8 The Geometry of Vector Spaces

INTRODUCTORY EXAMPLE

The Platonic Solids

In the city of Athens in 387 B.C., the Greek philosopher Plato founded an Academy, sometimes referred to as the world's first university. While the curriculum included astronomy, biology, political theory, and philosophy, the subject closest to his heart was geometry. Indeed, inscribed over the doors of his academy were these words: *"Let no one destitute of geometry enter my doors."*

The Greeks were greatly impressed by geometric patterns such as the regular solids. A polyhedron is called regular if its faces are congruent regular polygons and all the angles at the vertices are equal. As early as 100 years before Plato, the Pythagoreans knew at least three of the regular solids: the tetrahedron (4 triangular faces), the cube (6 square faces), and the octahedron (8 triangular faces). (See Figure 1.) These shapes occur naturally as crystals of common minerals. There are only five such regular solids, the remaining two being the dodecahedron (12 pentagonal faces) and the icosahedron (20 triangular faces).

Plato discussed the basic theory of these five solids in the dialogue *Timaeus*, and since then they have carried his name: the Platonic solids.

For centuries there was no need to envision geometric objects in more than three dimensions. But nowadays mathematicians regularly deal with objects in vector spaces having four, five, or even hundreds of dimensions. It is not necessarily clear what geometrical properties one might ascribe to these objects in higher dimensions.

For example, what properties do lines have in 2-space and planes have in 3-space that would be useful in higher dimensions? How can one characterize such objects? Sections 8.1 and 8.4 provide some answers. The hyperplanes of Section 8.4 will be important for understanding the multidimensional nature of the linear programming problems in Chapter 9.

What would the analogue of a polyhedron "look like" in more than three dimensions? A partial answer is provided by two-dimensional projections of the four-dimensional object, created in a manner analogous to two-dimensional projections of a three-dimensional object. Section 8.5 illustrates this idea for the four-dimensional "cube" and the four-dimensional "simplex."

The study of geometry in higher dimensions not only provides new ways of visualizing abstract algebraic concepts, but also creates tools that may be applied in $\mathbb{R}^3$. For instance, Sections 8.2 and 8.6 include applications to computer graphics, and Section 8.5 outlines a proof (in Exercise 22) that there are only five regular polyhedra in $\mathbb{R}^3$.

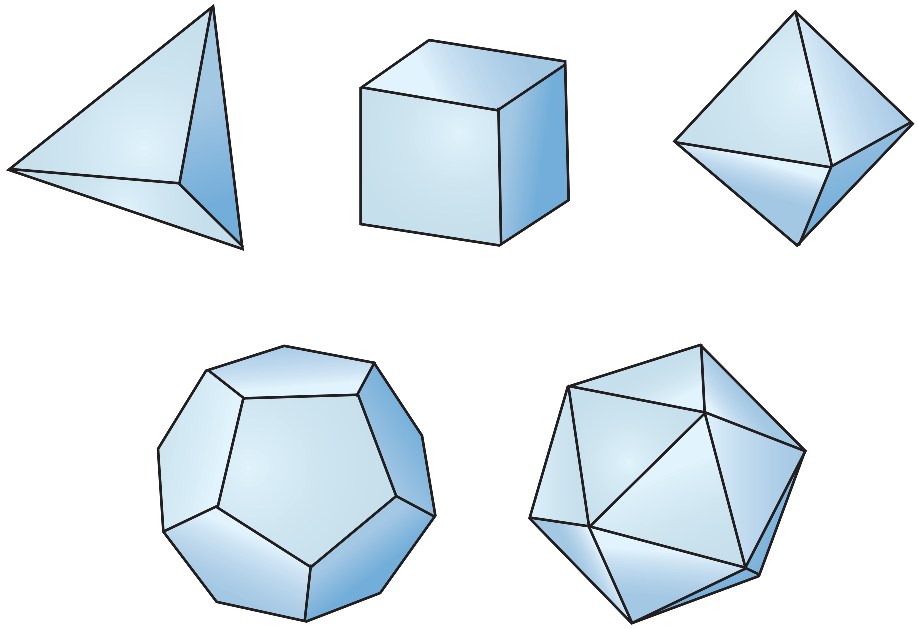


FIGURE 1 The five Platonic solids.

Most applications in earlier chapters involved algebraic calculations with subspaces and linear combinations of vectors. This chapter studies sets of vectors that can be visualized as geometric objects such as line segments, polygons, and solid objects. Individual vectors are viewed as points. The concepts introduced here are used in computer graphics, linear programming (in Chapter 9), and other areas of mathematics.[1]

Throughout the chapter, sets of vectors are described by linear combinations, but with various restrictions on the weights used in the combinations. For instance, in Section 8.1, the sum of the weights is 1, while in Section 8.2, the weights are positive and sum to 1. The visualizations are in $\mathbb{R}^2$ or $\mathbb{R}^3$, of course, but the concepts also apply to $\mathbb{R}^n$ and other vector spaces.

8.1 AFFINE COMBINATIONS

An affine combination of vectors is a special kind of linear combination. Given vectors (or "points") $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$ and scalars $c_1, \dots, c_p$, an **affine combination** of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ is a linear combination

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

such that the weights satisfy $c_1 + \cdots + c_p = 1$.

[1] See Foley, van Dam, Feiner, and Hughes, *Computer Graphics—Principles and Practice*, 2nd edition (Boston: Addison-Wesley, 1996), pp. 1083–1112. That material also discusses coordinate-free "affine spaces."

DEFINITION

The set of all affine combinations of points in a set S is called the **affine hull** (or **affine span**) of S, denoted by aff S.

The affine hull of a single point $\mathbf{v}_1$ is just the set $\{\mathbf{v}_1\}$, since it has the form $c_1\mathbf{v}_1$ where $c_1 = 1$. The affine hull of two distinct points is often written in a special way. Suppose $\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ with $c_1 + c_2 = 1$. Write t in place of c_2, so that $c_1 = 1 - c_2 = 1 - t$. Then the affine hull of $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the set

$$\mathbf{y} = (1-t)\mathbf{v}_1 + t\mathbf{v}_2, \quad \text{with } t \text{ in } \mathbb{R} \tag{1}$$

This set of points includes $\mathbf{v}_1$ (when $t = 0$) and $\mathbf{v}_2$ (when $t = 1$). If $\mathbf{v}_2 = \mathbf{v}_1$, then (1) again describes just one point. Otherwise, (1) describes the *line* through $\mathbf{v}_1$ and $\mathbf{v}_2$. To see this, rewrite (1) in the form

$$\mathbf{y} = \mathbf{v}_1 + t(\mathbf{v}_2 - \mathbf{v}_1) = \mathbf{p} + t\mathbf{u}, \quad \text{with } t \text{ in } \mathbb{R}$$

where $\mathbf{p}$ is $\mathbf{v}_1$ and $\mathbf{u}$ is $\mathbf{v}_2 - \mathbf{v}_1$. The set of all multiples of $\mathbf{u}$ is Span $\{\mathbf{u}\}$, the line through $\mathbf{u}$ and the origin. Adding $\mathbf{p}$ to each point on this line translates Span $\{\mathbf{u}\}$ into the line through $\mathbf{p}$ parallel to the line through $\mathbf{u}$ and the origin. See Figure 1. (Compare this figure with Figure 5 in Section 1.5.)

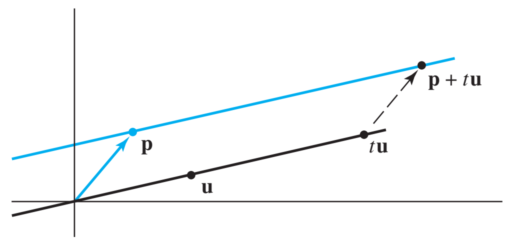


FIGURE 1

Figure 2 uses the original points $\mathbf{v}_1$ and $\mathbf{v}_2$, and displays aff $\{\mathbf{v}_1, \mathbf{v}_2\}$ as the line through $\mathbf{v}_1$ and $\mathbf{v}_2$.

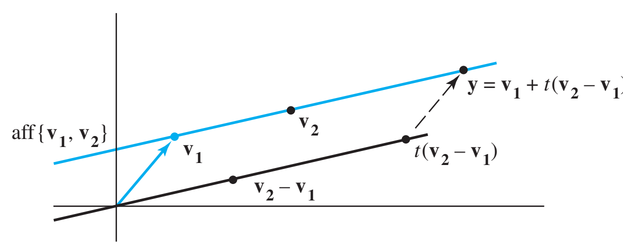


FIGURE 2

Notice that while the point $\mathbf{y}$ in Figure 2 is an affine combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, the point $\mathbf{y} - \mathbf{v}_1$ equals $t(\mathbf{v}_2 - \mathbf{v}_1)$, which is a linear combination (in fact, a multiple) of $\mathbf{v}_2 - \mathbf{v}_1$. This relation between $\mathbf{y}$ and $\mathbf{y} - \mathbf{v}_1$ holds for any affine combination of points, as the following theorem shows.

THEOREM 1

A point $\mathbf{y}$ in $\mathbb{R}^n$ is an affine combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$ if and only if $\mathbf{y} - \mathbf{v}_1$ is a linear combination of the translated points $\mathbf{v}_2 - \mathbf{v}_1, \dots, \mathbf{v}_p - \mathbf{v}_1$.

PROOF If $\mathbf{y}-\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2-\mathbf{v}_1, \dots, \mathbf{v}_p-\mathbf{v}_1$, there exist weights $c_2, \dots, c_p$ such that

$$\mathbf{y}-\mathbf{v}_1 = c_2(\mathbf{v}_2-\mathbf{v}_1)+\cdots+c_p(\mathbf{v}_p-\mathbf{v}_1) \tag{2}$$

Then

$$\mathbf{y} = (1-c_2-\cdots-c_p)\mathbf{v}_1 + c_2\mathbf{v}_2+\cdots+c_p\mathbf{v}_p \tag{3}$$

and the weights in this linear combination sum to 1. So $\mathbf{y}$ is an affine combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$. Conversely, suppose

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2+\cdots+c_p\mathbf{v}_p \tag{4}$$

where $c_1+\cdots+c_p = 1$. Since $c_1 = 1-c_2-\cdots-c_p$, equation (4) may be written as in (3), and this leads to (2), which shows that $\mathbf{y}-\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2-\mathbf{v}_1, \dots, \mathbf{v}_p-\mathbf{v}_1$. ■

In the statement of Theorem 1, the point $\mathbf{v}_1$ could be replaced by any of the other points in the list $\mathbf{v}_1, \dots, \mathbf{v}_p$. Only the notation in the proof would change.

EXAMPLE 1 Let $\mathbf{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}2\\5\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}1\\3\end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix}-2\\2\end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix}4\\1\end{bmatrix}$. If possible, write $\mathbf{y}$ as an affine combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$.

SOLUTION Compute the translated points

$$\mathbf{v}_2-\mathbf{v}_1 = \begin{bmatrix}1\\3\end{bmatrix},\quad \mathbf{v}_3-\mathbf{v}_1 = \begin{bmatrix}0\\1\end{bmatrix},\quad \mathbf{v}_4-\mathbf{v}_1 = \begin{bmatrix}-3\\0\end{bmatrix},\quad \mathbf{y}-\mathbf{v}_1 = \begin{bmatrix}3\\-1\end{bmatrix}$$

To find scalars c_2, c_3, and c_4 such that

$$c_2(\mathbf{v}_2-\mathbf{v}_1)+c_3(\mathbf{v}_3-\mathbf{v}_1)+c_4(\mathbf{v}_4-\mathbf{v}_1) = \mathbf{y}-\mathbf{v}_1 \tag{5}$$

row reduce the augmented matrix having these points as columns:

$$\begin{bmatrix}1 & 0 & -3 & 3\\ 3 & 1 & 0 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -3 & 3\\ 0 & 1 & 9 & -10\end{bmatrix}$$

This shows that equation (5) is consistent, and the general solution is $c_2 = 3c_4+3$, $c_3 = -9c_4-10$, with c_4 free. When $c_4 = 0$,

$$\mathbf{y}-\mathbf{v}_1 = 3(\mathbf{v}_2-\mathbf{v}_1)-10(\mathbf{v}_3-\mathbf{v}_1)+0(\mathbf{v}_4-\mathbf{v}_1)$$

and

$$\mathbf{y} = 8\mathbf{v}_1+3\mathbf{v}_2-10\mathbf{v}_3$$

As another example, take $c_4 = 1$. Then $c_2 = 6$ and $c_3 = -19$, so

$$\mathbf{y}-\mathbf{v}_1 = 6(\mathbf{v}_2-\mathbf{v}_1)-19(\mathbf{v}_3-\mathbf{v}_1)+1(\mathbf{v}_4-\mathbf{v}_1)$$

and

$$\mathbf{y} = 13\mathbf{v}_1+6\mathbf{v}_2-19\mathbf{v}_3+\mathbf{v}_4$$ ■

While the procedure in Example 1 works for arbitrary points $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$, the question can be answered more directly if the chosen points $\mathbf{v}_i$ are a basis for $\mathbb{R}^n$. For example, let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be such a basis. Then any $\mathbf{y}$ in $\mathbb{R}^n$ is a unique *linear* combination of $\mathbf{b}_1, \dots, \mathbf{b}_n$. This combination is an affine combination of the $\mathbf{b}$'s if and only if the weights sum to 1. (These weights are just the $\mathcal{B}$-coordinates of $\mathbf{y}$, as in Section 4.4.)

EXAMPLE 2 Let $\mathbf{b}_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. The set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbb{R}^3$. Determine whether the points $\mathbf{p}_1$ and $\mathbf{p}_2$ are affine combinations of the points in $\mathcal{B}$.

SOLUTION Find the $\mathcal{B}$-coordinates of $\mathbf{p}_1$ and $\mathbf{p}_2$. These two calculations can be combined by row reducing the matrix $[\,\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3 \;\; \mathbf{p}_1 \;\; \mathbf{p}_2\,]$, with two augmented columns:

$$\begin{bmatrix} 4 & 0 & 5 & 2 & 1 \\ 0 & 4 & 2 & 0 & 2 \\ 3 & 2 & 4 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & \frac{2}{3} \\ 0 & 1 & 0 & -1 & \frac{2}{3} \\ 0 & 0 & 1 & 2 & -\frac{1}{3} \end{bmatrix}$$

Read column 4 to build $\mathbf{p}_1$, and read column 5 to build $\mathbf{p}_2$:

$$\mathbf{p}_1 = -2\mathbf{b}_1 - \mathbf{b}_2 + 2\mathbf{b}_3 \quad \text{and} \quad \mathbf{p}_2 = \tfrac{2}{3}\mathbf{b}_1 + \tfrac{2}{3}\mathbf{b}_2 - \tfrac{1}{3}\mathbf{b}_3$$

The sum of the weights in the linear combination for $\mathbf{p}_1$ is -1, not 1, so $\mathbf{p}_1$ is *not* an affine combination of the **b**'s. However, $\mathbf{p}_2$ *is* an affine combination of the **b**'s, because the sum of the weights for $\mathbf{p}_2$ is 1. ■

DEFINITION

A set S is **affine** if $\mathbf{p}, \mathbf{q} \in S$ implies that $(1 - t)\mathbf{p} + t\mathbf{q} \in S$ for each real number t.

Geometrically, a set is affine if whenever two points are in the set, the entire line through these points is in the set. (If S contains only one point, $\mathbf{p}$, then the line through $\mathbf{p}$ and $\mathbf{p}$ is just a point, a "degenerate" line.) Algebraically, for a set S to be affine, the definition requires that every affine combination of two points of S belong to S. Remarkably, this is equivalent to requiring that S contain every affine combination of an arbitrary number of points of S.

THEOREM 2

A set S is affine if and only if every affine combination of points of S lies in S. That is, S is affine if and only if $S = \text{aff } S$.

Remark: See the remark prior to Theorem 5 in Chapter 3 regarding mathematical induction.

PROOF Suppose that S is affine and use induction on the number m of points of S occurring in an affine combination. When m is 1 or 2, an affine combination of m points of S lies in S, by the definition of an affine set. Now, assume that every affine combination of k or fewer points of S yields a point in S, and consider a combination of $k + 1$ points. Take $\mathbf{v}_i$ in S for $i = 1, \ldots, k + 1$, and let $\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1}$, where $c_1 + \cdots + c_{k+1} = 1$. Since the c_i's sum to 1, at least one of them must not be equal to 1. By reindexing the $\mathbf{v}_i$ and c_i, if necessary, we may assume that $c_{k+1} \neq 1$. Let $t = c_1 + \cdots + c_k$. Then $t = 1 - c_{k+1} \neq 0$, and

$$\mathbf{y} = (1 - c_{k+1})\left(\frac{c_1}{t}\mathbf{v}_1 + \cdots + \frac{c_k}{t}\mathbf{v}_k\right) + c_{k+1}\mathbf{v}_{k+1} \tag{6}$$

By the induction hypothesis, the point $\mathbf{z} = (c_1/t)\mathbf{v}_1 + \cdots + (c_k/t)\mathbf{v}_k$ is in S, since the coefficients sum to 1. Thus (6) displays $\mathbf{y}$ as an affine combination of two points in S, and so $\mathbf{y} \in S$. By the principle of induction, every affine combination of such points lies in S. That is, aff $S \subset S$. But the reverse inclusion, $S \subset \text{aff } S$, always applies. Thus, when S is affine, $S = \text{aff } S$. Conversely, if $S = \text{aff } S$, then affine combinations of two (or more) points of S lie in S, so S is affine. ■

The next definition provides terminology for affine sets that emphasizes their close connection with subspaces of $\mathbb{R}^n$.

DEFINITION

A translate of a set S in $\mathbb{R}^n$ by a vector $\mathbf{p}$ is the set $S + \mathbf{p} = \{\mathbf{s} + \mathbf{p} : \mathbf{s} \in S\}$.[2] A **flat** in $\mathbb{R}^n$ is a translate of a subspace of $\mathbb{R}^n$. Two flats are **parallel** if one is a translate of the other. The **dimension of a flat** is the dimension of the corresponding parallel subspace. The **dimension of a set** S, written as dim S, is the dimension of the smallest flat containing S. A **line** in $\mathbb{R}^n$ is a flat of dimension 1. A **hyperplane** in $\mathbb{R}^n$ is a flat of dimension $n - 1$.

In $\mathbb{R}^3$, the proper subspaces[3] consist of the origin $\mathbf{0}$, the set of all lines through $\mathbf{0}$, and the set of all planes through $\mathbf{0}$. Thus the proper flats in $\mathbb{R}^3$ are points (zero-dimensional), lines (one-dimensional), and planes (two-dimensional), which may or may not pass through the origin.

The next theorem shows that these geometric descriptions of lines and planes in $\mathbb{R}^3$ (as translates of subspaces) actually coincide with their earlier algebraic descriptions as sets of all affine combinations of two or three points, respectively.

THEOREM 3

A nonempty set S is affine if and only if it is a flat.

Remark: Notice the key role that definitions play in this proof. For example, the first part assumes that S is affine and seeks to show that S is a flat. By definition, a flat is a translate of a subspace. By choosing $\mathbf{p}$ in S and defining $W = S + (-\mathbf{p})$, the set S is translated to the origin and $S = W + \mathbf{p}$. It remains to show that W is a subspace, for then S will be a translate of a subspace and hence a flat.

PROOF Suppose that S is affine. Let $\mathbf{p}$ be any fixed point in S and let $W = S + (-\mathbf{p})$, so that $S = W + \mathbf{p}$. To show that S is a flat, it suffices to show that W is a subspace of $\mathbb{R}^n$. Since $\mathbf{p}$ is in S, the zero vector is in W. To show that W is closed under sums and scalar multiples, it suffices to show that if $\mathbf{u}_1$ and $\mathbf{u}_2$ are elements of W, then $\mathbf{u}_1 + t\mathbf{u}_2$ is in W for every real t. Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are in W, there exist $\mathbf{s}_1$ and $\mathbf{s}_2$ in S such that $\mathbf{u}_1 = \mathbf{s}_1 - \mathbf{p}$ and $\mathbf{u}_2 = \mathbf{s}_2 - \mathbf{p}$. So, for each real t,

$$\begin{aligned} \mathbf{u}_1 + t\mathbf{u}_2 &= (\mathbf{s}_1 - \mathbf{p}) + t(\mathbf{s}_2 - \mathbf{p}) \\ &= (1 - t)\mathbf{s}_1 + t(\mathbf{s}_1 + \mathbf{s}_2 - \mathbf{p}) - \mathbf{p} \end{aligned}$$

Let $\mathbf{y} = \mathbf{s}_1 + \mathbf{s}_2 - \mathbf{p}$. Then $\mathbf{y}$ is an affine combination of points in S. Since S is affine, $\mathbf{y}$ is in S (by Theorem 2). But then $(1 - t)\mathbf{s}_1 + t\mathbf{y}$ is also in S. So $\mathbf{u}_1 + t\mathbf{u}_2$ is in $-\mathbf{p} + S = W$. This shows that W is a subspace of $\mathbb{R}^n$. Thus S is a flat, because $S = W + \mathbf{p}$.

