THE HAUPTVERMUTUNG FOR GRAPHS

As indicated in `advancednotes2012.tex`, one fundamental question regarding the homology groups of simplicial complexes is whether homeomorphic polyhedra have isomorphic homology groups. At a very early point, mathematicians realized that if $(P, \mathbf{L})$ is a subdivision of $(P, \mathbf{K})$, then the homology groups of $(P, \mathbf{K})$ and $(P, \mathbf{L})$ are isomorphic; it follows that if $(P, \mathbf{K})$ and $(P', \mathbf{K}')$ are simplicial complexes which have isomorphic subdivisions, then the homology groups of $(P, \mathbf{K})$ and $(P', \mathbf{K}')$ must also be isomorphic. This means that the topological invariance of simplicial homology would follow directly if one could prove the following statement, which was formulated by E. Steinitz (1871–1928) and H. Tietze (1880–1964) around 1908:

Hauptvermutung (*i.e.*, Main or Central Conjecture). *If $(P, \mathbf{K})$ and $(P', \mathbf{K}')$ are simplicial complexes such that P is homeomorphic to P', then they have isomorphic subdivisions.*

In this document we shall prove that this statement is true for 1-dimensional complexes.

Here are three important facts about this conjecture:

1. Within about a decade, the invariance of simplicial homology under homeomorphism — and in fact under homotopy equivalence — was established by an argument which does not require the *Hauptvermutung*. Further information on this point appears on pages ????? of Eilenberg and Steenrod.
2. The *Hauptvermutung* is true for all complexes of dimension ≤ 3, false for all complexes of dimension ≥ 5, and false for some (and possibly all) complexes of dimension 4. The historical notes at the end of this document give further references.
3. Despite the preceding two points, the *Hauptvermutung* has had a major impact on geometric topology; once again, there are references in the historical notes at the end of this document.

A few preliminaries

For the most part the notation follows that of the 205B notes:

`http://math.ucr.edu/≃res/math205B-2012/algtopnotes2012.pdf`

Given a vertex p in a graph $(X, \mathcal{E})$, the *valency* (VAY-len-see) $\mathbf{V}(p)$ of p with respect to $(X, \mathcal{E})$ is the number of edges which have p as a vertex. Since a graph is a finite union of its edges, it follows that for each vertex p the valency $\mathbf{V}(p)$ is a positive integer. By Theorem VII.1.6 this number only depends upon the topological space and not on the choice of decomposition into edges because $H_1(X, X - \{p\})$ is free abelian on $\mathbf{V}(p) - 1$ generators. This has the following simple but far-reaching consequence:

Proposition 1. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two graph structures on the connected space X, and let $n \neq 2$ be a positive integer. Then $p \in X$ is a vertex of $(X, \mathcal{E})$ with valency n if and only if p is a vertex of $(X', \mathcal{E}')$ with valency n.*■

The next statement is a useful reduction of the 1-dimensional Hauptvermutung to a special case.

Theorem 2. *If the Hauptvermutung is true for connected graphs, then it is true for all simplicial complexes of dimension ≤ 1.*

Proof. Clearly the Hauptvermutung will be true for a complex if it is true for each connected component of that complex. Suppose that we have a connected complex $(P, \mathbf{K})$ such that P contains a vertex p which does not lie on a simplex of higher dimension. Then $P - \{p\}$ will be a union of all the remaining simplices, and as such it will be closed in P. Since P is $\mathbf{T}_1$, this means that $P - \{p\}$ is both open and closed, so by connectedness this proper subset of P must be empty. Now the Hauptvermutung is clearly true for a 0-dimensional complex consisting of a single point, so this reduces the proof of the Hauptvermutung for complexes of dimension ≤ 1 to the special case of a connected graph, for a 1-dimensional complex is a graph if every vertex lies on at least one edge.■

The main result(s)

For the sake of completeness, we include is a formal statement of the result to be proved. Since every graph is homeomorphic to a 1-dimensional polyhedron such that the edges and vertices are 1-simplices and 0-simplices, for the rest of this document we shall view graphs as special types of 1-dimensional simplicial complexes.

