$ into U_α. ∎

CLAIM p is a covering space projection.

Observe that the inverse image of U_α under p is the union of the pairwise disjoint open subsets $U_\alpha^*([\gamma])$ where $[\gamma]$ runs through all classes s.t. $p([\gamma]) = \gamma(1) = b$.

We now need to verify that each restriction mapping $p \mid U_\alpha^*([\gamma])$ is open. This is a little more complicated than it might seem at first. However, since p is 1-1 onto, it suffices to show that images of generating open sets are open.

A typical generating set has the form $U_\alpha^*([\gamma]) \cap \left(U_{\alpha'}^*([\gamma']) \cap p^{-1}[V]\right)$ where V is open in $U_{\alpha'}$ (we may have $p([\gamma']) \neq p([\gamma])$!). The image of such a set is equal to $U_\alpha \cap U_{\alpha'} \cap V = U_\alpha \cap V$ since $V \subseteq U_{\alpha'}$. This is an intersection of open sets in B and U_α, and hence it is open in U_α. ∎

HENCE $\Pi(B, b_0) \longrightarrow B$ IS A COVERING MAP.

Claim: $\Pi(B, b_0)$ is arcwise connected.

Let e_0 = constant curve at b_0. Let $[\gamma] \in \Pi(B, b_0)$, and define $\Gamma(t) = [\gamma_t]$, where $\gamma_t(s) = \gamma(ts)$. Continuity will follow if $p \circ \Gamma$ is continuous since p is locally a homeo. But $p \circ \Gamma(t) = \gamma_t(1) = \gamma(t)$, so $p \circ \Gamma$ is continuous. ∎

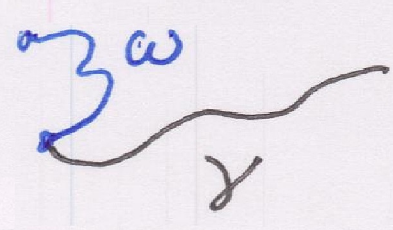
Only one more thing to check:

Claim: $\Pi(B, b_0)$ is simply connected.

By construction $\pi_1(B, b_0)$ acts on the right by covering space transformations

$g = [\omega] \in \pi_1(B, b_0)$ maps $[\gamma]$ to

$[\gamma] \cdot g = [(-\omega) + \gamma]$.



Then $[\gamma]\cdot g_1 g_2 =$

$$[-(\omega_1+\omega_2)+\gamma] = [(-\omega_2)+(-\omega_1)+\gamma] =$$

$$[(-\omega_1)+\gamma]\cdot g_2 = ([\gamma]\cdot g_1)\cdot g_2 .$$

This extends the usual right action on the fiber of b_0, and hence this fiber is $\pi_1(B, b_0)/\text{Image } p_*$.

HOWEVER by construction we know that the fiber is $\pi_1(B, b_0)$ with nothing factored out. This means that Image p_* $= \{1\}$, and since p_* is 1-1 it implies that $\pi(B, b_0)$ is simply connected.