Conversely, suppose S is a flat. That is, $S = W + \mathbf{p}$ for some $\mathbf{p} \in \mathbb{R}^n$ and some subspace W. To show that S is affine, it suffices to show that for any pair $\mathbf{s}_1$ and $\mathbf{s}_2$ of points in S, the line through $\mathbf{s}_1$ and $\mathbf{s}_2$ lies in S. By definition of W, there exist $\mathbf{u}_1$ and $\mathbf{u}_2$ in W such that $\mathbf{s}_1 = \mathbf{u}_1 + \mathbf{p}$ and $\mathbf{s}_2 = \mathbf{u}_2 + \mathbf{p}$. So, for each real t,

$$\begin{aligned} (1 - t)\mathbf{s}_1 + t\mathbf{s}_2 &= (1 - t)(\mathbf{u}_1 + \mathbf{p}) + t(\mathbf{u}_2 + \mathbf{p}) \\ &= (1 - t)\mathbf{u}_1 + t\mathbf{u}_2 + \mathbf{p} \end{aligned}$$

Since W is a subspace, $(1 - t)\mathbf{u}_1 + t\mathbf{u}_2 \in W$ and so $(1 - t)\mathbf{s}_1 + t\mathbf{s}_2 \in W + \mathbf{p} = S$. Thus S is affine. ∎

[2] If $\mathbf{p} = \mathbf{0}$, then the translate is just S itself. See Figure 4 in Section 1.5.

[3] A subset A of a set B is called a **proper** subset of B if $A \neq B$. The same condition applies to proper subspaces and proper flats in $\mathbb{R}^n$: they are not equal to $\mathbb{R}^n$.

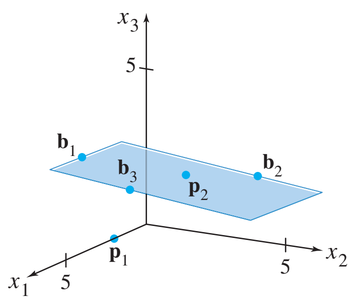


FIGURE 3

Theorem 3 provides a geometric way to view the affine hull of a set: it is the flat that consists of all the affine combinations of points in the set. For instance, Figure 3 shows the points studied in Example 2. Although the set of all *linear* combinations of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ is all of $\mathbb{R}^3$, the set of all *affine* combinations is only the plane through $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$. Note that $\mathbf{p}_2$ (from Example 2) is in the plane through $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$, while $\mathbf{p}_1$ is not in that plane. Also, see Exercise 14.

The next example takes a fresh look at a familiar set—the set of all solutions of a system $A\mathbf{x} = \mathbf{b}$.

EXAMPLE 3 Suppose that the solutions of an equation $A\mathbf{x} = \mathbf{b}$ are all of the form $\mathbf{x} = x_3\mathbf{u} + \mathbf{p}$, where $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$. Recall from Section 1.5 that this set is parallel to the solution set of $A\mathbf{x} = \mathbf{0}$, which consists of all points of the form $x_3\mathbf{u}$. Find points $\mathbf{v}_1$ and $\mathbf{v}_2$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ is aff $\{\mathbf{v}_1, \mathbf{v}_2\}$.

SOLUTION The solution set is a line through $\mathbf{p}$ in the direction of $\mathbf{u}$, as in Figure 1. Since aff $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a line through $\mathbf{v}_1$ and $\mathbf{v}_2$, identify two points on the line $\mathbf{x} = x_3\mathbf{u} + \mathbf{p}$. Two simple choices appear when $x_3 = 0$ and $x_3 = 1$. That is, take $\mathbf{v}_1 = \mathbf{p}$ and $\mathbf{v}_2 = \mathbf{u} + \mathbf{p}$, so that

$$\mathbf{v}_2 = \mathbf{u} + \mathbf{p} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -2 \end{bmatrix}.$$

In this case, the solution set is described as the set of all affine combinations of the form

$$\mathbf{x} = (1 - x_3)\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} + x_3\begin{bmatrix} 6 \\ -3 \\ -2 \end{bmatrix}.$$ ■

Earlier, Theorem 1 displayed an important connection between affine combinations and linear combinations. The next theorem provides another view of affine combinations, which for $\mathbb{R}^2$ and $\mathbb{R}^3$ is closely connected to applications in computer graphics, discussed in the next section (and in Section 2.7).

DEFINITION

For $\mathbf{v}$ in $\mathbb{R}^n$, the standard **homogeneous form** of $\mathbf{v}$ is the point $\tilde{\mathbf{v}} = \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}$ in $\mathbb{R}^{n+1}$.

THEOREM 4

A point $\mathbf{y}$ in $\mathbb{R}^n$ is an affine combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$ if and only if the homogeneous form of $\mathbf{y}$ is in Span $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_p\}$. In fact, $\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, with $c_1 + \cdots + c_p = 1$, if and only if $\tilde{\mathbf{y}} = c_1\tilde{\mathbf{v}}_1 + \cdots + c_p\tilde{\mathbf{v}}_p$.

PROOF A point $\mathbf{y}$ is in aff $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ if and only if there exist weights $c_1, \dots, c_p$ such that

$$\begin{bmatrix} \mathbf{y} \\ 1 \end{bmatrix} = c_1\begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} \mathbf{v}_2 \\ 1 \end{bmatrix} + \cdots + c_p\begin{bmatrix} \mathbf{v}_p \\ 1 \end{bmatrix}$$

This happens if and only if $\tilde{\mathbf{y}}$ is in Span $\{\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \dots, \tilde{\mathbf{v}}_p\}$. ■

EXAMPLE 4 Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$. Use Theorem 4 to write $\mathbf{p}$ as an affine combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, if possible.

SOLUTION Row reduce the augmented matrix for the equation

$$x_1\tilde{\mathbf{v}}_1 + x_2\tilde{\mathbf{v}}_2 + x_3\tilde{\mathbf{v}}_3 = \tilde{\mathbf{p}}$$

To simplify the arithmetic, move the fourth row of 1's to the top (equivalent to three row interchanges). After this, the number of arithmetic operations here is basically the same as the number needed for the method using Theorem 1.

$$[\,\tilde{\mathbf{v}}_1 \;\; \tilde{\mathbf{v}}_2 \;\; \tilde{\mathbf{v}}_3 \;\; \tilde{\mathbf{p}}\,] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 4 \\ 1 & 2 & 7 & 3 \\ 1 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 1 \\ 0 & 1 & 6 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 1.5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & .5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem 4, $1.5\mathbf{v}_1 - \mathbf{v}_2 + .5\mathbf{v}_3 = \mathbf{p}$. See Figure 4, which shows the plane that contains $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{p}$ (together with points on the coordinate axes). ■

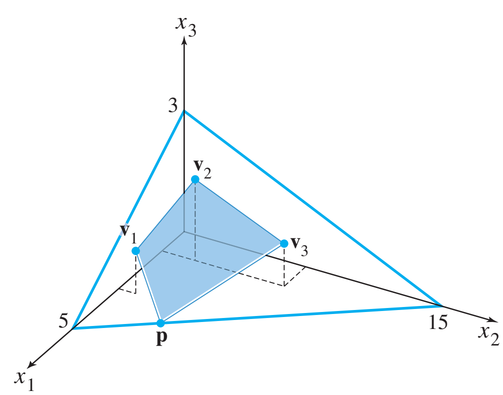


FIGURE 4

PRACTICE PROBLEM

Plot the points $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ on graph paper, and explain why $\mathbf{p}$ *must* be an affine combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Then find the affine combination for $\mathbf{p}$. [*Hint:* What is the dimension of aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?]

8.1 EXERCISES

In Exercises 1–4, write $\mathbf{y}$ as an affine combination of the other points listed, if possible.

1. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

2. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

3. $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 17 \\ 1 \\ 5 \end{bmatrix}$

4. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -6 \\ 7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -3 \\ 4 \\ -4 \end{bmatrix}$

In Exercises 5 and 6, let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$, and $S = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Note that S is an orthogonal basis for $\mathbb{R}^3$. Write each of the given points as an affine combination of the points in the set S, if possible. [*Hint:* Use Theorem 5 in Section 6.2 instead of row reduction to find the weights.]

5. a. $\mathbf{p}_1 = \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}$ b. $\mathbf{p}_2 = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$ c. $\mathbf{p}_3 = \begin{bmatrix} 0 \\ -1 \\ -5 \end{bmatrix}$

6. a. $\mathbf{p}_1 = \begin{bmatrix} 0 \\ -19 \\ -5 \end{bmatrix}$ b. $\mathbf{p}_2 = \begin{bmatrix} 1.5 \\ -1.3 \\ -.5 \end{bmatrix}$ c. $\mathbf{p}_3 = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}$

7. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{p}_1 = \begin{bmatrix} 5 \\ -3 \\ 5 \\ 3 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} -9 \\ 10 \\ 9 \\ -13 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} 4 \\ 2 \\ 8 \\ 5 \end{bmatrix},$$

and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. It can be shown that S is linearly independent.

a. Is $\mathbf{p}_1$ in Span S? Is $\mathbf{p}_1$ in aff S?

b. Is $\mathbf{p}_2$ in Span S? Is $\mathbf{p}_2$ in aff S?

c. Is $\mathbf{p}_3$ in Span S? Is $\mathbf{p}_3$ in aff S?

8. Repeat Exercise 7 when

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 6 \\ -5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ 12 \\ -6 \end{bmatrix},$$

$$\mathbf{p}_1 = \begin{bmatrix} 4 \\ -1 \\ 15 \\ -7 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} -5 \\ 3 \\ -8 \\ 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{p}_3 = \begin{bmatrix} 1 \\ 6 \\ -6 \\ -8 \end{bmatrix}.$$

9. Suppose that the solutions of an equation $A\mathbf{x} = \mathbf{b}$ are all of the form $\mathbf{x} = x_3\mathbf{u} + \mathbf{p}$, where $\mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$. Find points $\mathbf{v}_1$ and $\mathbf{v}_2$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ is aff $\{\mathbf{v}_1, \mathbf{v}_2\}$.

10. Suppose that the solutions of an equation $A\mathbf{x} = \mathbf{b}$ are all of the form $\mathbf{x} = x_3\mathbf{u} + \mathbf{p}$, where $\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$. Find points $\mathbf{v}_1$ and $\mathbf{v}_2$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ is aff $\{\mathbf{v}_1, \mathbf{v}_2\}$.

In Exercises 11 and 12, mark each statement True or False. Justify each answer.

11. a. The set of all affine combinations of points in a set S is called the affine hull of S.

b. If $\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is a linearly independent subset of $\mathbb{R}^n$ and if $\mathbf{p}$ is a linear combination of $\mathbf{b}_1, \dots, \mathbf{b}_k$, then $\mathbf{p}$ is an affine combination of $\mathbf{b}_1, \dots, \mathbf{b}_k$.

c. The affine hull of two distinct points is called a line.

d. A flat is a subspace.

e. A plane in $\mathbb{R}^3$ is a hyperplane.

12. a. If $S = \{\mathbf{x}\}$, then aff S is the empty set.

b. A set is affine if and only if it contains its affine hull.

c. A flat of dimension 1 is called a line.

d. A flat of dimension 2 is called a hyperplane.

e. A flat through the origin is a subspace.

13. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$. Show that Span $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$ is a plane in $\mathbb{R}^3$. [*Hint:* What can you say about $\mathbf{u}$ and $\mathbf{v}$ when Span $\{\mathbf{u}, \mathbf{v}\}$ is a plane?]

14. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$, then aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the plane through $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

15. Let A be an $m \times n$ matrix and, given $\mathbf{b}$ in $\mathbb{R}^m$, show that the set S of all solutions of $A\mathbf{x} = \mathbf{b}$ is an affine subset of $\mathbb{R}^n$.

16. Let $\mathbf{v} \in \mathbb{R}^n$ and let $k \in \mathbb{R}$. Prove that $S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{v} = k\}$ is an affine subset of $\mathbb{R}^n$.

17. Choose a set S of three points such that aff S is the plane in $\mathbb{R}^3$ whose equation is $x_3 = 5$. Justify your work.

18. Choose a set S of four distinct points in $\mathbb{R}^3$ such that aff S is the plane $2x_1 + x_2 - 3x_3 = 12$. Justify your work.

19. Let S be an affine subset of $\mathbb{R}^n$, suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and let $f(S)$ denote the set of images $\{f(\mathbf{x}) : \mathbf{x} \in S\}$. Prove that $f(S)$ is an affine subset of $\mathbb{R}^m$.

20. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, let T be an affine subset of $\mathbb{R}^m$, and let $S = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \in T\}$. Show that S is an affine subset of $\mathbb{R}^n$.

In Exercises 21–26, prove the given statement about subsets A and B of $\mathbb{R}^n$, or provide the required example in $\mathbb{R}^2$. A proof for an exercise may use results from earlier exercises (as well as theorems already available in the text).

21. If $A \subset B$ and B is affine, then aff $A \subset B$.

22. If $A \subset B$, then aff $A \subset$ aff B.

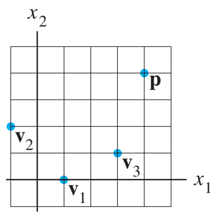
23. $[(\text{aff } A) \cup (\text{aff } B)] \subset \text{aff } (A \cup B)$. [*Hint:* To show that $D \cup E \subset F$, show that $D \subset F$ and $E \subset F$.]

24. Find an example in $\mathbb{R}^2$ to show that equality need not hold in the statement of Exercise 23. [*Hint:* Consider sets A and B, each of which contains only one or two points.]

25. $\text{aff } (A \cap B) \subset (\text{aff } A \cap \text{aff } B)$.

26. Find an example in $\mathbb{R}^2$ to show that equality need not hold in the statement of Exercise 25.

SOLUTION TO PRACTICE PROBLEM



Since the points $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are not collinear (that is, not on a single line), aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ cannot be one-dimensional. Thus, aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must equal $\mathbb{R}^2$. To find the actual weights used to express $\mathbf{p}$ as an affine combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, first compute

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{p} - \mathbf{v}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

To write $\mathbf{p} - \mathbf{v}_1$ as a linear combination of $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$, row reduce the matrix having these points as columns:

$$\begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 2 \end{bmatrix}$$

Thus $\mathbf{p} - \mathbf{v}_1 = \frac{1}{2}(\mathbf{v}_2 - \mathbf{v}_1) + 2(\mathbf{v}_3 - \mathbf{v}_1)$, which shows that

$$\mathbf{p} = \left(1 - \tfrac{1}{2} - 2\right)\mathbf{v}_1 + \tfrac{1}{2}\mathbf{v}_2 + 2\mathbf{v}_3 = -\tfrac{3}{2}\mathbf{v}_1 + \tfrac{1}{2}\mathbf{v}_2 + 2\mathbf{v}_3$$

This expresses $\mathbf{p}$ as an affine combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, because the coefficients sum to 1.

Alternatively, use the method of Example 4 and row reduce:

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{p} \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 3 & 4 \\ 0 & 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This shows that $\mathbf{p} = -\frac{3}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 + 2\mathbf{v}_3$.

8.2 AFFINE INDEPENDENCE

This section continues to explore the relation between linear concepts and affine concepts. Consider first a set of three vectors in $\mathbb{R}^3$, say $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. If S is linearly dependent, then one of the vectors is a linear combination of the other two vectors. What happens when one of the vectors is an *affine* combination of the others? For instance, suppose that

$$\mathbf{v}_3 = (1 - t)\mathbf{v}_1 + t\mathbf{v}_2, \quad \text{for some } t \text{ in } \mathbb{R}.$$

Then

$$(1 - t)\mathbf{v}_1 + t\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}.$$

This is a linear dependence relation because not all the weights are zero. But more is true—the weights in the dependence relation sum to 0:

$$(1 - t) + t + (-1) = 0.$$

This is the additional property needed to define *affine dependence*.

DEFINITION

An indexed set of points $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is **affinely dependent** if there exist real numbers $c_1, \dots, c_p$, not all zero, such that

$$c_1 + \cdots + c_p = 0 \quad \text{and} \quad c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

Otherwise, the set is **affinely independent**.

An affine combination is a special type of linear combination, and affine dependence is a restricted type of linear dependence. Thus, each affinely dependent set is automatically linearly dependent.

A set $\{\mathbf{v}_1\}$ of only one point (even the zero vector) must be affinely independent because the required properties of the coefficients c_i cannot be satisfied when there is only one coefficient. For $\{\mathbf{v}_1\}$, the first equation in (1) is just $c_1 = 0$, and yet at least one (the only one) coefficient must be nonzero.

Exercise 13 asks you to show that an indexed set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is affinely dependent if and only if $\mathbf{v}_1 = \mathbf{v}_2$. The following theorem handles the general case and shows how the concept of affine dependence is analogous to that of linear dependence. Parts (c) and (d) give useful methods for determining whether a set is affinely dependent. Recall from Section 8.1 that if $\mathbf{v}$ is in $\mathbb{R}^n$, then the vector $\tilde{\mathbf{v}}$ in $\mathbb{R}^{n+1}$ denotes the homogeneous form of $\mathbf{v}$.

THEOREM 5

Given an indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$, with $p \geq 2$, the following statements are logically equivalent. That is, either they are all true statements or they are all false.

a. S is affinely dependent.

b. One of the points in S is an affine combination of the other points in S.

c. The set $\{\mathbf{v}_2 - \mathbf{v}_1, \ldots, \mathbf{v}_p - \mathbf{v}_1\}$ in $\mathbb{R}^n$ is linearly dependent.

d. The set $\{\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_p\}$ of homogeneous forms in $\mathbb{R}^{n+1}$ is linearly dependent.

PROOF Suppose statement (a) is true, and let $c_1, \ldots, c_p$ satisfy (1). By renaming the points if necessary, one may assume that $c_1 \neq 0$ and divide both equations in (1) by c_1, so that $1 + (c_2/c_1) + \cdots + (c_p/c_1) = 0$ and

$$\mathbf{v}_1 = (-c_2/c_1)\mathbf{v}_2 + \cdots + (-c_p/c_1)\mathbf{v}_p \tag{2}$$

Note that the coefficients on the right side of (2) sum to 1. Thus (a) implies (b). Now, suppose that (b) is true. By renaming the points if necessary, one may assume that $\mathbf{v}_1 = c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$, where $c_2 + \cdots + c_p = 1$. Then

$$(c_2 + \cdots + c_p)\mathbf{v}_1 = c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p \tag{3}$$

and

$$c_2(\mathbf{v}_2 - \mathbf{v}_1) + \cdots + c_p(\mathbf{v}_p - \mathbf{v}_1) = \mathbf{0} \tag{4}$$

Not all of $c_2, \ldots, c_p$ can be zero because they sum to 1. So (b) implies (c).

Next, if (c) is true, then there exist weights $c_2, \ldots, c_p$, not all zero, such that (4) holds. Rewrite (4) as (3) and set $c_1 = -(c_2 + \cdots + c_p)$. Then $c_1 + \cdots + c_p = 0$. Thus (3) shows that (1) is true. So (c) implies (a), which proves that (a), (b), and (c) are logically equivalent. Finally, (d) is equivalent to (a) because the two equations in (1) are equivalent to the following equation involving the homogeneous forms of the points in S:

$$c_1\begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + \cdots + c_p\begin{bmatrix} \mathbf{v}_p \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix}$$

■

In statement (c) of Theorem 5, $\mathbf{v}_1$ could be replaced by any of the other points in the list $\mathbf{v}_1, \ldots, \mathbf{v}_p$. Only the notation in the proof would change. So, to test whether a set is affinely dependent, subtract one point in the set from the other points, and check whether the translated set of $p - 1$ points is linearly dependent.

EXAMPLE 1 The affine hull of two distinct points $\mathbf{p}$ and $\mathbf{q}$ is a line. If a third point $\mathbf{r}$ is on the line, then $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is an affinely dependent set. If a point $\mathbf{s}$ is not on the line through $\mathbf{p}$ and $\mathbf{q}$, then these three points are not collinear and $\{\mathbf{p}, \mathbf{q}, \mathbf{s}\}$ is an affinely independent set. See Figure 1. ■

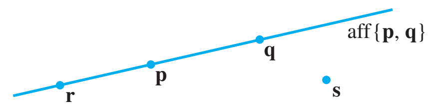


FIGURE 1 $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is affinely dependent.

EXAMPLE 2 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 7 \\ 6.5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 4 \\ 7 \end{bmatrix}$, and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Determine whether S is affinely independent.

SOLUTION Compute $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ -.5 \end{bmatrix}$ and $\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. These two points are not multiples and hence form a linearly independent set, S'. So all statements in Theorem 5 are false, and S is affinely independent. Figure 2 shows S and the translated set S'. Notice that Span S' is a plane through the origin and aff S is a parallel plane through $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. (Only a portion of each plane is shown here, of course.) ■

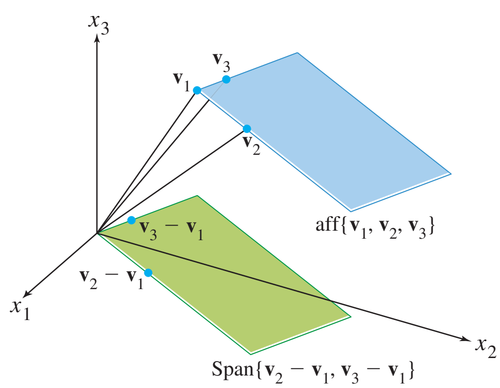


FIGURE 2 An affinely independent set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

EXAMPLE 3 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 7 \\ 6.5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 4 \\ 7 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 14 \\ 6 \end{bmatrix}$, and let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_4\}$. Is S affinely dependent?