Theorem 3. (*Hauptvermutung for graphs. If $(P, \mathbf{K})$ and $(P', \mathbf{K}')$ are connected graphs such that P is homeomorphic to P', then they have isomorphic subdivisions.*

Our proof will be based upon an examination of the subspace formed by removing all vertices of valency $\neq 2$. By the preceding comments, if $n \neq 2$ then a homeomorphism $h : P \to P'$ sends the vertices of valency n in P to the vertices of valency n in P'. We shall begin by disposing of an important special case which does not fit particularly well into our approach for the general case.

Proposition 4. *Suppose that $(P, \mathbf{K})$ and $(P', \mathbf{K}')$ are homeomorphic connected graphs such that all vertices in each have valency 2. Then P and P' are homeomorphic to S^1, and the graphs $(P, \mathbf{K})$ and $(P', \mathbf{K}')$ have isomorphic subdivisions.*

Proof of Proposition 4. Choose an arbitrary edge E_1 in $\mathbf{K}$, and let x_0 be one of its endpoints. If x_1 is the other vertex of E_0, then by the hypothesis there is an edge $E_1 \neq E_0$ which also has x_1 as one of its endpoints. Continuing in this manner we can recursively construct sequences of edges $E_1, \ E_2, \ \cdots$ and vertices $x_0, \ x_1, \ \cdots$ such that the endpoints of E_k are x_k and x_{k-1} and *no subsequence of three consecutive edges contains duplications*. The latter condition is equivalent to saying that for all k the edges E_{k-1} and E_{k+1} are distinct (and by construction neither is equal to E_k), and $E_{k-1} = E_{k+1}$ implies that x_k is a vertex of E_{k-1} and x_{k-1} is a vertex of E_{k+1}; since x_k is a vertex of E_{k+1} and x_{k-1} is a vertex of E_{k-1}, the condition $E_{k-1} = E_{k+1}$ would imply that vertices for the edges are the same as the vertices for E_k and hence all three edges would be equal, and we know that this is false by construction. The preceding discussion also implies that no subsequence of three consecutive vertices contains duplications (there is a similar argument in the 205B notes).

Since the complex $(P, \mathbf{K})$ has only finitely many edges, there is some pair of nonnegative integers $u < v$ such that $E_u = E_v$, and by the well-ordering of the positive integers we can choose u and v such that $v - u$ is the minimum value for all such differences. By the discussion in the preceding paragraph we must have $v - u \geq 3$, and since the vertices of the identical edges E_u and E_v are the same, we must have

$$(x_{u-1}, x_u) \quad = \quad (x_{v-1}, x_v) \quad \text{or} \quad (x_v, x_{v-1})$$

(where (y, z) denotes an ordered pair of vertices).

We claim that the first possibility always holds, so we shall assume the second holds and derive a contradiction. The assumption that all vertices have valency 2 implies that there are only two edges which have $x_u = x_{v-1}$ as an endpoint. On one hand, they are E_u and E_{u+1}, but on the other hand they are also E_{v-1} and E_v. Since $E_u = E_v$, this means that E_{u+1} must be the same as E_{v-1}, and since $v - u \geq 3$ it follows that

$$v - 1 \quad \geq \quad u + 2 \quad > \quad u + 1 \qquad \text{and} \qquad (v-1) \; - \; (u+1) \quad = \quad v - u - 2 \quad < \quad v - u \; .$$

This is a contradiction because $u' = u+1$ and $v' = v-1$ satisfy $u' < v'$, $E_{u'} = E_{v'}$, and $v' - u' < v - u$ and (u, v) was chosen so that $v - u$ was the minimum value for all pairs satisfying $s < t$, $E_s = E_t$.

