COVERING SPACES WITH CROSS SECTIONS

<u>Problem</u> Let $E \xrightarrow{p} B$ be a covering space where E & B are connected Hausdorff spaces and both are locally simply ($\Rightarrow$ arcwise) connected. If there is a continuous map $\sigma : B \to E$ such that $p \circ \sigma = id_B$ (i.e., σ is a <u>cross-section</u>), then p is a homeomorphism.

Pick basepts $\left\{ \begin{matrix} b_0 \in B \\ e_0 \in E \end{matrix} \right\}$ $p(e_0) = b_0$

<u>Solution</u> On the π_1 level

$$\pi_1(B, b_0) \xrightarrow{\sigma_*} \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0)$$

$p_* \sigma_* =$ identity on $\pi_1(B, b_0)$, which means σ_* is 1-1 and p_* is onto. Since p is a connected covering, p_* is also 1-1 and hence p_* is an isomorphism. By the classification of coverings $E \xrightarrow{p} B$ is equivalent to the trivial 1-sheeted covering $id_B : B \to B$. But this means that p is a homeomorphism. ∎