SOLUTION Compute $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ -.5 \end{bmatrix}$, $\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_4 - \mathbf{v}_1 = \begin{bmatrix} -1 \\ 11 \\ -1 \end{bmatrix}$, and row reduce the matrix:

$$\begin{bmatrix} 1 & -1 & -1 \\ 4 & 1 & 11 \\ -.5 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 5 & 15 \\ 0 & -.5 & -1.5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 5 & 15 \\ 0 & 0 & 0 \end{bmatrix}$$

Recall from Section 4.6 (or Section 2.8) that the columns are linearly dependent because not every column is a pivot column; so $\mathbf{v}_2 - \mathbf{v}_1$, $\mathbf{v}_3 - \mathbf{v}_1$, and $\mathbf{v}_4 - \mathbf{v}_1$ are linearly dependent. By statement (c) in Theorem 5, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is affinely dependent. This dependence can also be established using (d) in Theorem 5 instead of (c). ■

The calculations in Example 3 show that $\mathbf{v}_4 - \mathbf{v}_1$ is a linear combination of $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$, which means that $\mathbf{v}_4 - \mathbf{v}_1$ is in Span $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$. By Theorem 1 in Section 8.1, $\mathbf{v}_4$ is in aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. In fact, complete row reduction of the matrix in Example 3 would show that

$$\mathbf{v}_4 - \mathbf{v}_1 = 2(\mathbf{v}_2 - \mathbf{v}_1) + 3(\mathbf{v}_3 - \mathbf{v}_1) \tag{5}$$

$$\mathbf{v}_4 = -4\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3 \tag{6}$$

See Figure 3.

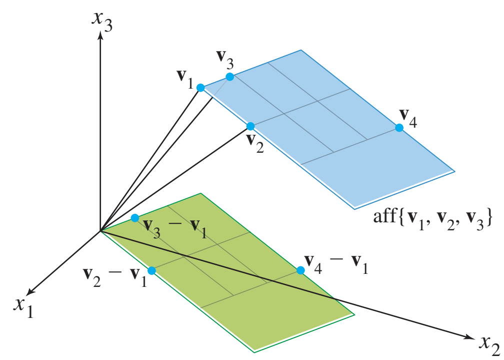


FIGURE 3 $\mathbf{v}_4$ is in the plane aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Figure 3 shows grids on both Span$\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$ and aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. The grid on aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is based on (5). Another "coordinate system" can be based on (6), in which the coefficients -4, 2, and 3 are called *affine* or *barycentric* coordinates of $\mathbf{v}_4$.

Barycentric Coordinates

The definition of barycentric coordinates depends on the following affine version of the Unique Representation Theorem in Section 4.4. See Exercise 17 in this section for the proof.

THEOREM 6

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an affinely independent set in $\mathbb{R}^n$. Then each $\mathbf{p}$ in aff S has a unique representation as an affine combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. That is, for each $\mathbf{p}$ there exists a unique set of scalars $c_1, \ldots, c_k$ such that

$$\mathbf{p} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \quad \text{and} \quad c_1 + \cdots + c_k = 1 \tag{7}$$

DEFINITION

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an affinely independent set. Then for each point $\mathbf{p}$ in aff S, the coefficients $c_1, \ldots, c_k$ in the unique representation (7) of $\mathbf{p}$ are called the **barycentric** (or, sometimes, **affine**) **coordinates** of $\mathbf{p}$.

Observe that (7) is equivalent to the single equation

$$\begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + \cdots + c_k \begin{bmatrix} \mathbf{v}_k \\ 1 \end{bmatrix} \tag{8}$$

involving the homogeneous forms of the points. Row reduction of the augmented matrix $\begin{bmatrix} \tilde{\mathbf{v}}_1 & \cdots & \tilde{\mathbf{v}}_k & \tilde{\mathbf{p}} \end{bmatrix}$ for (8) produces the barycentric coordinates of $\mathbf{p}$.

EXAMPLE 4 Let $\mathbf{a} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Find the barycentric coordinates of $\mathbf{p}$ determined by the affinely independent set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

SOLUTION Row reduce the augmented matrix of points in homogeneous form, moving the last row of ones to the top to simplify the arithmetic:

$$\begin{bmatrix} \tilde{\mathbf{a}} & \tilde{\mathbf{b}} & \tilde{\mathbf{c}} & \tilde{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 9 & 5 \\ 7 & 0 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 5 \\ 7 & 0 & 3 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{5}{12} \end{bmatrix}$$

The coordinates are $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{5}{12}$, so $\mathbf{p} = \frac{1}{4}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{5}{12}\mathbf{c}$. ■

Barycentric coordinates have both physical and geometric interpretations. They were originally defined by A. F. Moebius in 1827 for a point $\mathbf{p}$ inside a triangular region with vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. He wrote that the barycentric coordinates of $\mathbf{p}$ are three nonnegative numbers $m_{\mathbf{a}}$, $m_{\mathbf{b}}$, and $m_{\mathbf{c}}$ such that $\mathbf{p}$ is the center of mass of a system consisting of the triangle (with no mass) and masses $m_{\mathbf{a}}$, $m_{\mathbf{b}}$, and $m_{\mathbf{c}}$ at the corresponding vertices. The masses are uniquely determined by requiring that their sum be 1. This view is still useful in physics today.[1]

Figure 4 gives a geometric interpretation to the barycentric coordinates in Example 4, showing the triangle $\Delta\mathbf{abc}$ and three small triangles $\Delta\mathbf{pbc}$, $\Delta\mathbf{apc}$, and $\Delta\mathbf{abp}$. The areas of the small triangles are proportional to the barycentric coordinates of $\mathbf{p}$. In fact,

$$\begin{aligned} \text{area}(\Delta\mathbf{pbc}) &= \frac{1}{4} \cdot \text{area}(\Delta\mathbf{abc}) \\ \text{area}(\Delta\mathbf{apc}) &= \frac{1}{3} \cdot \text{area}(\Delta\mathbf{abc}) \\ \text{area}(\Delta\mathbf{abp}) &= \frac{5}{12} \cdot \text{area}(\Delta\mathbf{abc}) \end{aligned} \tag{9}$$

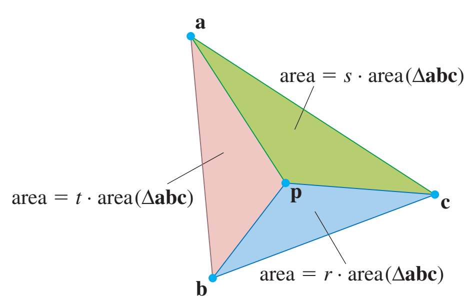


FIGURE 4 $\mathbf{p} = r\mathbf{a} + s\mathbf{b} + t\mathbf{c}$. Here, $r = \frac{1}{4}$, $s = \frac{1}{3}$, $t = \frac{5}{12}$.

The formulas in Figure 4 are verified in Exercises 21–23. Analogous equalities for volumes of tetrahedrons hold for the case when $\mathbf{p}$ is a point inside a tetrahedron in $\mathbb{R}^3$, with vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$.

[1] See Exercise 29 in Section 1.3. In astronomy, however, "barycentric coordinates" usually refer to ordinary $\mathbb{R}^3$ coordinates of points in what is now called the *International Celestial Reference System*, a Cartesian coordinate system for outer space, with the origin at the center of mass (the barycenter) of the solar system.

When a point is not inside the triangle (or tetrahedron), some of the barycentric coordinates will be negative. The case of a triangle is illustrated in Figure 5, for vertices **a**, **b**, **c**, and coordinate values r, s, t, as above. The points on the line through **b** and **c**, for instance, have $r = 0$ because they are affine combinations of only **b** and **c**. The parallel line through **a** identifies points with $r = 1$.

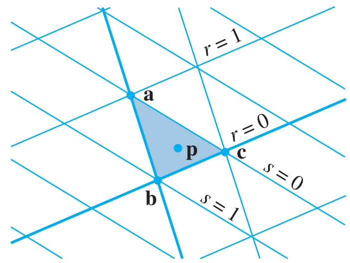


FIGURE 5 Barycentric coordinates for points in aff {**a**, **b**, **c**}.

Barycentric Coordinates in Computer Graphics

When working with geometric objects in a computer graphics program, a designer may use a "wire-frame" approximation to an object at certain key points in the process of creating a realistic final image.[2] For instance, if the surface of part of an object consists of small flat triangular surfaces, then a graphics program can easily add color, lighting, and shading to each small surface when that information is known only at the vertices. Barycentric coordinates provide the tool for smoothly interpolating the vertex information over the interior of a triangle. The interpolation at a point is simply the linear combination of the vertex values using the barycentric coordinates as weights.

Colors on a computer screen are often described by RGB coordinates. A triple (r, g, b) indicates the amount of each color—red, green, and blue—with the parameters varying from 0 to 1. For example, pure red is $(1, 0, 0)$, white is $(1, 1, 1)$, and black is $(0, 0, 0)$.

EXAMPLE 5 Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 3 \\ 3 \\ 3.5 \end{bmatrix}$. The colors at the vertices $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ of a triangle are magenta $(1, 0, 1)$, light magenta $(1, .4, 1)$, and purple $(.6, 0, 1)$, respectively. Find the interpolated color at $\mathbf{p}$. See Figure 6.

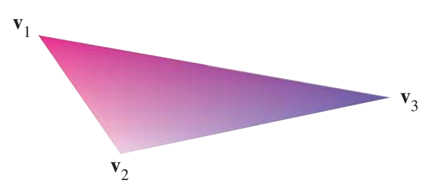


FIGURE 6 Interpolated colors.

[2] The Introductory Example for Chapter 2 shows a wire-frame model of a Boeing 777 airplane, used to visualize the flow of air over the surface of the plane.

SOLUTION First, find the barycentric coordinates of $\mathbf{p}$. Here is the calculation using homogeneous forms of the points, with the first step moving row 4 to row 1:

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{p}} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 1 & 3 \\ 1 & 3 & 5 & 3 \\ 5 & 4 & 1 & 3.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & .25 \\ 0 & 1 & 0 & .50 \\ 0 & 0 & 1 & .25 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\mathbf{p} = .25\mathbf{v}_1 + .5\mathbf{v}_2 + .25\mathbf{v}_3$. Use the barycentric coordinates of $\mathbf{p}$ to make a linear combination of the color data. The RGB values for $\mathbf{p}$ are

$$.25\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + .50\begin{bmatrix} 1 \\ .4 \\ 1 \end{bmatrix} + .25\begin{bmatrix} .6 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .9 \\ .2 \\ 1 \end{bmatrix} \begin{matrix} \text{red} \\ \text{green} \\ \text{blue} \end{matrix}$$ ■

One of the last steps in preparing a graphics scene for display on a computer screen is to remove "hidden surfaces" that should not be visible on the screen. Imagine the viewing screen as consisting of, say, a million pixels, and consider a ray or "line of sight" from the viewer's eye through a pixel and into the collection of objects that make up the 3D display. The color and other information displayed in the pixel on the screen should come from the object that the ray first intersects. See Figure 7. When the objects in the graphics scene are approximated by wire frames with triangular patches, the hidden surface problem can be solved using barycentric coordinates.

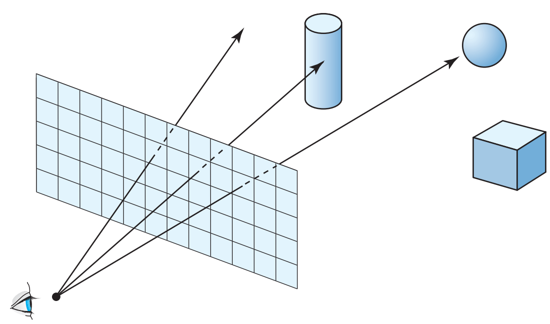


FIGURE 7 A ray from the eye through the screen to the nearest object.

The mathematics for finding the ray-triangle intersections can also be used to perform extremely realistic shading of objects. Currently, this *ray-tracing* method is too slow for real-time rendering, but recent advances in hardware implementation may change that in the future.[3]

EXAMPLE 6 Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 8 \\ 1 \\ -4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 11 \\ -2 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} .7 \\ .4 \\ -3 \end{bmatrix},$$

and $\mathbf{x}(t) = \mathbf{a} + t\mathbf{b}$ for $t \geq 0$. Find the point where the ray $\mathbf{x}(t)$ intersects the plane that contains the triangle with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Is this point inside the triangle?

[3] See Joshua Fender and Jonathan Rose, "A High-Speed Ray Tracing Engine Built on a Field-Programmable System," in *Proc. Int. Conf on Field-Programmable Technology*, IEEE (2003). (A single processor can calculate 600 million ray-triangle intersections per second.)

SOLUTION The plane is aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. A typical point in this plane may be written as $(1 - c_2 - c_3)\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ for some c_2 and c_3. (The weights in this combination sum to 1.) The ray $\mathbf{x}(t)$ intersects the plane when c_2, c_3, and t satisfy

$$(1 - c_2 - c_3)\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{a} + t\mathbf{b}$$

Rearrange this as $c_2(\mathbf{v}_2 - \mathbf{v}_1) + c_3(\mathbf{v}_3 - \mathbf{v}_1) + t(-\mathbf{b}) = \mathbf{a} - \mathbf{v}_1$. In matrix form,

$$\begin{bmatrix} \mathbf{v}_2 - \mathbf{v}_1 & \mathbf{v}_3 - \mathbf{v}_1 & -\mathbf{b} \end{bmatrix} \begin{bmatrix} c_2 \\ c_3 \\ t \end{bmatrix} = \mathbf{a} - \mathbf{v}_1$$

For the specific points given here,

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 7 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 4 \\ 10 \\ 4 \end{bmatrix}, \quad \mathbf{a} - \mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 16 \end{bmatrix}$$

Row reduction of the augmented matrix above produces

$$\begin{bmatrix} 7 & 4 & -.7 & -1 \\ 0 & 10 & -.4 & -1 \\ 2 & 4 & 3 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & .3 \\ 0 & 1 & 0 & .1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Thus $c_2 = .3$, $c_3 = .1$, and $t = 5$. Therefore, the intersection point is

$$\mathbf{x}(5) = \mathbf{a} + 5\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} + 5\begin{bmatrix} .7 \\ .4 \\ -3 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 2.0 \\ -5.0 \end{bmatrix}$$

Also,

$$\begin{aligned} \mathbf{x}(5) &= (1 - .3 - .1)\mathbf{v}_1 + .3\mathbf{v}_2 + .1\mathbf{v}_3 \\ &= .6\begin{bmatrix} 1 \\ 1 \\ -6 \end{bmatrix} + .3\begin{bmatrix} 8 \\ 1 \\ -4 \end{bmatrix} + .1\begin{bmatrix} 5 \\ 11 \\ -2 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 2.0 \\ -5.0 \end{bmatrix} \end{aligned}$$

The intersection point is inside the triangle because the barycentric weights for $\mathbf{x}(5)$ are all positive. ■

PRACTICE PROBLEMS

1. Describe a fast way to determine when three points are collinear.

2. The points $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ form an affinely dependent set. Find weights $c_1, \dots, c_4$ that produce an **affine dependence relation** $c_1\mathbf{v}_1 + \cdots + c_4\mathbf{v}_4 = \mathbf{0}$, where $c_1 + \cdots + c_4 = 0$ and not all c_i are zero. [*Hint:* See the end of the proof of Theorem 5.]

8.2 EXERCISES

In Exercises 1–6, determine if the set of points is affinely dependent. (See Practice Problem 2.) If so, construct an affine dependence relation for the points.

1. $\begin{bmatrix} 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

2. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

3. $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 15 \\ -9 \end{bmatrix}$

4. $\begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ -3 \end{bmatrix}$

5. $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -3 \end{bmatrix}$

6. $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$

In Exercises 7 and 8, find the barycentric coordinates of $\mathbf{p}$ with respect to the affinely independent set of points that precedes it.

7. $\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix}, \mathbf{p} = \begin{bmatrix} 5 \\ 4 \\ -2 \\ 2 \end{bmatrix}$

8. $\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -6 \\ 5 \end{bmatrix}, \mathbf{p} = \begin{bmatrix} -1 \\ 1 \\ -4 \\ 0 \end{bmatrix}$

In Exercises 9 and 10, mark each statement True or False. Justify each answer.

9. a. If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$ and if the set $\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_3 - \mathbf{v}_2, \ldots, \mathbf{v}_p - \mathbf{v}_2\}$ is linearly dependent, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is affinely dependent. (Read this carefully.)

   b. If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$ and if the set of homogeneous forms $\{\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_p\}$ in $\mathbb{R}^{n+1}$ is linearly independent, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is affinely dependent.

   c. A finite set of points $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is affinely dependent if there exist real numbers $c_1, \ldots, c_k$, not all zero, such that $c_1 + \cdots + c_k = 1$ and $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$.

   d. If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an affinely independent set in $\mathbb{R}^n$ and if $\mathbf{p}$ in $\mathbb{R}^n$ has a negative barycentric coordinate determined by S, then $\mathbf{p}$ is not in aff S.

   e. If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{a}$, and $\mathbf{b}$ are in $\mathbb{R}^3$ and if a ray $\mathbf{a} + t\mathbf{b}$ for $t \geq 0$ intersects the triangle with vertices $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$, then the barycentric coordinates of the intersection point are all nonnegative.

10. a. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an affinely dependent set in $\mathbb{R}^n$, then the set $\{\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_p\}$ in $\mathbb{R}^{n+1}$ of homogeneous forms may be linearly independent.

    b. If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are in $\mathbb{R}^3$ and if the set $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1, \mathbf{v}_4 - \mathbf{v}_1\}$ is linearly independent, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ is affinely independent.

    c. Given $S = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ in $\mathbb{R}^n$, each $\mathbf{p}$ in aff S has a unique representation as an affine combination of $\mathbf{b}_1, \ldots, \mathbf{b}_k$.

    d. When color information is specified at each vertex $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of a triangle in $\mathbb{R}^3$, then the color may be interpolated at a point $\mathbf{p}$ in aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ using the barycentric coordinates of $\mathbf{p}$.

    e. If T is a triangle in $\mathbb{R}^2$ and if a point $\mathbf{p}$ is on an edge of the triangle, then the barycentric coordinates of $\mathbf{p}$ (for this triangle) are not all positive.

11. Explain why any set of five or more points in $\mathbb{R}^3$ must be affinely dependent.

12. Show that a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is affinely dependent when $p \geq n + 2$.

13. Use only the definition of affine dependence to show that an indexed set $\{\mathbf{v}_1, \mathbf{v}_2\}$ in $\mathbb{R}^n$ is affinely dependent if and only if $\mathbf{v}_1 = \mathbf{v}_2$.

14. The conditions for affine dependence are stronger than those for linear dependence, so an affinely dependent set is automatically linearly dependent. Also, a linearly independent set cannot be affinely dependent and therefore must be affinely independent. Construct two linearly dependent indexed sets S_1 and S_2 in $\mathbb{R}^2$ such that S_1 is affinely dependent and S_2 is affinely independent. In each case, the set should contain either one, two, or three nonzero points.

15. Let $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

    a. Show that the set S is affinely independent.

    b. Find the barycentric coordinates of $\mathbf{p}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{p}_3 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\mathbf{p}_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\mathbf{p}_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, with respect to S.

    c. Let T be the triangle with vertices $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. When the sides of T are extended, the lines divide $\mathbb{R}^2$ into seven regions. See Figure 8. Note the signs of the barycentric coordinates of the points in each region. For example, $\mathbf{p}_5$ is inside the triangle T and all its barycentric coordinates are positive. Point $\mathbf{p}_1$ has coordinates $(-, +, +)$. Its third coordinate is positive because $\mathbf{p}_1$ is on the $\mathbf{v}_3$ side of the line through $\mathbf{v}_1$ and $\mathbf{v}_2$. Its first coordinate is negative because $\mathbf{p}_1$ is opposite the $\mathbf{v}_1$ side of the line through $\mathbf{v}_2$ and $\mathbf{v}_3$. Point $\mathbf{p}_2$ is on the $\mathbf{v}_2\mathbf{v}_3$ edge of T. Its coordinates are $(0, +, +)$. Without calculating the actual values, determine the signs of the barycentric coordinates of points $\mathbf{p}_6, \mathbf{p}_7$, and $\mathbf{p}_8$ as shown in Figure 8.

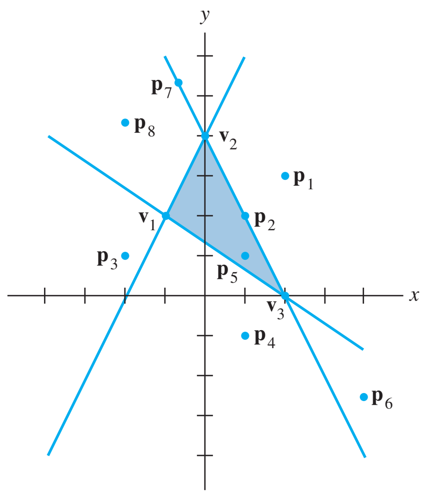


FIGURE 8

16. Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $\mathbf{p}_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{p}_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{p}_5 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $\mathbf{p}_6 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{p}_7 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

a. Show that the set S is affinely independent.

b. Find the barycentric coordinates of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ with respect to S.

c. On graph paper, sketch the triangle T with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, extend the sides as in Figure 8, and plot the points $\mathbf{p}_4$, $\mathbf{p}_5$, $\mathbf{p}_6$, and $\mathbf{p}_7$. Without calculating the actual values, determine the signs of the barycentric coordinates of points $\mathbf{p}_4$, $\mathbf{p}_5$, $\mathbf{p}_6$, and $\mathbf{p}_7$.

17. Prove Theorem 6 for an affinely independent set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ in $\mathbb{R}^n$. [*Hint:* One method is to mimic the proof of Theorem 7 in Section 4.4.]