Having shown that $(x_{u-1}, x_u) = (x_{v-1}, x_v)$, we can now conclude that $E_{u+1} = E_{v+1}$ and hence also $x_{u+1} = x_{v+1}$. To see this, recall that the edges containing $x_u = x_v$ are E_u and E_{u+1} on one hand and E_v and E_{v+1} on the other, and since $E_u = E_v$ we must also have $E_{u+1} = E_{v+1}$. More generally, in the same manner we can prove recursively that $E_{u+k} = E_{v+k}$ for all $k \geq 0$ and also that $E_{u-k} = E_{v-k}$ for all k satisfying $0 \leq k \leq u$. In other words, the sequence of edges is periodic with period $v - u$. It follows immediately that P is homeomorphic to a circle. Similar considerations yield the same conclusion for P'.

Finally, we need to show that $(P, \mathbf{K})$ and $(P', \mathbf{K}')$ have isomorphic subdivisions. Choose a vertex for each decomposition, and choose homeomorphisms $h : P \to S^1$ and $h' : P' \to S^1$ which send these vertices to 1 (the latter condition can be realized using the rotational symmetry of the circle). The images of all remaining vertex points for the decompositions all have the form $\exp(2\,\pi\, i\, t_\alpha)$ for unique real numbers $t_\alpha \in (0, 2\,\pi)$. Reorder these numbers in sequence as $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 2\pi$. If, $a < b$ such that the images of t_a and t_b in P are vertices but there is no c such that $a < c < b$ and the image of t_c is a vertex of P, then the counterclockwise open arc from the image of t_a to the image of t_b will be an open edge of P (by connectedness it is contained in an open edge since the complement of the vertices is the union of the pairwise disjoint open edges, and the union of the images of the open arcs is the complement of the vertices in P — since each of the subsets in question is connected and they are pairwise disjoint, the two collections must coincide), and likewise if P is replaced by P'. Therefore the images of the numbers t_k in P and P' determine subdivisions of both $(P, \mathbf{K})$ and $(P', \mathbf{K}')$, and these subdivisions are isomorphic.■

We are now ready to start working on the general case.

Proposition 5. *Suppose that $(P, \mathbf{K})$ is a connected graph, let $A \subset P$ be the set of all vertices with valency $\neq 2$, and suppose that A is nonempty. Let $\{U_\alpha \mid \alpha \in \Lambda\}$ denote the set of component for $U = P - A$. Then the following hold:*

(i) If E is an edge of $(P, \mathbf{K})$, then $E \cap U$ is contained in a component of U.

(ii) If U_α is a component of U such that $x \in U_\alpha$ is a vertex of $(P, \mathbf{K})$, and E is an edge such that $x \in E$, then $E \cap U \subset U_\alpha$.

(iii) For each component U_α of U, its closure $P_\alpha = \overline{U_\alpha}$ (in P) is a subcomplex of $(P, \mathbf{K})$.

(iv) Each closure P_α is either a simple circuit or a reduced edge path joining two distinct vertices. In the first case $P_\alpha - U_\alpha$ is a single vertex, and in the second case $P_\alpha - U_\alpha$ consists of two vertices.

(v) If $\alpha \neq \beta$ then $P_\alpha \cap P_\beta$ consists of 0, 1 or 2 vertices.

There is a drawing for the roof of this result in the course directory file `haupt4graphs2.pdf`.

NOTATION. Given an edge E in a graph P, the corresponding *open edge* E° is given by removing the endpoints (equivalently, vertices) of E; this set is open in P because its complement is the union of all (closed) edges other than E along with the vertices of E.

Note that since P is locally arcwise connected, the components of its open subsets are open in P.

Proof. (i) Let E be an edge, and let E° be the corresponding open edge. Since A is a finite set of vertices, it follows that $E^\circ \subset U = P - A$, and by connectedness E° lies in some component U_α of $U = P - A$. Now the closure of E° in U is equal to $E \cap U$ (since E is the closure of E° in P), and since the closure of a connected subset is connected we must also have $E \cap U \subset U_\alpha$.

(ii) Suppose that $x \in U$ is a vertex of the edge E, and let U_β be the component of U such that $E^\circ \subset U_\beta$. Then (i) implies that $x \in E \cap U$ and $x \in U_\beta$. Since the components form a pairwise disjoint decomposition of U, it follows that $U_\beta = U_\alpha$ and hence $E \cap U \subset U_\alpha$.