18. Let T be a tetrahedron in "standard" position, with three edges along the three positive coordinate axes in $\mathbb{R}^3$, and suppose the vertices are $a\mathbf{e}_1$, $b\mathbf{e}_2$, $c\mathbf{e}_3$, and $\mathbf{0}$, where $[\,\mathbf{e}_1 \;\; \mathbf{e}_2 \;\; \mathbf{e}_3\,] = I_3$. Find formulas for the barycentric coordinates of an arbitrary point $\mathbf{p}$ in $\mathbb{R}^3$.

19. Let $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ be an affinely dependent set of points in $\mathbb{R}^n$ and let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that $\{f(\mathbf{p}_1), f(\mathbf{p}_2), f(\mathbf{p}_3)\}$ is affinely dependent in $\mathbb{R}^m$.

20. Suppose that $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is an affinely independent set in $\mathbb{R}^n$ and $\mathbf{q}$ is an arbitrary point in $\mathbb{R}^n$. Show that the translated set $\{\mathbf{p}_1 + \mathbf{q}, \mathbf{p}_2 + \mathbf{q}, \mathbf{p}_3 + \mathbf{q}\}$ is also affinely independent.

In Exercises 21–24, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are noncollinear points in $\mathbb{R}^2$ and $\mathbf{p}$ is any other point in $\mathbb{R}^2$. Let $\Delta\mathbf{abc}$ denote the closed triangular region determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, and let $\Delta\mathbf{pbc}$ be the region determined by $\mathbf{p}$, $\mathbf{b}$, and $\mathbf{c}$. For convenience, assume that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are arranged so that $\det[\,\tilde{\mathbf{a}} \;\; \tilde{\mathbf{b}} \;\; \tilde{\mathbf{c}}\,]$ is positive, where $\tilde{\mathbf{a}}$, $\tilde{\mathbf{b}}$, and $\tilde{\mathbf{c}}$ are the standard homogeneous forms for the points.

21. Show that the area of $\Delta\mathbf{abc}$ is $\det[\,\tilde{\mathbf{a}} \;\; \tilde{\mathbf{b}} \;\; \tilde{\mathbf{c}}\,]/2$. [*Hint:* Consult Sections 3.2 and 3.3, including the Exercises.]

22. Let $\mathbf{p}$ be a point on the line through $\mathbf{a}$ and $\mathbf{b}$. Show that $\det[\,\tilde{\mathbf{a}} \;\; \tilde{\mathbf{b}} \;\; \tilde{\mathbf{p}}\,] = 0$.

23. Let $\mathbf{p}$ be any point in the interior of $\Delta\mathbf{abc}$, with barycentric coordinates (r, s, t), so that

$$[\,\tilde{\mathbf{a}} \;\; \tilde{\mathbf{b}} \;\; \tilde{\mathbf{c}}\,]\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \tilde{\mathbf{p}}$$

Use Exercise 21 and a fact about determinants (Chapter 3) to show that

$r = (\text{area of } \Delta\mathbf{pbc})/(\text{area of } \Delta\mathbf{abc})$

$s = (\text{area of } \Delta\mathbf{apc})/(\text{area of } \Delta\mathbf{abc})$

$t = (\text{area of } \Delta\mathbf{abp})/(\text{area of } \Delta\mathbf{abc})$

24. Take $\mathbf{q}$ on the line segment from $\mathbf{b}$ to $\mathbf{c}$ and consider the line through $\mathbf{q}$ and $\mathbf{a}$, which may be written as $\mathbf{p} = (1 - x)\mathbf{q} + x\mathbf{a}$ for all real x. Show that, for each x, $\det[\,\tilde{\mathbf{p}} \;\; \tilde{\mathbf{b}} \;\; \tilde{\mathbf{c}}\,] = x \cdot \det[\,\tilde{\mathbf{a}} \;\; \tilde{\mathbf{b}} \;\; \tilde{\mathbf{c}}\,]$. From this and earlier work, conclude that the parameter x is the first barycentric coordinate of $\mathbf{p}$. However, by construction, the parameter x also determines the relative distance between $\mathbf{p}$ and $\mathbf{q}$ along the segment from $\mathbf{q}$ to $\mathbf{a}$. (When $x = 1$, $\mathbf{p} = \mathbf{a}$.) When this fact is applied to Example 5, it shows that the colors at vertex $\mathbf{a}$ and the point $\mathbf{q}$ are smoothly interpolated as $\mathbf{p}$ moves along the line between $\mathbf{a}$ and $\mathbf{q}$.

25. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 7 \\ 3 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 9 \\ -2 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1.4 \\ 1.5 \\ -3.1 \end{bmatrix}$, and $\mathbf{x}(t) = \mathbf{a} + t\mathbf{b}$ for $t \geq 0$. Find the point where the ray $\mathbf{x}(t)$ intersects the plane that contains the triangle with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Is this point inside the triangle?

26. Repeat Exercise 25 with $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 8 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 10 \\ -2 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} .9 \\ 2.0 \\ -3.7 \end{bmatrix}$.

SOLUTIONS TO PRACTICE PROBLEMS

1. From Example 1, the problem is to determine if the points are affinely dependent. Use the method of Example 2 and subtract one point from the other two. If one of these two new points is a multiple of the other, the original three points lie on a line.

2. The proof of Theorem 5 essentially points out that an affine dependence relation among points corresponds to a linear dependence relation among the homogeneous forms of the points, using the *same* weights. So, row reduce:

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{v}}_4 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 5 & 1 \\ 1 & 0 & 4 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 5 & 1 \\ 1 & 0 & 4 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1.25 \\ 0 & 0 & 1 & .75 \end{bmatrix}$$

View this matrix as the coefficient matrix for $A\mathbf{x} = \mathbf{0}$ with four variables. Then x_4 is free, $x_1 = x_4$, $x_2 = -1.25x_4$, and $x_3 = -.75x_4$. One solution is $x_1 = x_4 = 4$, $x_2 = -5$, and $x_3 = -3$. A linear dependence among the homogeneous forms is $4\tilde{\mathbf{v}}_1 - 5\tilde{\mathbf{v}}_2 - 3\tilde{\mathbf{v}}_3 + 4\tilde{\mathbf{v}}_4 = \mathbf{0}$. So $4\mathbf{v}_1 - 5\mathbf{v}_2 - 3\mathbf{v}_3 + 4\mathbf{v}_4 = \mathbf{0}$.

Another solution method is to translate the problem to the origin by subtracting $\mathbf{v}_1$ from the other points, find a linear dependence relation among the translated points, and then rearrange the terms. The amount of arithmetic involved is about the same as in the approach shown above.

8.3 CONVEX COMBINATIONS

Section 8.1 considered special linear combinations of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k, \quad \text{where } c_1 + c_2 + \cdots + c_k = 1$$

This section further restricts the weights to be nonnegative.

DEFINITION

A **convex combination** of points $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ is a linear combination of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

such that $c_1 + c_2 + \cdots + c_k = 1$ and $c_i \geq 0$ for all i. The set of all convex combinations of points in a set S is called the **convex hull** of S, denoted by conv S.

The convex hull of a single point $\mathbf{v}_1$ is just the set $\{\mathbf{v}_1\}$, the same as the affine hull. In other cases, the convex hull is properly contained in the affine hull. Recall that the affine hull of distinct points $\mathbf{v}_1$ and $\mathbf{v}_2$ is the line

$$\mathbf{y} = (1 - t)\mathbf{v}_1 + t\mathbf{v}_2, \quad \text{with } t \text{ in } \mathbb{R}$$

Because the weights in a convex combination are nonnegative, the points in conv $\{\mathbf{v}_1, \mathbf{v}_2\}$ may be written as

$$\mathbf{y} = (1 - t)\mathbf{v}_1 + t\mathbf{v}_2, \quad \text{with } 0 \leq t \leq 1$$

which is the **line segment** between $\mathbf{v}_1$ and $\mathbf{v}_2$, hereafter denoted by $\overline{\mathbf{v}_1\mathbf{v}_2}$.

If a set S is affinely independent and if $\mathbf{p} \in \text{aff } S$, then $\mathbf{p} \in \text{conv } S$ if and only if the barycentric coordinates of $\mathbf{p}$ are nonnegative. Example 1 shows a special situation in which S is much more than just affinely independent.

EXAMPLE 1 Let

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 6 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -6 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{p}_1 = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} -10 \\ 5 \\ 11 \\ -4 \end{bmatrix},$$

and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Note that S is an orthogonal set. Determine whether $\mathbf{p}_1$ is in Span S, aff S, and conv S. Then do the same for $\mathbf{p}_2$.

SOLUTION If $\mathbf{p}_1$ is at least a *linear* combination of the points in S, then the weights are easily found, because S is an orthogonal set. Let W be the subspace spanned by S. A calculation as in Section 6.3 shows that the orthogonal projection of $\mathbf{p}_1$ onto W is $\mathbf{p}_1$ itself:

$$\begin{aligned} \text{proj}_W \mathbf{p}_1 &= \frac{\mathbf{p}_1 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{p}_1 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{p}_1 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 \\ &= \frac{18}{54}\mathbf{v}_1 + \frac{18}{54}\mathbf{v}_2 + \frac{18}{54}\mathbf{v}_3 \\ &= \frac{1}{3}\begin{bmatrix} 3 \\ 0 \\ 6 \\ -3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -6 \\ 3 \\ 3 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 3 \\ 6 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix} = \mathbf{p}_1 \end{aligned}$$

This shows that $\mathbf{p}_1$ is *in* Span S. Also, since the coefficients sum to 1, $\mathbf{p}_1$ is in aff S. In fact, $\mathbf{p}_1$ is in conv S, because the coefficients are also nonnegative.

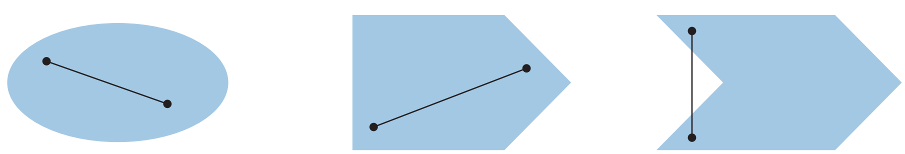
For $\mathbf{p}_2$, a similar calculation shows that $\text{proj}_W \mathbf{p}_2 \neq \mathbf{p}_2$. Since $\text{proj}_W \mathbf{p}_2$ is the closest point in Span S to $\mathbf{p}_2$, the point $\mathbf{p}_2$ is not in Span S. In particular, $\mathbf{p}_2$ cannot be in aff S or conv S. ■

Recall that a set S is affine if it contains all lines determined by pairs of points in S. When attention is restricted to convex combinations, the appropriate condition involves line segments rather than lines.

DEFINITION

A set S is **convex** if for each $\mathbf{p}, \mathbf{q} \in S$, the line segment $\overline{\mathbf{pq}}$ is contained in S.

Intuitively, a set S is convex if every two points in the set can "see" each other without the line of sight leaving the set. Figure 1 illustrates this idea.



Convex Convex Not convex

FIGURE 1

The next result is analogous to Theorem 2 for affine sets.

THEOREM 7

A set S is convex if and only if every convex combination of points of S lies in S. That is, S is convex if and only if $S = \text{conv } S$.

PROOF The argument is similar to the proof of Theorem 2. The only difference is in the induction step. When taking a convex combination of $k + 1$ points, consider $\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1}$, where $c_1 + \cdots + c_{k+1} = 1$ and $0 \le c_i \le 1$ for all i. If $c_{k+1} = 1$, then $\mathbf{y} = \mathbf{v}_{k+1}$, which belongs to S, and there is nothing further to prove. If $c_{k+1} < 1$, let $t = c_1 + \cdots + c_k$. Then $t = 1 - c_{k+1} > 0$ and

$$\mathbf{y} = (1 - c_{k+1})\left(\frac{c_1}{t}\mathbf{v}_1 + \cdots + \frac{c_k}{t}\mathbf{v}_k\right) + c_{k+1}\mathbf{v}_{k+1} \quad (1)$$

By the induction hypothesis, the point $\mathbf{z} = (c_1/t)\mathbf{v}_1 + \cdots + (c_k/t)\mathbf{v}_k$ is in S, since the nonnegative coefficients sum to 1. Thus equation (1) displays $\mathbf{y}$ as a convex combination of two points in S. By the principle of induction, every convex combination of such points lies in S. ■

Theorem 9 below provides a more geometric characterization of the convex hull of a set. It requires a preliminary result on intersections of sets. Recall from Section 4.1 (Exercise 32) that the intersection of two subspaces is itself a subspace. In fact, the intersection of any collection of subspaces is itself a subspace. A similar result holds for affine sets and convex sets.

THEOREM 8

Let $\{S_\alpha : \alpha \in \mathcal{A}\}$ be any collection of convex sets. Then $\cap_{\alpha\in\mathcal{A}} S_\alpha$ is convex. If $\{T_\beta : \beta \in \mathcal{B}\}$ is any collection of affine sets, then $\cap_{\beta\in\mathcal{B}} T_\beta$ is affine.

PROOF If $\mathbf{p}$ and $\mathbf{q}$ are in $\cap S_\alpha$, then $\mathbf{p}$ and $\mathbf{q}$ are in each S_α. Since each S_α is convex, the line segment between $\mathbf{p}$ and $\mathbf{q}$ is in S_α for all α and hence that segment is contained in $\cap S_\alpha$. The proof of the affine case is similar. ■

THEOREM 9

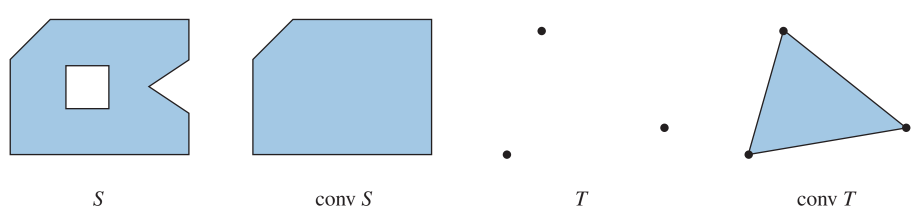
For any set S, the convex hull of S is the intersection of all the convex sets that contain S.

PROOF Let T denote the intersection of all the convex sets containing S. Since conv S is a convex set containing S, it follows that $T \subset \text{conv } S$. On the other hand, let C be any convex set containing S. Then C contains every convex combination of points of C (Theorem 7), and hence also contains every convex combination of points of the subset S. That is, conv $S \subset C$. Since this is true for every convex set C containing S, it is also true for the intersection of them all. That is, conv $S \subset T$. ■

Theorem 9 shows that conv S is in a natural sense the "smallest" convex set containing S. For example, consider a set S that lies inside some large rectangle in $\mathbb{R}^2$, and imagine stretching a rubber band around the outside of S. As the rubber band contracts around S, it outlines the boundary of the convex hull of S. Or to use another analogy, the convex hull of S fills *in* all the holes in the inside of S and fills *out* all the dents in the boundary of S.

EXAMPLE 2

a. The convex hulls of sets S and T in $\mathbb{R}^2$ are shown below.



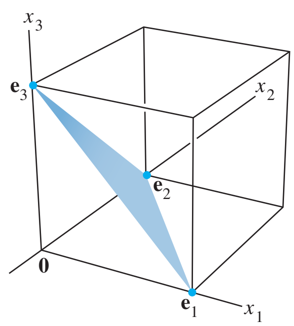


FIGURE 2

b. Let S be the set consisting of the standard basis for $\mathbb{R}^3$, $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then conv S is a triangular surface in $\mathbb{R}^3$, with vertices $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. See Figure 2. ■

EXAMPLE 3 Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0 \text{ and } y = x^2 \right\}$. Show that the convex hull of S is the union of the origin and $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x > 0 \text{ and } y \geq x^2 \right\}$. See Figure 3.

SOLUTION Every point in conv S must lie on a line segment that connects two points of S. The dashed line in Figure 3 indicates that, except for the origin, the positive y-axis is not in conv S, because the origin is the only point of S on the y-axis. It may seem reasonable that Figure 3 does show conv S, but how can you be sure that the point $(10^{-2}, 10^4)$, for example, is on a line segment from the origin to a point on the curve in S?

Consider any point $\mathbf{p}$ in the shaded region of Figure 3, say

$$\mathbf{p} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \text{with } a > 0 \text{ and } b \geq a^2$$

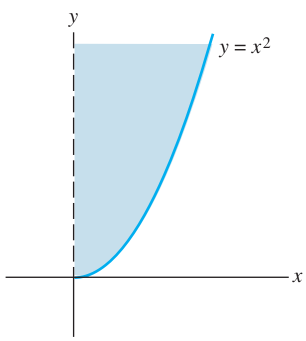


FIGURE 3

The line through $\mathbf{0}$ and $\mathbf{p}$ has the equation $y = (b/a)t$ for t real. That line intersects S where t satisfies $(b/a)t = t^2$, that is, when $t = b/a$. Thus, $\mathbf{p}$ is on the line segment from $\mathbf{0}$ to $\begin{bmatrix} b/a \\ b^2/a^2 \end{bmatrix}$, which shows that Figure 3 is correct. ■

The following theorem is basic in the study of convex sets. It was first proved by Constantin Caratheodory in 1907. If $\mathbf{p}$ is in the convex hull of S, then, by definition, $\mathbf{p}$ must be a convex combination of points of S. But the definition makes no stipulation as to how many points of S are required to make the combination. Caratheodory's remarkable theorem says that in an n-dimensional space, the number of points of S in the convex combination never has to be more than $n + 1$.

THEOREM 10

(Caratheodory) If S is a nonempty subset of $\mathbb{R}^n$, then every point in conv S can be expressed as a convex combination of $n + 1$ or fewer points of S.

PROOF Given $\mathbf{p}$ in conv S, one may write $\mathbf{p} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$, where $\mathbf{v}_i \in S$, $c_1 + \cdots + c_k = 1$, and $c_i \geq 0$, for some k and $i = 1, \ldots, k$. The goal is to show that such an expression exists for $\mathbf{p}$ with $k \leq n + 1$.

If $k > n + 1$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is affinely dependent, by Exercise 12 in Section 8.2. Thus there exist scalars $d_1, \ldots, d_k$, not all zero, such that

$$\sum_{i=1}^{k} d_i\mathbf{v}_i = \mathbf{0} \quad \text{and} \quad \sum_{i=1}^{k} d_i = 0$$

Consider the two equations

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{p}$$

and

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_k\mathbf{v}_k = \mathbf{0}$$

By subtracting an appropriate multiple of the second equation from the first, we now eliminate one of the $\mathbf{v}_i$ terms and obtain a convex combination of fewer than k elements of S that is equal to $\mathbf{p}$.

Since not all of the d_i coefficients are zero, we may assume (by reordering subscripts if necessary) that $d_k > 0$ and that $c_k/d_k \le c_i/d_i$ for all those i for which $d_i > 0$. For $i = 1, \dots, k$, let $b_i = c_i - (c_k/d_k)d_i$. Then $b_k = 0$ and

$$\sum_{i=1}^{k} b_i = \sum_{i=1}^{k} c_i - \frac{c_k}{d_k}\sum_{i=1}^{k} d_i = 1 - 0 = 1$$

Furthermore, each $b_i \ge 0$. Indeed, if $d_i \le 0$, then $b_i \ge c_i \ge 0$. If $d_i > 0$, then $b_i = d_i(c_i/d_i - c_k/d_k) \ge 0$. By construction,

$$\begin{aligned}\sum_{i=1}^{k-1} b_i\mathbf{v}_i &= \sum_{i=1}^{k} b_i\mathbf{v}_i = \sum_{i=1}^{k}\left(c_i - \frac{c_k}{d_k}d_i\right)\mathbf{v}_i \\ &= \sum_{i=1}^{k} c_i\mathbf{v}_i - \frac{c_k}{d_k}\sum_{i=1}^{k} d_i\mathbf{v}_i = \sum_{i=1}^{k} c_i\mathbf{v}_i = \mathbf{p}\end{aligned}$$

Thus $\mathbf{p}$ is now a convex combination of $k-1$ of the points $\mathbf{v}_1, \dots, \mathbf{v}_k$. This process may be repeated until $\mathbf{p}$ is expressed as a convex combination of at most $n+1$ of the points of S. ■

The following example illustrates the calculations in the proof above.

EXAMPLE 4 Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} \frac{10}{3} \\ \frac{5}{2} \end{bmatrix},$$

and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Then

$$\tfrac{1}{4}\mathbf{v}_1 + \tfrac{1}{6}\mathbf{v}_2 + \tfrac{1}{2}\mathbf{v}_3 + \tfrac{1}{12}\mathbf{v}_4 = \mathbf{p} \tag{2}$$

Use the procedure in the proof of Caratheodory's Theorem to express $\mathbf{p}$ as a convex combination of three points of S.

SOLUTION The set S is affinely dependent. Use the technique of Section 8.2 to obtain an affine dependence relation

$$-5\mathbf{v}_1 + 4\mathbf{v}_2 - 3\mathbf{v}_3 + 4\mathbf{v}_4 = \mathbf{0} \tag{3}$$

Next, choose the points $\mathbf{v}_2$ and $\mathbf{v}_4$ in (3), whose coefficients are positive. For each point, compute the ratio of the coefficients in equations (2) and (3). The ratio for $\mathbf{v}_2$ is $\frac{1}{6} \div 4 = \frac{1}{24}$, and that for $\mathbf{v}_4$ is $\frac{1}{12} \div 4 = \frac{1}{48}$. The ratio for $\mathbf{v}_4$ is smaller, so subtract $\frac{1}{48}$ times equation (3) from equation (2) to eliminate $\mathbf{v}_4$:

$$\left(\tfrac{1}{4} + \tfrac{5}{48}\right)\mathbf{v}_1 + \left(\tfrac{1}{6} - \tfrac{4}{48}\right)\mathbf{v}_2 + \left(\tfrac{1}{2} + \tfrac{3}{48}\right)\mathbf{v}_3 + \left(\tfrac{1}{12} - \tfrac{4}{48}\right)\mathbf{v}_4 = \mathbf{p}$$

$$\tfrac{17}{48}\mathbf{v}_1 + \tfrac{4}{48}\mathbf{v}_2 + \tfrac{27}{48}\mathbf{v}_3 = \mathbf{p}$$ ■

This result cannot, in general, be improved by decreasing the required number of points. Indeed, given any three non-collinear points in $\mathbb{R}^2$, the centroid of the triangle formed by them is in the convex hull of all three, but is not in the convex hull of any two.

PRACTICE PROBLEMS

1. Let $\mathbf{v}_1 = \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, and $\mathbf{p}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Determine whether $\mathbf{p}_1$ and $\mathbf{p}_2$ are in conv S.
2. Let S be the set of points on the curve $y = 1/x$ for $x > 0$. Explain geometrically why conv S consists of all points on and above the curve S.

8.3 EXERCISES

1. In $\mathbb{R}^2$, let $S = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} : 0 \le y < 1 \right\} \cup \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$. Describe (or sketch) the convex hull of S.

2. Describe the convex hull of the set S of points $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ that satisfy the given conditions. Justify your answers. (Show that an arbitrary point $\mathbf{p}$ in S belongs to conv S.)
   a. $y = 1/x$ and $x \ge 1/2$
   b. $y = \sin x$
   c. $y = x^{1/2}$ and $x \ge 0$

3. Consider the points in Exercise 5 in Section 8.1. Which of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are in conv S?

4. Consider the points in Exercise 6 in Section 8.1. Which of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are in conv S?

5. Let

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix},$$

and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Determine whether $\mathbf{p}_1$ and $\mathbf{p}_2$ are in conv S.

6. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 2 \\ -\frac{3}{2} \\ \frac{5}{2} \end{bmatrix}$,

$\mathbf{p}_2 = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{4} \\ \frac{7}{4} \end{bmatrix}$, $\mathbf{p}_3 = \begin{bmatrix} 6 \\ -4 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{p}_4 = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 4 \end{bmatrix}$, and let S be the orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Determine whether each $\mathbf{p}_i$ is in Span S, aff S, or conv S.
   a. $\mathbf{p}_1$ b. $\mathbf{p}_2$ c. $\mathbf{p}_3$ d. $\mathbf{p}_4$

Exercises 7–10 use the terminology from Section 8.2.

7. a. Let $T = \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$, and let

$$\mathbf{p}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{p}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \text{ and } \mathbf{p}_4 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

   Find the barycentric coordinates of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, and $\mathbf{p}_4$ with respect to T.

   b. Use your answers in part (a) to determine whether each of $\mathbf{p}_1, \ldots, \mathbf{p}_4$ in part (a) is inside, outside, or on the edge of conv T, a triangular region.

8. Repeat Exercise 7 for $T = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and

$$\mathbf{p}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{p}_3 = \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}, \text{ and } \mathbf{p}_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

9. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be an affinely independent set. Consider the points $\mathbf{p}_1, \ldots, \mathbf{p}_5$ whose barycentric coordinates with respect to S are given by $(2, 0, 0, -1)$, $\left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$, $\left(\frac{1}{2}, 0, \frac{3}{2}, -1\right)$, $\left(\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}\right)$, and $\left(\frac{1}{3}, 0, \frac{2}{3}, 0\right)$, respectively. Determine whether each of $\mathbf{p}_1, \ldots, \mathbf{p}_5$ is inside, outside, or on the surface of conv S, a tetrahedron. Are any of these points on an edge of conv S?

10. Repeat Exercise 9 for the points $\mathbf{q}_1, \ldots, \mathbf{q}_5$ whose barycentric coordinates with respect to S are given by $\left(\frac{1}{8}, \frac{1}{4}, \frac{1}{8}, \frac{1}{2}\right)$, $\left(\frac{3}{4}, -\frac{1}{4}, 0, \frac{1}{2}\right)$, $\left(0, \frac{3}{4}, \frac{1}{4}, 0\right)$, $(0, -2, 0, 3)$, and $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$, respectively.

In Exercises 11 and 12, mark each statement True or False. Justify each answer.

11. a. If $\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ and $c_1 + c_2 + c_3 = 1$, then $\mathbf{y}$ is a convex combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.
    b. If S is a nonempty set, then conv S contains some points that are not in S.
    c. If S and T are convex sets, then $S \cup T$ is also convex.

12. a. A set is convex if $\mathbf{x}, \mathbf{y} \in S$ implies that the line segment between $\mathbf{x}$ and $\mathbf{y}$ is contained in S.
    b. If S and T are convex sets, then $S \cap T$ is also convex.

c. If S is a nonempty subset of $\mathbb{R}^5$ and $\mathbf{y} \in \text{conv } S$, then there exist distinct points $\mathbf{v}_1, \dots, \mathbf{v}_6$ in S such that $\mathbf{y}$ is a convex combination of $\mathbf{v}_1, \dots, \mathbf{v}_6$.

13. Let S be a convex subset of $\mathbb{R}^n$ and suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Prove that the set $f(S) = \{f(\mathbf{x}) : \mathbf{x} \in S\}$ is a convex subset of $\mathbb{R}^m$.

14. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let T be a convex subset of $\mathbb{R}^m$. Prove that the set $S = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \in T\}$ is a convex subset of $\mathbb{R}^n$.

15. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Confirm that

$$\mathbf{p} = \tfrac{1}{3}\mathbf{v}_1 + \tfrac{1}{3}\mathbf{v}_2 + \tfrac{1}{6}\mathbf{v}_3 + \tfrac{1}{6}\mathbf{v}_4 \quad \text{and} \quad \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4 = \mathbf{0}.$$

Use the procedure in the proof of Caratheodory's Theorem to express $\mathbf{p}$ as a convex combination of three of the $\mathbf{v}_i$'s. Do this in *two* ways.

16. Repeat Exercise 15 for points $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, given that

$$\mathbf{p} = \tfrac{1}{121}\mathbf{v}_1 + \tfrac{72}{121}\mathbf{v}_2 + \tfrac{37}{121}\mathbf{v}_3 + \tfrac{1}{11}\mathbf{v}_4$$

and

$$10\mathbf{v}_1 - 6\mathbf{v}_2 + 7\mathbf{v}_3 - 11\mathbf{v}_4 = \mathbf{0}.$$

In Exercises 17–20, prove the given statement about subsets A and B of $\mathbb{R}^n$. A proof for an exercise may use results of earlier exercises.

17. If $A \subset B$ and B is convex, then $\text{conv } A \subset B$.

18. If $A \subset B$, then $\text{conv } A \subset \text{conv } B$.

19. a. $[(\text{conv } A) \cup (\text{conv } B)] \subset \text{conv } (A \cup B)$

b. Find an example in $\mathbb{R}^2$ to show that equality need not hold in part (a).

20. a. $\text{conv } (A \cap B) \subset [(\text{conv } A) \cap (\text{conv } B)]$

b. Find an example in $\mathbb{R}^2$ to show that equality need not hold in part (a).

21. Let $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ be points in $\mathbb{R}^n$, and define $\mathbf{f}_0(t) = (1-t)\mathbf{p}_0 + t\mathbf{p}_1$, $\mathbf{f}_1(t) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$, and $\mathbf{g}(t) = (1-t)\mathbf{f}_0(t) + t\mathbf{f}_1(t)$ for $0 \le t \le 1$. For the points as shown below, draw a picture that shows $\mathbf{f}_0\left(\frac{1}{2}\right)$, $\mathbf{f}_1\left(\frac{1}{2}\right)$, and $\mathbf{g}\left(\frac{1}{2}\right)$.

22. Repeat Exercise 21 for $\mathbf{f}_0\left(\frac{3}{4}\right)$, $\mathbf{f}_1\left(\frac{3}{4}\right)$, and $\mathbf{g}\left(\frac{3}{4}\right)$.

23. Let $\mathbf{g}(t)$ be defined as in Exercise 21. Its graph is called a *quadratic Bézier curve*, and it is used in some computer graphics designs. The points $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ are called the *control points* for the curve. Compute a formula for $\mathbf{g}(t)$ that involves only $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$. Then show that $\mathbf{g}(t)$ is in $\text{conv}\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$ for $0 \le t \le 1$.

24. Given control points $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ in $\mathbb{R}^n$, let $\mathbf{g}_1(t)$ for $0 \le t \le 1$ be the quadratic Bézier curve from Exercise 23 determined by $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$, and let $\mathbf{g}_2(t)$ be defined similarly for $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. For $0 \le t \le 1$, define $\mathbf{h}(t) = (1-t)\mathbf{g}_1(t) + t\mathbf{g}_2(t)$. Show that the graph of $\mathbf{h}(t)$ lies in the convex hull of the four control points. This curve is called a *cubic Bézier curve*, and its definition here is one step in an algorithm for constructing Bézier curves (discussed later in Section 8.6). A Bézier curve of degree k is determined by $k+1$ control points, and its graph lies in the convex hull of these control points.

SOLUTIONS TO PRACTICE PROBLEMS

1. The points $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are not orthogonal, so compute

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} -8 \\ 2 \\ -3 \end{bmatrix}, \quad \mathbf{p}_1 - \mathbf{v}_1 = \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}, \text{ and } \mathbf{p}_2 - \mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix}$$

Augment the matrix $[\,\mathbf{v}_2 - \mathbf{v}_1 \quad \mathbf{v}_3 - \mathbf{v}_1\,]$ with both $\mathbf{p}_1 - \mathbf{v}_1$ and $\mathbf{p}_2 - \mathbf{v}_1$, and row reduce:

$$\begin{bmatrix} 1 & -8 & -5 & -3 \\ -1 & 2 & 1 & 0 \\ 3 & -3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{3} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{5}{2} \end{bmatrix}$$

The third column shows that $\mathbf{p}_1 - \mathbf{v}_1 = \frac{1}{3}(\mathbf{v}_2 - \mathbf{v}_1) + \frac{2}{3}(\mathbf{v}_3 - \mathbf{v}_1)$, which leads to $\mathbf{p}_1 = 0\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3$. Thus $\mathbf{p}_1$ is in conv S. In fact, $\mathbf{p}_1$ is in $\text{conv}\{\mathbf{v}_2, \mathbf{v}_3\}$.

The last column of the matrix shows that $\mathbf{p}_2 - \mathbf{v}_1$ is not a linear combination of $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$. Thus $\mathbf{p}_2$ is not an affine combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, so $\mathbf{p}_2$ cannot possibly be in conv S.

An alternative method of solution is to row reduce the augmented matrix of homogeneous forms:

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{p}}_1 & \tilde{\mathbf{p}}_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. If $\mathbf{p}$ is a point above S, then the line through $\mathbf{p}$ with slope -1 will intersect S at two points before it reaches the positive x- and y-axes.

8.4 HYPERPLANES

Hyperplanes play a special role in the geometry of $\mathbb{R}^n$ because they divide the space into two disjoint pieces, just as a plane separates $\mathbb{R}^3$ into two parts and a line cuts through $\mathbb{R}^2$. The key to working with hyperplanes is to use simple *implicit* descriptions, rather than the *explicit* or parametric representations of lines and planes used in the earlier work with affine sets.[1]

An implicit equation of a line in $\mathbb{R}^2$ has the form $ax + by = d$. An implicit equation of a plane in $\mathbb{R}^3$ has the form $ax + by + cz = d$. Both equations describe the line or plane as the set of all points at which a linear expression (also called a *linear functional*) has a fixed value, d.

DEFINITION

A **linear functional** on $\mathbb{R}^n$ is a linear transformation f from $\mathbb{R}^n$ into $\mathbb{R}$. For each scalar d in $\mathbb{R}$, the symbol $[f : d]$ denotes the set of all $\mathbf{x}$ in $\mathbb{R}^n$ at which the value of f is d. That is,

$$[f : d] \quad \text{is the set} \quad \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) = d\}$$

The **zero** *functional* is the transformation such that $f(\mathbf{x}) = 0$ for all $\mathbf{x}$ in $\mathbb{R}^n$. All other linear functionals on $\mathbb{R}^n$ are said to be **nonzero**.

EXAMPLE 1 In $\mathbb{R}^2$, the line $x - 4y = 13$ is a hyperplane in $\mathbb{R}^2$, and it is the set of points at which the linear functional $f(x, y) = x - 4y$ has the value 13. That is, the line is the set $[f : 13]$. ■

EXAMPLE 2 In $\mathbb{R}^3$, the plane $5x - 2y + 3z = 21$ is a hyperplane, the set of points at which the linear functional $g(x, y, z) = 5x - 2y + 3z$ has the value 21. This hyperplane is the set $[g : 21]$. ■

If f is a linear functional on $\mathbb{R}^n$, then the standard matrix of this linear transformation f is a $1 \times n$ matrix A, say $A = [\,a_1 \quad a_2 \quad \cdots \quad a_n\,]$. So

$$[f : 0] \quad \text{is the same as} \quad \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = 0\} = \text{Nul}\, A \tag{1}$$

[1] Parametric representations were introduced in Section 1.5.

If f is a nonzero functional, then rank $A = 1$, and dim Nul $A = n - 1$, by the Rank Theorem.[2] Thus, the subspace $[f : 0]$ has dimension $n - 1$ and so is a hyperplane. Also, if d is any number in $\mathbb{R}$, then

$$[f : d] \quad \text{is the same as} \quad \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = d\} \tag{2}$$

Recall from Theorem 6 in Section 1.5 that the set of solutions of $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, using any particular solution $\mathbf{p}$ of $A\mathbf{x} = \mathbf{b}$. When A is the standard matrix of the transformation f, this theorem says that

$$[f : d] = [f : 0] + \mathbf{p} \quad \text{for any } \mathbf{p} \text{ in } [f : d] \tag{3}$$

Thus the sets $[f : d]$ are hyperplanes parallel to $[f : 0]$. See Figure 1.

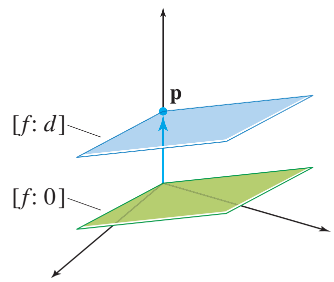


FIGURE 1 Parallel hyperplanes, with $f(\mathbf{p}) = d$.

When A is a $1 \times n$ matrix, the equation $A\mathbf{x} = d$ may be written with an inner product $\mathbf{n} \cdot \mathbf{x}$, using $\mathbf{n}$ in $\mathbb{R}^n$ with the same entries as A. Thus, from (2),

$$[f : d] \quad \text{is the same as} \quad \{\mathbf{x} \in \mathbb{R}^n : \mathbf{n} \cdot \mathbf{x} = d\} \tag{4}$$

Then $[f : 0] = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{n} \cdot \mathbf{x} = 0\}$, which shows that $[f : 0]$ is the orthogonal complement of the subspace spanned by $\mathbf{n}$. In the terminology of calculus and geometry for $\mathbb{R}^3$, $\mathbf{n}$ is called a **normal** vector to $[f : 0]$. (A "normal" vector in this sense need not have unit length.) Also, $\mathbf{n}$ is said to be **normal** to each parallel hyperplane $[f : d]$, even though $\mathbf{n} \cdot \mathbf{x}$ is not zero when $d \neq 0$.

Another name for $[f : d]$ is a *level set* of f, and $\mathbf{n}$ is sometimes called the *gradient* of f when $f(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x}$ for each $\mathbf{x}$.

EXAMPLE 3 Let $\mathbf{n} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$, and let $H = \{\mathbf{x} : \mathbf{n} \cdot \mathbf{x} = 12\}$, so $H = [f : 12]$, where $f(x, y) = 3x + 4y$. Thus H is the line $3x + 4y = 12$. Find an implicit description of the parallel hyperplane (line) $H_1 = H + \mathbf{v}$.

SOLUTION First, find a point $\mathbf{p}$ in H_1. To do this, find a point in H and add $\mathbf{v}$ to it. For instance, $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ is in H, so $\mathbf{p} = \begin{bmatrix} 1 \\ -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is in H_1. Now, compute $\mathbf{n} \cdot \mathbf{p} = -9$. This shows that $H_1 = [f : -9]$. See Figure 2, which also shows the subspace $H_0 = \{\mathbf{x} : \mathbf{n} \cdot \mathbf{x} = 0\}$. ■

The next three examples show connections between implicit and explicit descriptions of hyperplanes. Example 4 begins with an implicit form.

[2] See Theorem 14 in Section 2.9 or Theorem 14 in Section 4.6.

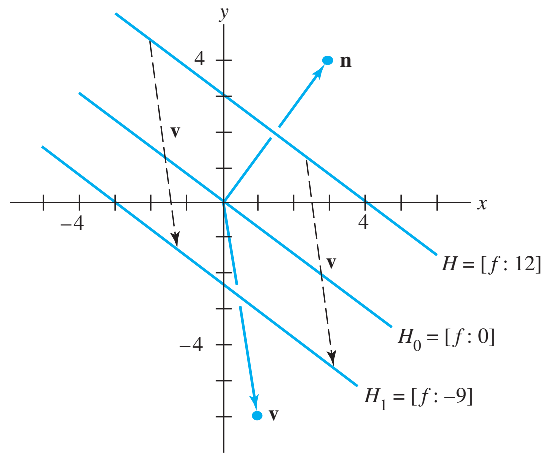


FIGURE 2

EXAMPLE 4 In $\mathbb{R}^2$, give an explicit description of the line $x - 4y = 13$ in parametric vector form.

SOLUTION This amounts to solving a nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$, where $A = [\,1 \quad -4\,]$ and $\mathbf{b}$ is the number 13 in $\mathbb{R}$. Write $x = 13 + 4y$, where y is a free variable. In parametric form, the solution is

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 + 4y \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \end{bmatrix} + y\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \mathbf{p} + y\mathbf{q}, \quad y \in \mathbb{R}$$ ■

Converting an explicit description of a line into implicit form is more involved. The basic idea is to construct $[f:0]$ and then find d for $[f:d]$.

EXAMPLE 5 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, and let L_1 be the line through $\mathbf{v}_1$ and $\mathbf{v}_2$. Find a linear functional f and a constant d such that $L_1 = [f:d]$.

SOLUTION The line L_1 is parallel to the translated line L_0 through $\mathbf{v}_2 - \mathbf{v}_1$ and the origin. The defining equation for L_0 has the form

$$[\,a \quad b\,]\begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \text{or} \quad \mathbf{n}\cdot\mathbf{x} = 0, \quad \text{where} \quad \mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{5}$$

Since $\mathbf{n}$ is orthogonal to the subspace L_0, which contains $\mathbf{v}_2 - \mathbf{v}_1$, compute

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 6 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

and solve

$$[\,a \quad b\,]\begin{bmatrix} 5 \\ -2 \end{bmatrix} = 0$$

By inspection, a solution is $[\,a \quad b\,] = [\,2 \quad 5\,]$. Let $f(x, y) = 2x + 5y$. From (5), $L_0 = [f:0]$, and $L_1 = [f:d]$ for some d. Since $\mathbf{v}_1$ is on line L_1, $d = f(\mathbf{v}_1) = 2(1) + 5(2) = 12$. Thus, the equation for L_1 is $2x + 5y = 12$. As a check, note that $f(\mathbf{v}_2) = f(6, 0) = 2(6) + 5(0) = 12$, so $\mathbf{v}_2$ is on L_1, too. ■

EXAMPLE 6 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. Find an implicit description $[f:d]$ of the plane H_1 that passes through $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

SOLUTION H_1 is parallel to a plane H_0 through the origin that contains the translated points

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Since these two points are linearly independent, $H_0 = \text{Span}\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$. Let $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the normal to H_0. Then $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$ are each orthogonal to $\mathbf{n}$. That is, $(\mathbf{v}_2 - \mathbf{v}_1)\cdot\mathbf{n} = 0$ and $(\mathbf{v}_3 - \mathbf{v}_1)\cdot\mathbf{n} = 0$. These two equations form a system whose augmented matrix can be row reduced:

$$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0, \quad \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0, \quad \begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix}$$

Row operations yield $a = (-\frac{2}{4})c$, $b = (\frac{5}{4})c$, with c free. Set $c = 4$, for instance. Then $\mathbf{n} = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$ and $H_0 = [f:0]$, where $f(\mathbf{x}) = -2x_1 + 5x_2 + 4x_3$.

The parallel hyperplane H_1 is $[f:d]$. To find d, use the fact that $\mathbf{v}_1$ is in H_1, and compute $d = f(\mathbf{v}_1) = f(1, 1, 1) = -2(1) + 5(1) + 4(1) = 7$. As a check, compute $f(\mathbf{v}_2) = f(2, -1, 4) = -2(2) + 5(-1) + 4(4) = 16 - 9 = 7$. Observe $f(\mathbf{v}_3) =$ 7 also. ■

The procedure in Example 6 generalizes to higher dimensions. However, for the special case of $\mathbb{R}^3$, one can also use the **cross-product** formula to compute $\mathbf{n}$, using a symbolic determinant as a mnemonic device:

$$\begin{aligned} \mathbf{n} &= (\mathbf{v}_2 - \mathbf{v}_1) \times (\mathbf{v}_3 - \mathbf{v}_1) \\ &= \begin{vmatrix} 1 & 2 & \mathbf{i} \\ -2 & 0 & \mathbf{j} \\ 3 & 1 & \mathbf{k} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix}\mathbf{k} \\ &= -2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} \end{aligned}$$

If only the formula for f is needed, the cross-product calculation may be written as an ordinary determinant:

$$\begin{aligned} f(x_1, x_2, x_3) &= \begin{vmatrix} 1 & 2 & x_1 \\ -2 & 0 & x_2 \\ 3 & 1 & x_3 \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 3 & 1 \end{vmatrix}x_1 - \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}x_2 + \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix}x_3 \\ &= -2x_1 + 5x_2 + 4x_3 \end{aligned}$$

So far, every hyperplane examined has been described as $[f:d]$ for some linear functional f and some d in $\mathbb{R}$, or equivalently as $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{n}\cdot\mathbf{x} = d\}$ for some $\mathbf{n}$ in $\mathbb{R}^n$. The following theorem shows that *every* hyperplane has these equivalent descriptions.