(iii) Let $\{E_t \mid t \in T\}$ be the set of all edges such that $E_t^\circ \subset U_\alpha$. Then by (ii) we have

$$\cup_t \, E_t^\circ \quad \subset \quad U_\alpha \quad \subset \quad \cup_t \, (E_t \cap U) \quad \subset \quad \cup_t \, E_t \; .$$

Taking closures, we see that

$$\cup_t \, E_t \quad \subset \quad \cup_t \, \overline{E_t^\circ} \quad \subset \quad \overline{U_\alpha} \quad \subset \quad \cup_t \, E_t$$

and hence $P_\alpha = \overline{U_\alpha}$ is a subcomplex.

(iv) This is the most complicated part of the proof. The drawing in `haupt4graphs2.pdf` may provide some helpful insight; in particular, it depicts many of the different possibilities which are mentioned at various points in the argument.

If A is empty then the statements of (iv) and (v) follow from Proposition 4, so for the rest of this proof we shall assume A is nonempty.

STEP 1. Given a component U_α of $U = P - A$, we claim that *there is an edge E such that $E^\circ \subset U_\alpha$ but at least one vertex of E does not belong to U_α.* — We shall assume this is false and derive a contradiction.

The negation of the assertion is that $E^\circ \subset U_\alpha$ implies $E \subset U_\alpha$ for all E, so this is what we are assuming. By (iii) we know that U_α is equal to a finite union of the compact (hence closed) subsets E_t such that $E_t^\circ \subset U_\alpha$, and hence U_α is closed and equal to the subcomplex $P_\alpha = \overline{U_\alpha}$. Since a vertex of the graph has valency 2 if it lies in $U = P - A$, we know this condition holds for P_α, and therefore P_α is a simple circuit by Proposition 4. If we can show that $U_\alpha = P_\alpha = P$ in this case, then it will follow that P has no vertices of valency $\neq 2$, contradicting the assumption that A is nonempty.

Suppose that E^* is an edge of P which is not in P_α. Since he vertices of P_α all have valency 2 in P and each such vertex lies on two edges in P_α, it follows that no vertex of E^* can belong to P_α. Therefore, if $S \subset P$ is the union of all edges which are not in P_α, then $S \cap P = \emptyset$. By construction S is closed in P; since P is connected, it follows that S and P_α cannot both be nonempty, and since P_α is nonempty it follows that S must be empty, so that $P = P_\alpha$. As noted in the preceding paragraph, the assertion for Step 1 follows immediately.

STEP 2. Given U_α, by the preceding step we can choose an edge E_1 such that $E_1^\circ \subset U_\alpha$ but the vertex $x_0 \in E_1$ does not lie in U_α. We now claim that *there is a simple edge path or simple circuit $E_1 E_2 \;\cdots\; E_m$ in P_α — with endpoints $x_j \in E_j \cap E_{j-1}$ for $2 \leq j \leq m-1$ and a second endpoint x_m for E_m — such that $x_j \in U_\alpha$ for $2 \leq j \leq m-1$ but $x_m \notin U_\alpha$.*

As the drawing in `haupt4graphs2.pdf` suggests, the notation is meant to include the cases $k = 1$ (for which the edge path sequence is just E_1) and $k = 2$ (for which there are no closed edges E_j completely contained in U_α). The drawing also gives examples where the edge sequence $E_1 E_2 \ \cdots \ E_m$ can be either a simple edge path or a simple circuit depending upon whether $x_0 \neq x_m$ or $x_0 = x_m$.

To prove the claim, consider all *admissible* edge sequences $E_1 E_2 \ \cdots \ E_k$ starting with E_1 and satisfying the following conditions:

(1) Each open edge E_j° is contained in U_α.

(2) There are no duplications in the sequence.

(3) For each j such that $2 \leq j \leq k$ the consecutive pair of edges $\{E_{j-1}, E_j\}$ has a vertex x_j in common, and this common vertex belongs to U_α (if $k = 1$ this is an empty statement).

The one term sequence E_1 is a trivial example of such a sequence, and since the graph has only finitely many edges there is a maximal sequence of this type.