THEOREM 11

A subset H of $\mathbb{R}^n$ is a hyperplane if and only if $H = [f:d]$ for some nonzero linear functional f and some scalar d in $\mathbb{R}$. Thus, if H is a hyperplane, there exist a nonzero vector $\mathbf{n}$ and a real number d such that $H = \{\mathbf{x} : \mathbf{n}\cdot\mathbf{x} = d\}$.

PROOF Suppose that H is a hyperplane, take $\mathbf{p} \in H$, and let $H_0 = H - \mathbf{p}$. Then H_0 is an $(n-1)$-dimensional subspace. Next, take any point $\mathbf{y}$ that is not in H_0. By the Orthogonal Decomposition Theorem in Section 6.3,

$$\mathbf{y} = \mathbf{y}_1 + \mathbf{n}$$

where $\mathbf{y}_1$ is a vector in H_0 and $\mathbf{n}$ is orthogonal to every vector in H_0. The function f defined by

$$f(\mathbf{x}) = \mathbf{n}\cdot\mathbf{x} \quad \text{for } \mathbf{x} \in \mathbb{R}^n$$

is a linear functional, by properties of the inner product. Now, $[f:0]$ is a hyperplane that contains H_0, by construction of $\mathbf{n}$. It follows that

$$H_0 = [f:0]$$

[*Argument:* H_0 contains a basis S of $n-1$ vectors, and since S is in the $(n-1)$-dimensional subspace $[f:0]$, S must also be a basis for $[f:0]$, by the Basis Theorem.] Finally, let $d = f(\mathbf{p}) = \mathbf{n}\cdot\mathbf{p}$. Then, as in (3) shown earlier,

$$[f:d] = [f:0] + \mathbf{p} = H_0 + \mathbf{p} = H$$

The converse statement that $[f:d]$ is a hyperplane follows from (1) and (3) above. ■

Many important applications of hyperplanes depend on the possibility of "separating" two sets by a hyperplane. Intuitively, this means that one of the sets is on one side of the hyperplane and the other set is on the other side. The following terminology and notation will help to make this idea more precise.

TOPOLOGY IN $\mathbb{R}^n$: TERMS AND FACTS

For any point $\mathbf{p}$ in $\mathbb{R}^n$ and any real $\delta > 0$, the **open ball** $B(\mathbf{p}, \delta)$ with center $\mathbf{p}$ and radius δ is given by

$$B(\mathbf{p}, \delta) = \{\mathbf{x} : \|\mathbf{x} - \mathbf{p}\| < \delta\}$$

Given a set S in $\mathbb{R}^n$, a point $\mathbf{p}$ is an **interior point** of S if there exists a $\delta > 0$ such that $B(\mathbf{p}, \delta) \subset S$. If every open ball centered at $\mathbf{p}$ intersects both S and the complement of S, then $\mathbf{p}$ is called a **boundary point** of S. A set is **open** if it contains none of its boundary points. (This is equivalent to saying that all of its points are interior points.) A set is **closed** if it contains all of its boundary points. (If S contains some but not all of its boundary points, then S is neither open nor closed.) A set S is **bounded** if there exists a $\delta > 0$ such that $S \subset B(\mathbf{0}, \delta)$. A set in $\mathbb{R}^n$ is **compact** if it is closed and bounded.

Theorem: The convex hull of an open set is open, and the convex hull of a compact set is compact. (The convex hull of a closed set need not be closed. See Exercise 27.)

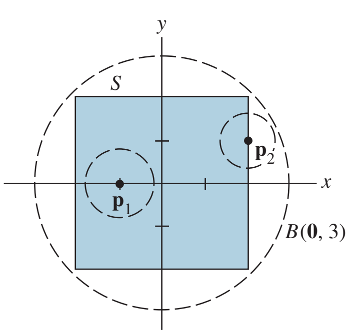


FIGURE 3
The set S is closed and bounded.

EXAMPLE 7 Let

$$S = \text{conv}\left\{\begin{bmatrix}-2\\2\end{bmatrix}, \begin{bmatrix}-2\\-2\end{bmatrix}, \begin{bmatrix}2\\-2\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}, \quad \mathbf{p}_1 = \begin{bmatrix}-1\\0\end{bmatrix}, \quad \text{and} \quad \mathbf{p}_2 = \begin{bmatrix}2\\1\end{bmatrix},$$

as shown in Figure 3. Then $\mathbf{p}_1$ is an interior point since $B\left(\mathbf{p}, \frac{3}{4}\right) \subset S$. The point $\mathbf{p}_2$ is a boundary point since every open ball centered at $\mathbf{p}_2$ intersects both S and the complement of S. The set S is closed since it contains all its boundary points. The set S is bounded since $S \subset B(\mathbf{0}, 3)$. Thus S is also compact. ■

Notation: If f is a linear functional, then $f(A) \le d$ means $f(\mathbf{x}) \le d$ for each $\mathbf{x} \in A$. Corresponding notations will be used when the inequalities are reversed or when they are strict.

DEFINITION

The hyperplane $H = [f:d]$ **separates** two sets A and B if one of the following holds:

(i) $f(A) \le d$ and $f(B) \ge d$, or
(ii) $f(A) \ge d$ and $f(B) \le d$.

If in the conditions above all the weak inequalities are replaced by strict inequalities, then H is said to **strictly separate** A and B.

Notice that strict separation requires that the two sets be disjoint, while mere separation does not. Indeed, if two circles in the plane are externally tangent, then their common tangent line separates them (but does not separate them strictly).

Although it is necessary that two sets be disjoint in order to strictly separate them, this condition is not sufficient, even for closed convex sets. For example, let

$$A = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \ge \frac{1}{2} \text{ and } \frac{1}{x} \le y \le 2 \right\} \quad \text{and} \quad B = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \ge 0 \text{ and } y = 0 \right\}$$

Then A and B are disjoint closed convex sets, but they cannot be strictly separated by a hyperplane (line in $\mathbb{R}^2$). See Figure 4. Thus the problem of separating (or strictly separating) two sets by a hyperplane is more complex than it might at first appear.

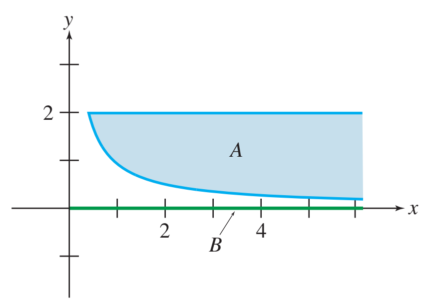


FIGURE 4 Disjoint closed convex sets.

There are many interesting conditions on the sets A and B that imply the existence of a separating hyperplane, but the following two theorems are sufficient for this section. The proof of the first theorem requires quite a bit of preliminary material,[3] but the second theorem follows easily from the first.

THEOREM 12

Suppose A and B are nonempty convex sets such that A is compact and B is closed. Then there exists a hyperplane H that strictly separates A and B if and only if $A \cap B = \varnothing$.

THEOREM 13

Suppose A and B are nonempty compact sets. Then there exists a hyperplane that strictly separates A and B if and only if $(\text{conv } A) \cap (\text{conv } B) = \varnothing$.

[3] A proof of Theorem 12 is given in Steven R. Lay, *Convex Sets and Their Applications* (New York: John Wiley & Sons, 1982; Mineola, NY: Dover Publications, 2007), pp. 34–39.

PROOF Suppose that $(\text{conv } A) \cap (\text{conv } B) = \varnothing$. Since the convex hull of a compact set is compact, Theorem 12 ensures that there is a hyperplane H that strictly separates conv A and conv B. Clearly, H also strictly separates the smaller sets A and B.

Conversely, suppose the hyperplane $H = [f:d]$ strictly separates A and B. Without loss of generality, assume that $f(A) < d$ and $f(B) > d$. Let $\mathbf{x} = c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k$ be any convex combination of elements of A. Then

$$f(\mathbf{x}) = c_1 f(\mathbf{x}_1) + \cdots + c_k f(\mathbf{x}_k) < c_1 d + \cdots + c_k d = d$$

since $c_1 + \cdots + c_k = 1$. Thus $f(\text{conv } A) < d$. Likewise, $f(\text{conv } B) > d$, so $H = [f:d]$ strictly separates conv A and conv B. By Theorem 12, conv A and conv B must be disjoint. ■

EXAMPLE 8 Let

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix},$$

and let $A = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Show that the hyperplane $H = [f:5]$, where $f(x_1, x_2, x_3) = 2x_1 - 3x_2 + x_3$, does not separate A and B. Is there a hyperplane parallel to H that does separate A and B? Do the convex hulls of A and B intersect?

SOLUTION Evaluate the linear functional f at each of the points in A and B:

$$f(\mathbf{a}_1) = 2, \quad f(\mathbf{a}_2) = -11, \quad f(\mathbf{a}_3) = -6, \quad f(\mathbf{b}_1) = 4, \quad \text{and} \quad f(\mathbf{b}_2) = 12$$

Since $f(\mathbf{b}_1) = 4$ is less than 5 and $f(\mathbf{b}_2) = 12$ is greater than 5, points of B lie on both sides of $H = [f:5]$ and so H does not separate A and B.

Since $f(A) < 3$ and $f(B) > 3$, the parallel hyperplane $[f:3]$ strictly separates A and B. By Theorem 13, $(\text{conv } A) \cap (\text{conv } B) = \varnothing$.

Caution: If there were no hyperplane parallel to H that strictly separated A and B, this would *not* necessarily imply that their convex hulls intersect. It might be that some other hyperplane not parallel to H would strictly separate them. ■

PRACTICE PROBLEM

Let $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{n}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, and $\mathbf{n}_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$; let H_1 be the hyperplane (plane) in $\mathbb{R}^3$ passing through the point $\mathbf{p}_1$ and having normal vector $\mathbf{n}_1$; and let H_2 be the hyperplane passing through the point $\mathbf{p}_2$ and having normal vector $\mathbf{n}_2$. Give an explicit description of $H_1 \cap H_2$ by a formula that shows how to generate all points in $H_1 \cap H_2$.

8.4 EXERCISES

1. Let L be the line in $\mathbb{R}^2$ through the points $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find a linear functional f and a real number d such that $L = [f:d]$.

2. Let L be the line in $\mathbb{R}^2$ through the points $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$. Find a linear functional f and a real number d such that $L = [f:d]$.

In Exercises 3 and 4, determine whether each set is open or closed or neither open nor closed.

3. a. $\{(x, y) : y > 0\}$
b. $\{(x, y) : x = 2 \text{ and } 1 \le y \le 3\}$
c. $\{(x, y) : x = 2 \text{ and } 1 < y < 3\}$
d. $\{(x, y) : xy = 1 \text{ and } x > 0\}$
e. $\{(x, y) : xy \ge 1 \text{ and } x > 0\}$

4. a. $\{(x, y) : x^2 + y^2 = 1\}$
b. $\{(x, y) : x^2 + y^2 > 1\}$
c. $\{(x, y) : x^2 + y^2 \le 1 \text{ and } y > 0\}$
d. $\{(x, y) : y \ge x^2\}$
e. $\{(x, y) : y < x^2\}$

In Exercises 5 and 6, determine whether or not each set is compact and whether or not it is convex.

5. Use the sets from Exercise 3.

6. Use the sets from Exercise 4.

In Exercises 7–10, let H be the hyperplane through the listed points. (a) Find a vector $\mathbf{n}$ that is normal to the hyperplane. (b) Find a linear functional f and a real number d such that $H = [f : d]$.

7. $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ -4 \\ 4 \end{bmatrix}$

9. $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

10. $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \\ -1 \end{bmatrix}$

11. Let $\mathbf{p} = \begin{bmatrix} 1 \\ -3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 4 \end{bmatrix}$, and let H be the hyperplane in $\mathbb{R}^4$ with normal $\mathbf{n}$ and passing through $\mathbf{p}$. Which of the points $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are on the same side of H as the origin, and which are not?

12. Let $\mathbf{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} -1 \\ 6 \\ 0 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{n} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$, and let $A = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Find a hyperplane H with normal $\mathbf{n}$ that separates A and B. Is there a hyperplane parallel to H that strictly separates A and B?

13. Let $\mathbf{p}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{n}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 2 \end{bmatrix}$, and $\mathbf{n}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 5 \end{bmatrix}$; let H_1 be the hyperplane in $\mathbb{R}^4$ through $\mathbf{p}_1$ with normal $\mathbf{n}_1$; and let H_2 be the hyperplane through $\mathbf{p}_2$ with normal $\mathbf{n}_2$. Give an explicit description of $H_1 \cap H_2$. [*Hint:* Find a point $\mathbf{p}$ in $H_1 \cap H_2$ and two linearly independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ that span a subspace parallel to the 2-dimensional flat $H_1 \cap H_2$.]

14. Let F_1 and F_2 be 4-dimensional flats in $\mathbb{R}^6$, and suppose that $F_1 \cap F_2 \neq \varnothing$. What are the possible dimensions of $F_1 \cap F_2$?

In Exercises 15–20, write a formula for a linear functional f and specify a number d, so that $[f : d]$ is the hyperplane H described in the exercise.

15. Let A be the 1×4 matrix $\begin{bmatrix} 1 & -3 & 4 & -2 \end{bmatrix}$ and let $b = 5$. Let $H = \{\mathbf{x} \text{ in } \mathbb{R}^4 : A\mathbf{x} = b\}$.

16. Let A be the 1×5 matrix $\begin{bmatrix} 2 & 5 & -3 & 0 & 6 \end{bmatrix}$. Note that Nul A is in $\mathbb{R}^5$. Let $H = \text{Nul } A$.

17. Let H be the plane in $\mathbb{R}^3$ spanned by the rows of $B = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \end{bmatrix}$. That is, $H = \text{Row } B$. [*Hint:* How is H related to Nul B? See Section 6.1.]

18. Let H be the plane in $\mathbb{R}^3$ spanned by the rows of $B = \begin{bmatrix} 1 & 4 & -5 \\ 0 & -2 & 8 \end{bmatrix}$. That is, $H = \text{Row } B$.

19. Let H be the column space of the matrix $B = \begin{bmatrix} 1 & 0 \\ 4 & 2 \\ -7 & -6 \end{bmatrix}$.
That is, $H = \text{Col } B$. [*Hint:* How is Col B related to Nul B^T? See Section 6.1.]

20. Let H be the column space of the matrix $B = \begin{bmatrix} 1 & 0 \\ 5 & 2 \\ -4 & -4 \end{bmatrix}$.
That is, $H = \text{Col } B$.

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. A linear transformation from $\mathbb{R}$ to $\mathbb{R}^n$ is called a linear functional.
b. If f is a linear functional defined on $\mathbb{R}^n$, then there exists a real number k such that $f(\mathbf{x}) = k\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.
c. If a hyperplane strictly separates sets A and B, then $A \cap B = \varnothing$.
d. If A and B are closed convex sets and $A \cap B = \varnothing$, then there exists a hyperplane that strictly separates A and B.

22. a. If d is a real number and f is a nonzero linear functional defined on $\mathbb{R}^n$, then $[f:d]$ is a hyperplane in $\mathbb{R}^n$.

b. Given any vector $\mathbf{n}$ and any real number d, the set $\{\mathbf{x} : \mathbf{n}\cdot\mathbf{x} = d\}$ is a hyperplane.

c. If A and B are nonempty disjoint sets such that A is compact and B is closed, then there exists a hyperplane that strictly separates A and B.

d. If there exists a hyperplane H such that H does not strictly separate two sets A and B, then $(\text{conv } A) \cap (\text{conv } B) \neq \varnothing$.

23. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Find a hyperplane $[f:d]$ (in this case, a line) that strictly separates $\mathbf{p}$ from $\text{conv}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

24. Repeat Exercise 23 for $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

25. Let $\mathbf{p} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Find a hyperplane $[f:d]$ that strictly separates $B(\mathbf{0}, 3)$ and $B(\mathbf{p}, 1)$. [*Hint:* After finding f, show that the point $\mathbf{v} = (1 - .75)\mathbf{0} + .75\mathbf{p}$ is neither in $B(\mathbf{0}, 3)$ nor in $B(\mathbf{p}, 1)$.]

26. Let $\mathbf{q} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$. Find a hyperplane $[f:d]$ that strictly separates $B(\mathbf{q}, 3)$ and $B(\mathbf{p}, 1)$.

27. Give an example of a closed subset S of $\mathbb{R}^2$ such that conv S is not closed.

28. Give an example of a compact set A and a closed set B in $\mathbb{R}^2$ such that $(\text{conv } A) \cap (\text{conv } B) = \varnothing$ but A and B cannot be strictly separated by a hyperplane.

29. Prove that the open ball $B(\mathbf{p}, \delta) = \{\mathbf{x} : \|\mathbf{x} - \mathbf{p}\| < \delta\}$ is a convex set. [*Hint:* Use the Triangle Inequality.]

30. Prove that the convex hull of a bounded set is bounded.

SOLUTION TO PRACTICE PROBLEM

First, compute $\mathbf{n}_1 \cdot \mathbf{p}_1 = -3$ and $\mathbf{n}_2 \cdot \mathbf{p}_2 = 7$. The hyperplane H_1 is the solution set of the equation $x_1 + x_2 - 2x_3 = -3$, and H_2 is the solution set of the equation $-2x_1 + x_2 + 3x_3 = 7$. Then

$$H_1 \cap H_2 = \{\mathbf{x} : x_1 + x_2 - 2x_3 = -3 \text{ and } -2x_1 + x_2 + 3x_3 = 7\}$$

This is an implicit description of $H_1 \cap H_2$. To find an explicit description, solve the system of equations by row reduction:

$$\begin{bmatrix} 1 & 1 & -2 & -3 \\ -2 & 1 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{5}{3} & -\frac{10}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Thus $x_1 = -\frac{10}{3} + \frac{5}{3}x_3$, $x_2 = \frac{1}{3} + \frac{1}{3}x_3$, $x_3 = x_3$. Let $\mathbf{p} = \begin{bmatrix} -\frac{10}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$. The general solution can be written as $\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$. Thus $H_1 \cap H_2$ is the line through $\mathbf{p}$ in the direction of $\mathbf{v}$. Note that $\mathbf{v}$ is orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$.

8.5 POLYTOPES

This section studies geometric properties of an important class of compact convex sets called polytopes. These sets arise in all sorts of applications, including game theory (Section 9.1), linear programming (Sections 9.2 to 9.4), and more general optimization problems, such as the design of feedback controls for engineering systems.

A **polytope** in $\mathbb{R}^n$ is the convex hull of a finite set of points. In $\mathbb{R}^2$, a polytope is simply a polygon. In $\mathbb{R}^3$, a polytope is called a polyhedron. Important features of a polyhedron are its faces, edges, and vertices. For example, the cube has 6 square faces, 12 edges, and 8 vertices. The following definitions provide terminology for higher dimensions as well as $\mathbb{R}^2$ and $\mathbb{R}^3$. Recall that the dimension of a set in $\mathbb{R}^n$ is the dimension of the smallest flat that contains it. Also, note that a polytope is a special type of compact convex set, because a finite set in $\mathbb{R}^n$ is compact and the convex hull of this set is compact, by the theorem in the topology terms and facts box in Section 8.4.

DEFINITION

Let S be a compact convex subset of $\mathbb{R}^n$. A nonempty subset F of S is called a (proper) **face** of S if $F \neq S$ and there exists a hyperplane $H = [f : d]$ such that $F = S \cap H$ and either $f(S) \leq d$ or $f(S) \geq d$. The hyperplane H is called a **supporting hyperplane** to S. If the dimension of F is k, then F is called a ***k*-face** of S.

If P is a polytope of dimension k, then P is called a ***k*-polytope**. A 0-face of P is called a **vertex** (plural: **vertices**), a 1-face is an **edge**, and a $(k-1)$-dimensional face is a **facet** of S.

EXAMPLE 1 Suppose S is a cube in $\mathbb{R}^3$. When a plane H is translated through $\mathbb{R}^3$ until it just touches (supports) the cube but does not cut through the interior of the cube, there are three possibilities for $H \cap S$, depending on the orientation of H. (See Figure 1.)

$H \cap S$ may be a 2-dimensional square face (facet) of the cube.

$H \cap S$ may be a 1-dimensional edge of the cube.

$H \cap S$ may be a 0-dimensional vertex of the cube. ■

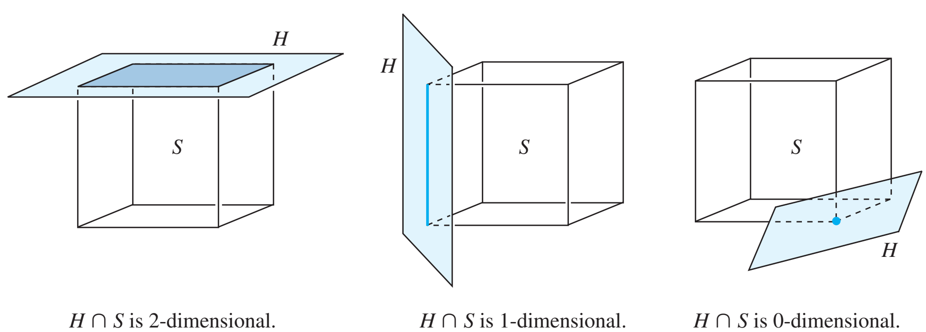


FIGURE 1

Most applications of polytopes involve the vertices in some way, because they have a special property that is identified in the following definition.

DEFINITION

Let S be a convex set. A point $\mathbf{p}$ in S is called an **extreme point** of S if $\mathbf{p}$ is not in the interior of any line segment that lies in S. More precisely, if $\mathbf{x}, \mathbf{y} \in S$ and $\mathbf{p} \in \overline{\mathbf{x}\mathbf{y}}$, then $\mathbf{p} = \mathbf{x}$ or $\mathbf{p} = \mathbf{y}$. The set of all extreme points of S is called the **profile** of S.

A vertex of any compact convex set S is automatically an extreme point of S. This fact is proved during the proof of Theorem 14, below. In working with a polytope, say $P = \text{conv}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ for $\mathbf{v}_1, \dots, \mathbf{v}_k$ in $\mathbb{R}^n$, it is usually helpful to know that $\mathbf{v}_1, \dots, \mathbf{v}_k$ are the extreme points of P. However, such a list might contain extraneous points. For example, some vector $\mathbf{v}_i$ could be the midpoint of an edge of the polytope. Of course, in this case $\mathbf{v}_i$ is not really needed to generate the convex hull. The following definition describes the property of the vertices that will make them all extreme points.

DEFINITION

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a **minimal representation** of the polytope P if $P = \text{conv}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and for each $i = 1, \dots, k$, $\mathbf{v}_i \notin \text{conv}\{\mathbf{v}_j : j \neq i\}$.

Every polytope has a minimal representation. For if $P = \text{conv}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and if some $\mathbf{v}_i$ is a convex combination of the other points, then $\mathbf{v}_i$ may be deleted from the set of points without changing the convex hull. This process may be repeated until the minimal representation is left. It can be shown that the minimal representation is unique.

THEOREM 14

Suppose $M = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is the minimal representation of the polytope P. Then the following three statements are equivalent:

a. $\mathbf{p} \in M$.

b. $\mathbf{p}$ is a vertex of P.

c. $\mathbf{p}$ is an extreme point of P.

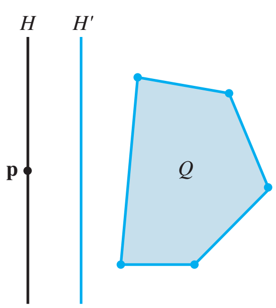


FIGURE 2

PROOF (a) $\Rightarrow$ (b) Suppose $\mathbf{p} \in M$ and let $Q = \text{conv}\{\mathbf{v} : \mathbf{v} \in M \text{ and } \mathbf{v} \neq \mathbf{p}\}$. It follows from the definition of M that $\mathbf{p} \notin Q$, and since Q is compact, Theorem 13 implies the existence of a hyperplane H' that strictly separates $\{\mathbf{p}\}$ and Q. Let H be the hyperplane through $\mathbf{p}$ parallel to H'. See Figure 2.

Then Q lies in one of the closed half-spaces H^+ bounded by H and so $P \subseteq H^+$. Thus H supports P at $\mathbf{p}$. Furthermore, $\mathbf{p}$ is the only point of P that can lie on H, so $H \cap P = \{\mathbf{p}\}$ and $\mathbf{p}$ is a vertex of P.

(b) $\Rightarrow$ (c) Let $\mathbf{p}$ be a vertex of P. Then there exists a hyperplane $H = [f : d]$ such that $H \cap P = \{\mathbf{p}\}$ and $f(P) \geq d$. If $\mathbf{p}$ were not an extreme point, then there would exist points $\mathbf{x}$ and $\mathbf{y}$ in P such that $\mathbf{p} = (1-c)\mathbf{x} + c\mathbf{y}$ with $0 < c < 1$. That is,

$$c\mathbf{y} = \mathbf{p} - (1-c)\mathbf{x} \quad \text{and} \quad \mathbf{y} = \left(\frac{1}{c}\right)(\mathbf{p}) - \left(\frac{1}{c} - 1\right)(\mathbf{x})$$

It follows that $f(\mathbf{y}) = \dfrac{1}{c} f(\mathbf{p}) - \left(\dfrac{1}{c} - 1\right) f(\mathbf{x})$. But $f(\mathbf{p}) = d$ and $f(\mathbf{x}) \geq d$, so

$$f(\mathbf{y}) \leq \left(\frac{1}{c}\right)(d) - \left(\frac{1}{c} - 1\right)(d) = d$$

On the other hand, $\mathbf{y} \in P$, so $f(\mathbf{y}) \geq d$. It follows that $f(\mathbf{y}) = d$ and that $\mathbf{y} \in H \cap P$. This contradicts the fact that $\mathbf{p}$ is a vertex. So $\mathbf{p}$ must be an extreme point. (Note that this part of the proof does not depend on P being a polytope. It holds for any compact convex set.)

(c) $\Rightarrow$ (a) It is clear that any extreme point of P must be a member of M. ■

EXAMPLE 2 Recall that the profile of a set S is the set of extreme points of S. Theorem 14 shows that the profile of a polygon in $\mathbb{R}^2$ is the set of vertices. (See Figure 3.) The profile of a closed ball is its boundary. An open set has no extreme points, so its profile is empty. A closed half-space has no extreme points, so its profile is empty. ■

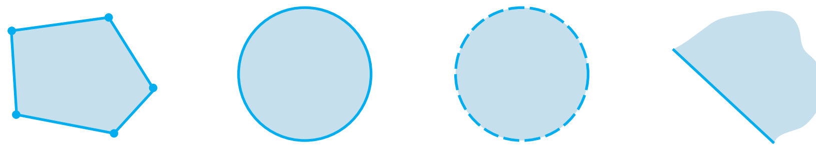


FIGURE 3

Exercise 18 asks you to show that a point $\mathbf{p}$ in a convex set S is an extreme point of S if and only if, when $\mathbf{p}$ is removed from S, the remaining points still form a convex set. It follows that if S^* is any subset of S such that conv S^* is equal to S, then S^* must contain the profile of S. The sets in Example 2 show that in general S^* may have to be larger than the profile of S. It is true, however, that when S is compact, we may actually take S^* to be the profile of S, as Theorem 15 will show. Thus every nonempty compact convex set S has an extreme point, and the set of all extreme points is the smallest subset of S whose convex hull is equal to S.

THEOREM 15

Let S be a nonempty compact convex set. Then S is the convex hull of its profile (the set of extreme points of S).

PROOF The proof is by induction on the dimension of the set S.[1] ■

One important application of Theorem 15 is the following theorem. It is one of the key theoretical results in the development of linear programming. Linear functionals are continuous, and continuous functions always attain their maximum and minimum on a compact set. The significance of Theorem 16 is that for compact convex sets, the maximum (and minimum) is actually attained at an extreme point of S.

THEOREM 16

Let f be a linear functional defined on a nonempty compact convex set S. Then there exist extreme points $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ of S such that

$$f(\hat{\mathbf{v}}) = \max_{\mathbf{v}\in S} f(\mathbf{v}) \quad \text{and} \quad f(\hat{\mathbf{w}}) = \min_{\mathbf{v}\in S} f(\mathbf{v})$$

PROOF Assume that f attains its maximum m on S at some point $\mathbf{v}'$ in S. That is, $f(\mathbf{v}') = m$. We wish to show that there exists an extreme point in S with the same property. By Theorem 15, $\mathbf{v}'$ is a convex combination of the extreme points of S. That is, there exist extreme points $\mathbf{v}_1, \dots, \mathbf{v}_k$ of S and nonnegative $c_1, \dots, c_k$ such that

$$\mathbf{v}' = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \quad \text{with } c_1 + \cdots + c_k = 1$$

If none of the extreme points of S satisfies $f(\mathbf{v}) = m$, then

$$f(\mathbf{v}_i) < m \quad \text{for } i = 1, \dots, k$$

[1] The details may be found in Steven R. Lay, *Convex Sets and Their Applications* (New York: John Wiley & Sons, 1982; Mineola, NY: Dover Publications, 2007), p. 43.

since m is the maximum of f on S. But then, because f is linear,

$$\begin{aligned} m = f(\mathbf{v}') &= f(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) \\ &= c_1 f(\mathbf{v}_1) + \cdots + c_k f(\mathbf{v}_k) \\ &< c_1 m + \cdots + c_k m = m(c_1 + \cdots + c_k) = m \end{aligned}$$

This contradiction implies that some extreme point $\hat{\mathbf{v}}$ of S must satisfy $f(\hat{\mathbf{v}}) = m$.

The proof for $\hat{\mathbf{w}}$ is similar. ■

EXAMPLE 3 Given points $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^2$, let $S = \text{conv}\,\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$. For each linear functional f, find the maximum value m of f on the set S, and find all points $\mathbf{x}$ in S at which $f(\mathbf{x}) = m$.

a. $f_1(x_1, x_2) = x_1 + x_2$ b. $f_2(x_1, x_2) = -3x_1 + x_2$ c. $f_3(x_1, x_2) = x_1 + 2x_2$

SOLUTION By Theorem 16, the maximum value is attained at one of the extreme points of S. So to find m, evaluate f at each extreme point and select the largest value.

a. $f_1(\mathbf{p}_1) = -1$, $f_1(\mathbf{p}_2) = 4$, and $f_1(\mathbf{p}_3) = 3$, so $m_1 = 4$. Graph the line $f_1(x_1, x_2) = m_1$, that is, $x_1 + x_2 = 4$, and note that $\mathbf{x} = \mathbf{p}_2$ is the only point in S at which $f_1(\mathbf{x}) = 4$. See Figure 4(a).

b. $f_2(\mathbf{p}_1) = 3$, $f_2(\mathbf{p}_2) = -8$, and $f_2(\mathbf{p}_3) = -1$, so $m_2 = 3$. Graph the line $f_2(x_1, x_2) = m_2$, that is, $-3x_1 + x_2 = 3$, and note that $\mathbf{x} = \mathbf{p}_1$ is the only point in S at which $f_2(\mathbf{x}) = 3$. See Figure 4(b).

c. $f_3(\mathbf{p}_1) = -1$, $f_3(\mathbf{p}_2) = 5$, and $f_3(\mathbf{p}_3) = 5$, so $m_3 = 5$. Graph the line $f_3(x_1, x_2) = m_3$, that is, $x_1 + 2x_2 = 5$. Here, f_3 attains its maximum value at $\mathbf{p}_2$, at $\mathbf{p}_3$, and at every point in the convex hull of $\mathbf{p}_2$ and $\mathbf{p}_3$. See Figure 4(c). ■

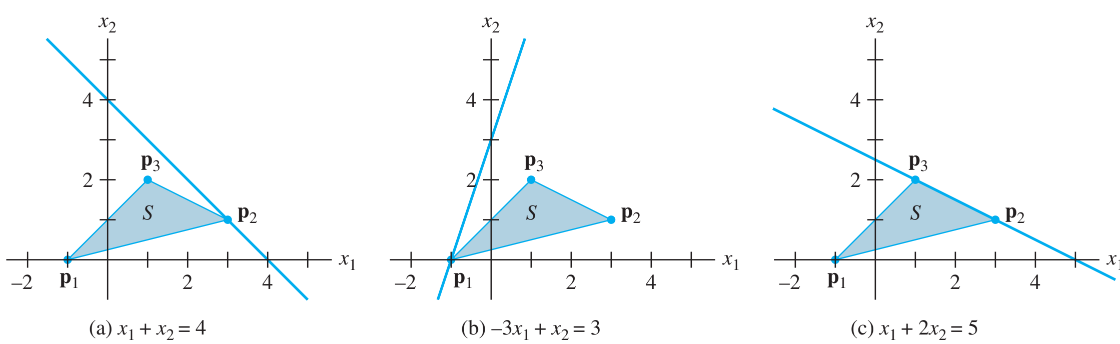


FIGURE 4

The situation illustrated in Example 3 for $\mathbb{R}^2$ also applies in higher dimensions. The maximum value of a linear functional f on a polytope P occurs at the intersection of a supporting hyperplane and P. This intersection is either a single extreme point of P, or the convex hull of 2 or more extreme points of P. In either case, the intersection is a polytope, and its extreme points form a subset of the extreme points of P.

By definition, a polytope is the convex hull of a finite set of points. This is an explicit representation of the polytope since it identifies points in the set. A polytope may also be represented implicitly as the intersection of a finite number of closed half-spaces. Example 4 illustrates this in $\mathbb{R}^2$.

EXAMPLE 4 Let

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{p}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

in $\mathbb{R}^2$, and let $S = \text{conv}\,\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$. Simple algebra shows that the line through $\mathbf{p}_1$ and $\mathbf{p}_2$ is given by $x_1 + x_2 = 1$, and S is on the side of this line where

$$x_1 + x_2 \geq 1 \quad \text{or, equivalently,} \quad -x_1 - x_2 \leq -1.$$

Similarly, the line through $\mathbf{p}_2$ and $\mathbf{p}_3$ is $x_1 - x_2 = 1$, and S is on the side where

$$x_1 - x_2 \leq 1$$

Also, the line through $\mathbf{p}_3$ and $\mathbf{p}_1$ is $-x_1 + 3x_2 = 3$, and S is on the side where

$$-x_1 + 3x_2 \leq 3.$$

See Figure 5. It follows that S can be described as the solution set of the system of linear inequalities

$$\begin{aligned} -x_1 - x_2 &\leq -1 \\ x_1 - x_2 &\leq 1 \\ -x_1 + 3x_2 &\leq 3 \end{aligned}$$

This system may be written as $A\mathbf{x} \leq \mathbf{b}$, where

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \\ -1 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

Note that an inequality between two vectors, such as $A\mathbf{x}$ and $\mathbf{b}$, applies to each of the corresponding coordinates in those vectors. ■

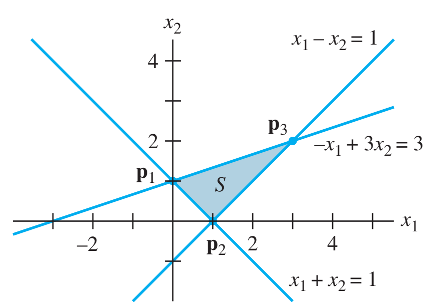


FIGURE 5

In Chapter 9, it will be necessary to replace an implicit description of a polytope by a minimal representation of the polytope, listing all the extreme points of the polytope. In simple cases, a graphical solution is feasible. The following example shows how to handle the situation when several points of interest are too close to identify easily on a graph.

EXAMPLE 5 Let P be the set of points in $\mathbb{R}^2$ that satisfy $A\mathbf{x} \leq \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 18 \\ 8 \\ 21 \end{bmatrix}$$

and $\mathbf{x} \geq \mathbf{0}$. Find the minimal representation of P.

SOLUTION The condition $\mathbf{x} \geq \mathbf{0}$ places P in the first quadrant of $\mathbb{R}^2$, a typical condition in linear programming problems. The three inequalities in $A\mathbf{x} \leq \mathbf{b}$ involve three boundary lines:

$$(1)\ \ x_1 + 3x_2 = 18 \quad (2)\ \ x_1 + x_2 = 8 \quad (3)\ \ 3x_1 + 2x_2 = 21$$

All three lines have negative slopes, so a general idea of the shape of P is easy to visualize. Even a rough sketch of the graphs of these lines will reveal that $(0, 0)$, $(7, 0)$, and $(0, 6)$ are vertices of the polytope P.

What about the intersections of the lines (1), (2), and (3)? Sometimes it is clear from the graph which intersections to include. But if not, then the following algebraic procedure will work well:

> When an intersection point is found that corresponds to two inequalities, test it in the other inequalities to see whether the point is in the polytope.

The intersection of (1) and (2) is $\mathbf{p}_{12} = (3, 5)$. Both coordinates are nonnegative, so $\mathbf{p}_{12}$ satisfies all inequalities except possibly the third inequality. Test this:

$$3(3) + 2(5) = 19 < 21$$

This intersection point satisfies the inequality for (3), so $\mathbf{p}_{12}$ is in the polytope.

The intersection of (2) and (3) is $\mathbf{p}_{23} = (5, 3)$. This satisfies all inequalities except possibly the inequality for (1). Test this:

$$1(5) + 3(3) = 14 < 18$$

This shows that $\mathbf{p}_{23}$ is in the polytope.

Finally, the intersection of (1) and (3) is $\mathbf{p}_{13} = \left(\frac{27}{7}, \frac{33}{7}\right)$. Test this in the inequality for (2):

$$1\left(\tfrac{27}{7}\right) + 1\left(\tfrac{33}{7}\right) = \tfrac{60}{7} \approx 8.6 > 8$$

Thus $\mathbf{p}_{13}$ does **not** satisfy the second inequality, which shows that $\mathbf{p}_{13}$ is **not** in P. In conclusion, the minimal representation of the polytope P is

$$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \end{bmatrix} \right\}.$$

■

The remainder of this section discusses the construction of two basic polytopes in $\mathbb{R}^3$ (and higher dimensions). The first appears in linear programming problems, the subject of Chapter 9. Both polytopes provide opportunities to visualize $\mathbb{R}^4$ in a remarkable way.

Simplex

A **simplex** is the convex hull of an affinely independent finite set of vectors. To construct a k-dimensional simplex (or k-simplex), proceed as follows:

$$\begin{aligned}
&\text{0-simplex } S^0: && \text{a single point } \{\mathbf{v}_1\} \\
&\text{1-simplex } S^1: && \text{conv}(S^0 \cup \{\mathbf{v}_2\}), \text{ with } \mathbf{v}_2 \text{ not in aff } S^0 \\
&\text{2-simplex } S^2: && \text{conv}(S^1 \cup \{\mathbf{v}_3\}), \text{ with } \mathbf{v}_3 \text{ not in aff } S^1 \\
&\qquad\vdots && \\
&k\text{-simplex } S^k: && \text{conv}(S^{k-1} \cup \{\mathbf{v}_{k+1}\}), \text{ with } \mathbf{v}_{k+1} \text{ not in aff } S^{k-1}
\end{aligned}$$

The simplex S^1 is a line segment. The triangle S^2 comes from choosing a point $\mathbf{v}_3$ that is not in the line containing S^1 and then forming the convex hull with S^1.

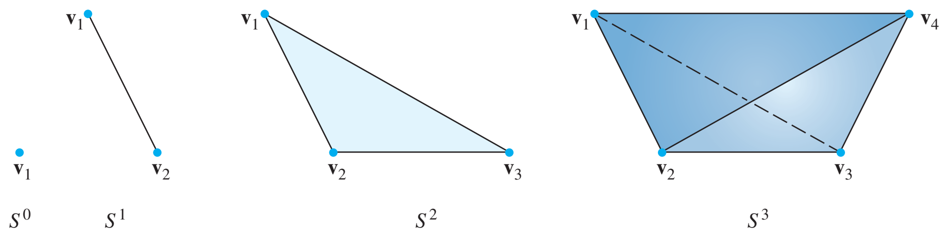


FIGURE 6

(See Figure 6.) The tetrahedron S^3 is produced by choosing a point $\mathbf{v}_4$ not in the plane of S^2 and then forming the convex hull with S^2.

Before continuing, consider some of the patterns that are appearing. The triangle S^2 has three edges. Each of these edges is a line segment like S^1. Where do these three line segments come from? One of them is S^1. One of them comes by joining the endpoint $\mathbf{v}_2$ to the new point $\mathbf{v}_3$. The third comes from joining the other endpoint $\mathbf{v}_1$ to $\mathbf{v}_3$. You might say that each endpoint in S^1 is stretched out into a line segment in S^2.

The tetrahedron S^3 in Figure 6 has four triangular faces. One of these is the original triangle S^2, and the other three come from stretching the edges of S^2 out to the new point $\mathbf{v}_4$. Notice too that the vertices of S^2 get stretched out into edges in S^3. The other edges in S^3 come from the edges in S^2. This suggests how to "visualize" the four-dimensional S^4.

The construction of S^4, called a pentatope, involves forming the convex hull of S^3 with a point $\mathbf{v}_5$ not in the 3-space of S^3. A complete picture is impossible, of course, but Figure 7 is suggestive: S^4 has five vertices, and any four of the vertices determine a facet in the shape of a tetrahedron. For example, the figure emphasizes the facet with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_4$, and $\mathbf{v}_5$ and the facet with vertices $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, and $\mathbf{v}_5$. There are five

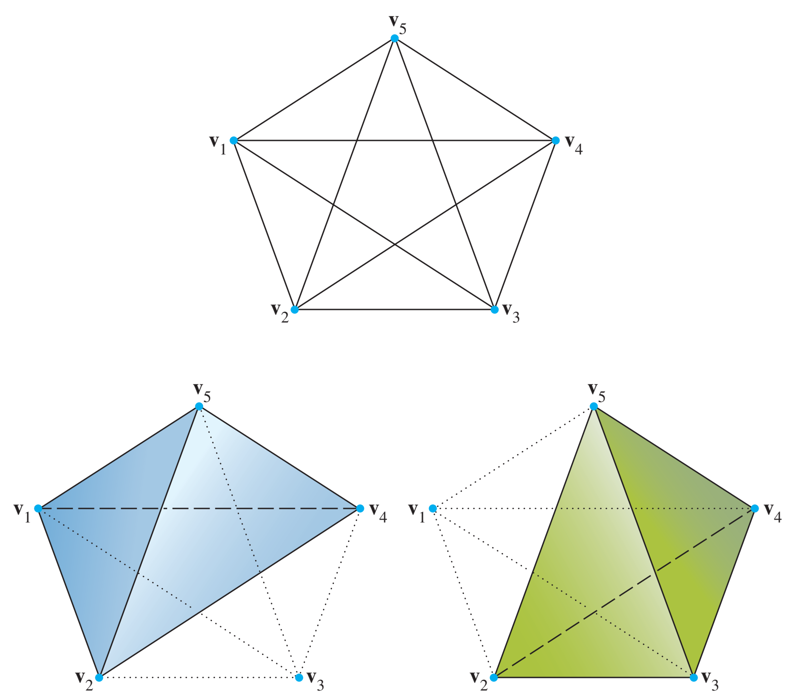


FIGURE 7 The 4-dimensional simplex S^4 projected onto $\mathbb{R}^2$, with two tetrahedral facets emphasized.

such facets. Figure 7 identifies all ten edges of S^4, and these can be used to visualize the ten triangular faces.