The proof of the claim reduces to showing that if $E_1 E_2 \ \cdots \ E_m$ is a maximal sequence then the second vertex x_m of E_m is not in U_α. We shall prove the contrapositive statement: *If $x_m \in U_\alpha$ then the sequence is not maximal.*

If $x_m \in U_\alpha$, then it has valency 2 in P, and hence there is a unique edge $E_{m+1} \neq E_m$ such that $x_m \in E_{m+1}$. By (ii) we know that $E_{m+1} \cap U \subset U_\alpha$.

Using homeomorphisms from E_{m+1} and E_m to the standard closed interval $[0, 1]$, we can find small half-open intervals $N_- \subset E_m$ and $N_+ \subset E_{m+1}$ with endpoint x_m such that N_- and N_+ are open neighborhoods of x_m in $E_1 \cup \ \cdots \ E_m$ and E_{m+1} respectively, and we can do this so that $N_+ \cup N_-$ is an open neighborhood of x_m in P because x_m has valency 2 in P. The sequence $E_1 \ \cdots \ E_m E_{m+1}$ will satisfy the admissibility conditions if and only if $E_{m+1} \neq E_j$ for $j \leq m$. Assume to the contrary that $E_{m+1} = E_j$ for some such j. Then N_- is an open neighborhood of x_m in $E_1 \cup \ \cdots \ \cup E_m \cup E_{m+1} = E_1 \cup \ \cdots \ \cup E_m$. On the other hand, N_- is not open in the open subset

$$N_+ \cup N_- \quad \subset \quad E_1 \cup \ \cdots \ \cup E_m \cup E_{m+1}$$

because if $a < 0 < b$ then the half open interval $(a, 0]$ is not open in the open interval (a, b). This contradicts the following elementary observation:

> If X is a topological space and $V_1 \subset V_2 \subset X$ such that each V_i is open in X, then V_1 is also open in V_2.

As noted above, this completes Step 2.

In the final step of the proof it will be convenient to write

$$W \quad = \quad \left(\bigcup_{j=1}^{m} E_j \right) \ - \ \{x_0, x_m\} \quad = \quad \left(\bigcup_{j=1}^{m} E_j \right) \ \cap \ U_\alpha \ .$$

By construction W is closed in U_α.

STEP 3. The preceding steps reduce the proof of Proposition 5 to showing that $W = U_\alpha$. — Since U_α is connected and W is nonempty, it suffices to prove that W is open in $u - \alpha$.

Suppose that E^* is an edge such that $(E^*)^\circ \subset U_\alpha$ but E^* does not appear in the maximal sequence $E_1 E_2 \ \cdots \ E_m$. Since the vertex x_j has valency 2 for $1 \leq j \leq k - 1$ and x_j is an endpoint of both E_j and E_{j-1}, it follows that x_j cannot be an endpoint for E^*. Since distinct edges can only meet in a common endpoint, it follows that $E^* \cap W = \emptyset$. If $\{F_\gamma \mid \gamma \in \Gamma\}$ is the set of all such edges and S is the union of these edges, then S is compact and $S \cap W = \emptyset$. Therefore $S \cap U_\alpha = W'$ is closed in U_α; by construction we know that $W \cup W' = U_\alpha$ and $W \cap W' = \emptyset$, so W and W' are disjoint closed subsets of U_α whose union is U_α. Since U_α is connected and $W \neq \emptyset$, it follows that W' is empty and $W = U_\alpha$, which is what we needed to prove.■

Proof of Theorem 2

Let $(P, \mathbf{K})$ and $(P', \mathbf{K}')$ be connected graphs, and assume that there is a homeomorphism $h : P \to P'$. If $A \subset P$ and $A' \subset P'$ are the sets of vertices with valency $\neq 2$, then we have already observed that $h[A] = A'$, and of course it follows that $h[P - A] = P' - A'$. If $\{U_\alpha \mid \alpha \in \Lambda\}$ and $\{V_\beta \mid \beta \in \Theta\}$ are the components of $P - A$ and $P' - A'$ respectively, then h induces a 1–1 correspondence between these sets of components such that for all α we have $h[U_\alpha] = V_{\beta(\alpha)}$. Furthermore, if we denote the closures of these components by P_α and P'_β respectively, then it follows that $h[P_\alpha] = P'_{\beta(\alpha)}$. Each of these complexes is homeomorphic to either S^1 or $[0, 1]$ by Proposition 5, and of course h preserves the homeomorphism types of the subspaces P_α. Furthermore, in each case h sends the vertices in $P_\alpha \cap A$ to the vertices in $P'_{\beta(\alpha)}$. If P_α is homeomorphic to S^1, then the last paragraph in the proof of Proposition 4 implies that the subcomplexes $P_\alpha]$ and $P'_{\beta(\alpha)}$ have isomorphic subdivisions. A similar argument proves there are also isomorphic subdivisions if P_α is homeomorphic to $[0, 1]$ (the details are left to the reader).

If we combine the isomorphic subdivisions described in the preceding two sentences, we obtain isomorphic subdivisions of the entire complexes P and P'.■

Historical notes

The validity of the *Hauptvermutung* for 1-dimensional complexes was understood known well before Steinitz and Tietze formulated the general statement explicitly, but there does not seem to be definitive information about who discovered it first and when this was done. The 2-dimensional and 3-dimensional cases were respectively established by C. D. Papkyriakopoulos in the 1940s and E. M. Brown in the 1960s, and the cited paper by Brown contains proofs of both cases; in particular, the 2-dimensional case is Theorem 4.6 in that paper (for the record, E. M. Brown and the algebraic topologist E. H. Brown, who proved the representability theorem often found in algebraic topology books, are not the same person). In 1961 J. W. Milnor produced explicit 7-dimensional counterexamples to the *Hauptvermutung*, and in 1969 R. Kirby and L. Siebenmann constructed counterexamples in each dimension greater than or equal to 5 (see the paper by Siebenmann). Subsequent work of R. D. Edwards and J. W. Cannon produced infinite families of counterexamples for which the underlying spaces are all spheres of dimension ≥ 5 (see pp. 833–834 of the article by Cannon), and one can use their results to construct infinite families of counterexamples for which the underlying space is an arbitrary polyhedron of dimension ≥ 5 (the proof is fairly straightforward, but does not seem to be stated explicitly in the literature; we shall not try to outline a proof because it requires a considerable amount of background material from a subject called *piecewise linear topology*). Apparently the first potential 4-dimensional counterexamples were found by S. Cappell and J. Shaneson in the mid 1970s but not shown to be counterexamples until later work of M. H. Freedman in the 1980s (see the bottom of the first page in the Cappell-Shaneson paper and Section 11.3 in the Freedman-Quinn book). Many other examples have been constructed since then, and in fact one can combine the work of Freedman with later work of S. Donaldson to show that there can be infinitely many inequivalent examples within some homeomorphism classes. However, in contrast to higher dimensions it is not known if there are counterexamples for which the underlying space is an arbitrary 4-dimensional polyhedron (in particular, this is not known for S^4).

The *Hauptvermutung* has had a very strong influence on the development of geometric topology. Before Milnor's discovery of counterexamples, much of the emphasis was on efforts to prove the conjecture, and this succeeded in low dimensions and under some regularity conditions for the simplicial decomposition (*e.g.*, the results in the second half of Munkres, *Elementary Differential Topology*). The existence of counterexamples came as a surprise to many topologists; although the construction used some techniques that had been around for two decades, the latter were neither widely known or well understood at the time.

Although the discovery of counterexamples to the *Hauptvermutung* clearly changed the direction of work on this issue, there were also other factors which shaped subsequent research in the area. Around the time when the counterexamples were discovered, topologists had also discovered that a weaker version of *Hauptvermutung* (the *manifold Hauptvermutung* discussed in the first article of the book edited by Ranicki) was true for many examples, and the conjecture was one of the central motivating questions for the breakthroughs in the general theory of topological manifolds which was constructed mainly in the 1960s and 1970s (part of this is described in Siebenmann's paper). Subsequent work has extended that theory to study suitably defined manifolds with singularities (see the book by S. Weinberger).

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