Figure 8 shows another representation of the 4-dimensional simplex S^4. This time the fifth vertex appears "inside" the tetrahedron S^3. The highlighted tetrahedral facets also appear to be "inside" S^3.

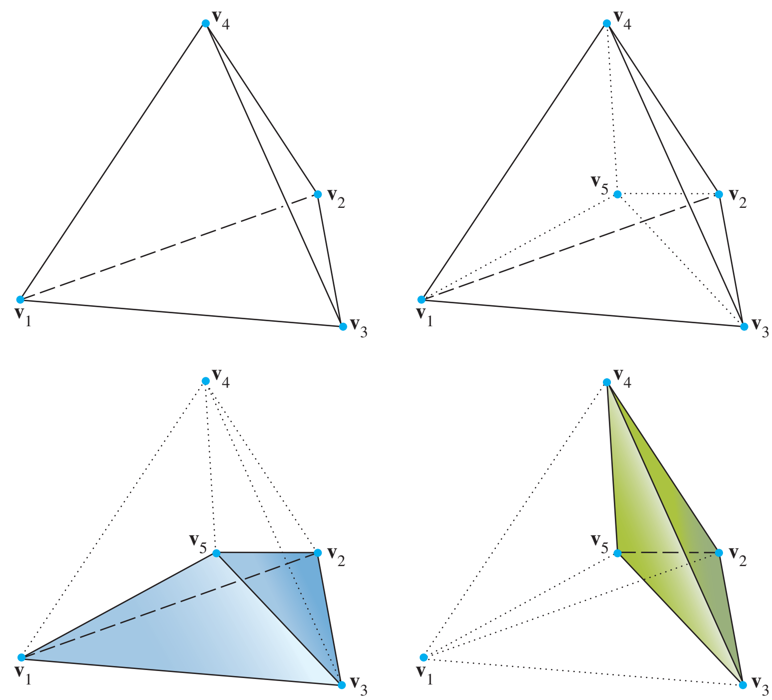


FIGURE 8 The fifth vertex of S^4 is "inside" S^3.

Hypercube

Let $I_i = \overline{\mathbf{0e}_i}$ be the line segment from the origin $\mathbf{0}$ to the standard basis vector $\mathbf{e}_i$ in $\mathbb{R}^n$. Then for k such that $1 \leq k \leq n$, the vector sum[2]

$$C^k = I_1 + I_2 + \cdots + I_k$$

is called a k-dimensional **hypercube**.

To visualize the construction of C^k, start with the simple cases. The hypercube C^1 is the line segment I_1. If C^1 is translated by $\mathbf{e}_2$, the convex hull of its initial and final positions describes a square C^2. (See Figure 9.) Translating C^2 by $\mathbf{e}_3$ creates the cube C^3. A similar translation of C^3 by the vector $\mathbf{e}_4$ yields the 4-dimensional hypercube C^4.

Again, this is hard to visualize, but Figure 10 shows a 2-dimensional projection of C^4. Each of the edges of C^3 is stretched into a square face of C^4. And each of the square faces of C^3 is stretched into a cubic face of C^4. Figure 11 shows three facets of C^4. Part (a) highlights the cube that comes from the left square face of C^3. Part (b) shows the cube that comes from the front square face of C^3. And part (c) emphasizes the cube that comes from the top square face of C^3.

[2] The vector sum of two sets A and B is defined by $A + B = \{\mathbf{c} : \mathbf{c} = \mathbf{a} + \mathbf{b} \text{ for some } \mathbf{a} \in A \text{ and } \mathbf{b} \in B\}$.

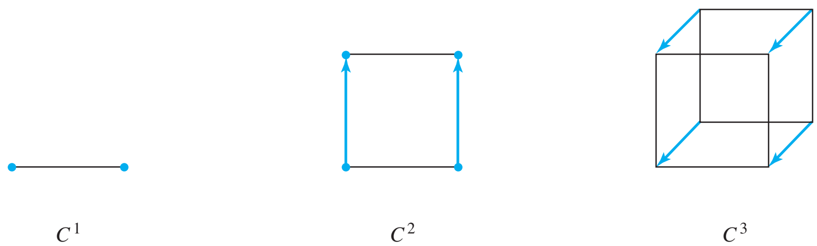


FIGURE 9 Constructing the cube C^3.

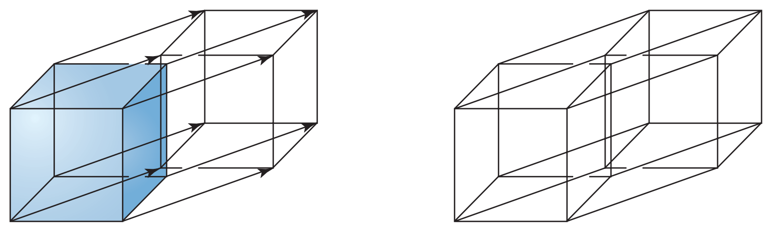


FIGURE 10 C^4 projected onto $\mathbb{R}^2$.

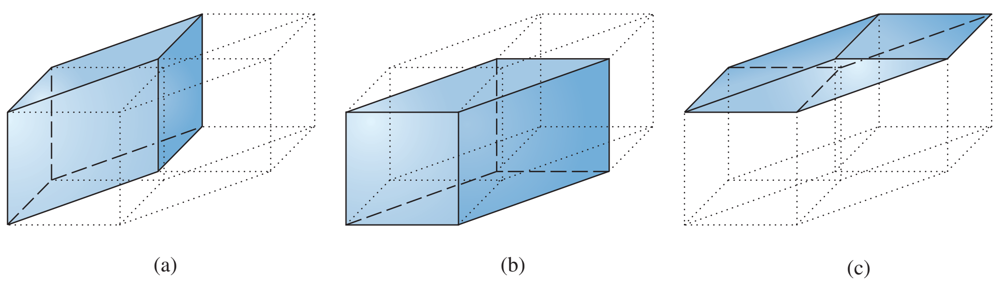


FIGURE 11 Three of the cubic facets of C^4.

Figure 12 shows another representation of C^4 in which the translated cube is placed "inside" C^3. This makes it easier to visualize the cubic facets of C^4, since there is less distortion.

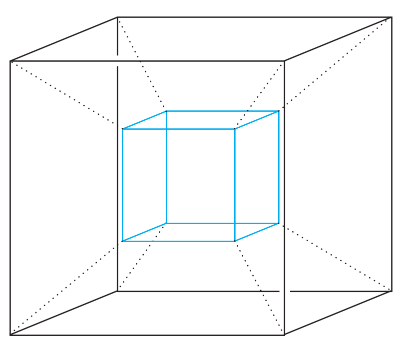


FIGURE 12 The translated image of C^3 is placed "inside" C^3 to obtain C^4.

Altogether, the 4-dimensional cube C^4 has eight cubic faces. Two come from the original and translated images of C^3, and six come from the square faces of C^3 that are stretched into cubes. The square 2-dimensional faces of C^4 come from the square faces of C^3 and its translate, and the edges of C^3 that are stretched into squares. Thus there are $2 \times 6 + 12 = 24$ square faces. To count the edges, take 2 times the number of edges in C^3 and add the number of vertices in C^3. This makes $2 \times 12 + 8 = 32$ edges in C^4. The vertices in C^4 all come from C^3 and its translate, so there are $2 \times 8 = 16$ vertices.

One of the truly remarkable results in the study of polytopes is the following formula, first proved by Leonard Euler (1707–1783). It establishes a simple relationship between the number of faces of different dimensions in a polytope. To simplify the statement of the formula, let $f_k(P)$ denote the number of k-dimensional faces of an n-dimensional polytope P.[3]

$$\text{Euler's formula:} \qquad \sum_{k=0}^{n-1} (-1)^k f_k(P) = 1 + (-1)^{n-1}$$

In particular, when $n = 3$, $v - e + f = 2$, where v, e, and f denote the number of vertices, edges, and facets (respectively) of P.

PRACTICE PROBLEM

Find the minimal representation of the polytope P defined by the inequalities $A\mathbf{x} \le \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$, when $A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 12 \\ 9 \\ 12 \end{bmatrix}$.

8.5 EXERCISES

1. Given points $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $\mathbf{p}_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^2$, let $S = \text{conv}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$. For each linear functional f, find the maximum value m of f on the set S, and find all points $\mathbf{x}$ in S at which $f(\mathbf{x}) = m$.

a. $f(x_1, x_2) = x_1 - x_2$
b. $f(x_1, x_2) = x_1 + x_2$
c. $f(x_1, x_2) = -3x_1 + x_2$

2. Given points $\mathbf{p}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^2$, let $S = \text{conv}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$. For each linear functional f, find the maximum value m of f on the set S, and find all points $\mathbf{x}$ in S at which $f(\mathbf{x}) = m$.

a. $f(x_1, x_2) = x_1 + x_2$
b. $f(x_1, x_2) = x_1 - x_2$
c. $f(x_1, x_2) = -2x_1 + x_2$

3. Repeat Exercise 1 where m is the *minimum* value of f on S instead of the maximum value.

4. Repeat Exercise 2 where m is the *minimum* value of f on S instead of the maximum value.

In Exercises 5–8, find the minimal representation of the polytope defined by the inequalities $A\mathbf{x} \le \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$.

5. $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 18 \\ 16 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 4 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 18 \\ 10 \\ 28 \end{bmatrix}$

8. $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 8 \\ 6 \\ 7 \end{bmatrix}$

9. Let $S = \{(x, y) : x^2 + (y-1)^2 \le 1\} \cup \{(3, 0)\}$. Is the origin an extreme point of conv S? Is the origin a vertex of conv S?

10. Find an example of a closed convex set S in $\mathbb{R}^2$ such that its profile P is nonempty but conv $P \ne S$.

11. Find an example of a bounded convex set S in $\mathbb{R}^2$ such that its profile P is nonempty but conv $P \ne S$.

12. a. Determine the number of k-faces of the 5-dimensional simplex S^5 for $k = 0, 1, \dots, 4$. Verify that your answer satisfies Euler's formula.

b. Make a chart of the values of $f_k(S^n)$ for $n = 1, \dots, 5$ and $k = 0, 1, \dots, 4$. Can you see a pattern? Guess a general formula for $f_k(S^n)$.

[3] A proof when $n = 3$ is presented in Steven R. Lay, *Convex Sets and Their Applications* (New York: John Wiley & Sons, 1982; Mineola, NY: Dover Publications, 2007), p. 131.

13. a. Determine the number of k-faces of the 5-dimensional hypercube C^5 for $k = 0, 1, \ldots, 4$. Verify that your answer satisfies Euler's formula.

b. Make a chart of the values of $f_k(C^n)$ for $n = 1, \ldots, 5$ and $k = 0, 1, \ldots, 4$. Can you see a pattern? Guess a general formula for $f_k(C^n)$.

14. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent vectors in $\mathbb{R}^n$ $(1 \le k \le n)$. Then the set $X^k = \text{conv}\{\pm\mathbf{v}_1, \ldots, \pm\mathbf{v}_k\}$ is called a ***k*-crosspolytope**.

a. Sketch X^1 and X^2.

b. Determine the number of k-faces of the 3-dimensional crosspolytope X^3 for $k = 0, 1, 2$. What is another name for X^3?

c. Determine the number of k-faces of the 4-dimensional crosspolytope X^4 for $k = 0, 1, 2, 3$. Verify that your answer satisfies Euler's formula.

d. Find a formula for $f_k(X^n)$, the number of k-faces of X^n, for $0 \le k \le n - 1$.

15. A ***k*-pyramid** P^k is the convex hull of a $(k-1)$-polytope Q and a point $\mathbf{x} \notin \text{aff}\, Q$. Find a formula for each of the following in terms of $f_j(Q)$, $j = 0, \ldots, n-1$.

a. The number of vertices of P^n: $f_0(P^n)$.

b. The number of k-faces of P^n: $f_k(P^n)$, for $1 \le k \le n-2$.

c. The number of $(n-1)$-dimensional facets of P^n: $f_{n-1}(P^n)$.

In Exercises 16 and 17, mark each statement True or False. Justify each answer.

16. a. A polytope is the convex hull of a finite set of points.

b. Let $\mathbf{p}$ be an extreme point of a convex set S. If $\mathbf{u}, \mathbf{v} \in S$, $\mathbf{p} \in \overline{\mathbf{uv}}$, and $\mathbf{p} \ne \mathbf{u}$, then $\mathbf{p} = \mathbf{v}$.

c. If S is a nonempty convex subset of $\mathbb{R}^n$, then S is the convex hull of its profile.

d. The 4-dimensional simplex S^4 has exactly five facets, each of which is a 3-dimensional tetrahedron.

17. a. A cube in $\mathbb{R}^3$ has exactly five facets.

b. A point $\mathbf{p}$ is an extreme point of a polytope P if and only if $\mathbf{p}$ is a vertex of P.

c. If S is a nonempty compact convex set and a linear functional attains its maximum at a point $\mathbf{p}$, then $\mathbf{p}$ is an extreme point of S.

d. A 2-dimensional polytope always has the same number of vertices and edges.

18. Let $\mathbf{v}$ be an element of the convex set S. Prove that $\mathbf{v}$ is an extreme point of S if and only if the set $\{\mathbf{x} \in S : \mathbf{x} \ne \mathbf{v}\}$ is convex.

19. If $c \in \mathbb{R}$ and S is a set, define $cS = \{c\mathbf{x} : \mathbf{x} \in S\}$. Let S be a convex set and suppose $c > 0$ and $d > 0$. Prove that $cS + dS = (c+d)S$.

20. Find an example to show that the convexity of S is necessary in Exercise 19.

21. If A and B are convex sets, prove that $A + B$ is convex.

22. A polyhedron (3-polytope) is called **regular** if all its facets are congruent regular polygons and all the angles at the vertices are equal. Supply the details in the following proof that there are only five regular polyhedra.

a. Suppose that a regular polyhedron has r facets, each of which is a k-sided regular polygon, and that s edges meet at each vertex. Letting v and e denote the numbers of vertices and edges in the polyhedron, explain why $kr = 2e$ and $sv = 2e$.

b. Use Euler's formula to show that $\dfrac{1}{s} + \dfrac{1}{k} = \dfrac{1}{2} + \dfrac{1}{e}$.

c. Find all the integral solutions of the equation in part (b) that satisfy the geometric constraints of the problem. (How small can k and s be?)

For your information, the five regular polyhedra are the tetrahedron (4, 6, 4), the cube (8, 12, 6), the octahedron (6, 12, 8), the dodecahedron (20, 30, 12), and the icosahedron (12, 30, 20). (The numbers in parentheses indicate the numbers of vertices, edges, and faces, respectively.)

SOLUTION TO PRACTICE PROBLEM

The matrix inequality $A\mathbf{x} \le \mathbf{b}$ yields the following system of inequalities:

$$\begin{aligned} &(\text{a})\ \ x_1 + 3x_2 \le 12 \\ &(\text{b})\ \ x_1 + 2x_2 \le 9 \\ &(\text{c})\ \ 2x_1 + x_2 \le 12 \end{aligned}$$

The condition $\mathbf{x} \ge \mathbf{0}$, places the polytope in the first quadrant of the plane. One vertex is $(0, 0)$. The x_1-intercepts of the three lines (when $x_2 = 0$) are 12, 9, and 6, so $(6, 0)$ is a vertex. The x_2-intercepts of the three lines (when $x_1 = 0$) are 4, 4.5, and 12, so $(0, 4)$ is a vertex.

How do the three boundary lines intersect for positive values of x_1 and x_2? The intersection of (a) and (b) is at $\mathbf{p}_{ab} = (3, 3)$. Testing $\mathbf{p}_{ab}$ in (c) gives $2(3) + 1(3) = 9 < 12$, so $\mathbf{p}_{ab}$ is in P. The intersection of (b) and (c) is at $\mathbf{p}_{bc} = (5, 2)$. Testing $\mathbf{p}_{bc}$ in (a) gives $1(5) + 3(2) = 11 < 12$, so $\mathbf{p}_{bc}$ is in P. The intersection of (a) and (c) is at $\mathbf{p}_{ac} = (4.8, 2.4)$. Testing $\mathbf{p}_{ac}$ in (b) gives $1(4.8) + 2(2.4) = 9.6 > 9$. So $\mathbf{p}_{ac}$ is not in P.

Finally, the five vertices (extreme points) of the polytope are $(0, 0)$, $(6, 0)$, $(5, 2)$ $(3, 3)$, and $(0, 4)$. These points form the minimal representation of P. This is displayed graphically in Figure 13.

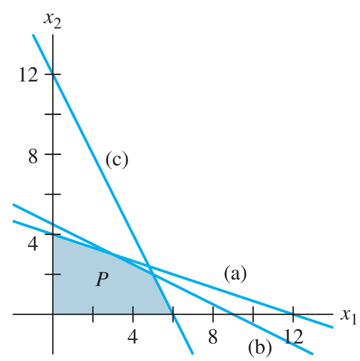


FIGURE 13

8.6 CURVES AND SURFACES

For thousands of years, builders used long thin strips of wood to create the hull of a boat. In more recent times, designers used long, flexible metal strips to lay out the surfaces of cars and airplanes. Weights and pegs shaped the strips into smooth curves called *natural cubic splines*. The curve between two successive control points (pegs or weights) has a parametric representation using cubic polynomials. Unfortunately, such curves have the property that moving one control point affects the shape of the entire curve, because of physical forces that the pegs and weights exert on the strip. Design engineers had long wanted local control of the curve—in which movement of one control point would affect only a small portion of the curve. In 1962, a French automotive engineer, Pierre Bézier, solved this problem by adding extra control points and using a class of curves now called by his name.

Bézier Curves

The curves described below play an important role in computer graphics as well as engineering. For example, they are used in Adobe Illustrator and Macromedia Freehand, and in application programming languages such as OpenGL. These curves permit a program to store exact information about curved segments and surfaces in a relatively small number of control points. All graphics commands for the segments and surfaces have only to be computed for the control points. The special structure of these curves also speeds up other calculations in the "graphics pipeline" that creates the final display on the viewing screen.

Exercises in Section 8.3 introduced quadratic Bézier curves and showed one method for constructing Bézier curves of higher degree. The discussion here focuses on quadratic and cubic Bézier curves, which are determined by three or four control points, denoted by $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. These points can be in $\mathbb{R}^2$ or $\mathbb{R}^3$, or they can be represented by homogeneous forms in $\mathbb{R}^3$ or $\mathbb{R}^4$. The standard parametric descriptions of these curves, for $0 \leq t \leq 1$, are

$$\mathbf{w}(t) = (1-t)^2\mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2\mathbf{p}_2 \tag{1}$$
$$\mathbf{x}(t) = (1-t)^3\mathbf{p}_0 + 3t(1-t)^2\mathbf{p}_1 + 3t^2(1-t)\mathbf{p}_2 + t^3\mathbf{p}_3 \tag{2}$$

Figure 1 shows two typical curves. Usually, the curves pass through only the initial and terminal control points, but a Bézier curve is always in the convex hull of its control points. (See Exercises 21–24 in Section 8.3.)

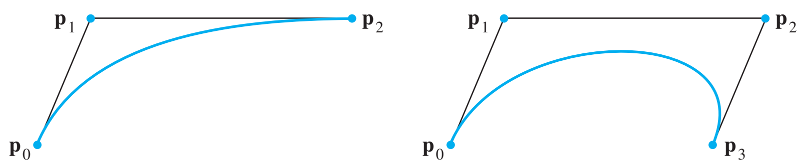


FIGURE 1 Quadratic and cubic Bézier curves.

Bézier curves are useful in computer graphics because their essential properties are preserved under the action of linear transformations and translations. For instance, if A is a matrix of appropriate size, then from the linearity of matrix multiplication, for $0 \leq t \leq 1$,

$$\begin{aligned} A\mathbf{x}(t) &= A[(1-t)^3\mathbf{p}_0 + 3t(1-t)^2\mathbf{p}_1 + 3t^2(1-t)\mathbf{p}_2 + t^3\mathbf{p}_3] \\ &= (1-t)^3A\mathbf{p}_0 + 3t(1-t)^2A\mathbf{p}_1 + 3t^2(1-t)A\mathbf{p}_2 + t^3A\mathbf{p}_3 \end{aligned}$$

The new control points are $A\mathbf{p}_0, \ldots, A\mathbf{p}_3$. Translations of Bézier curves are considered in Exercise 1.

The curves in Figure 1 suggest that the control points determine the tangent lines to the curves at the initial and terminal control points. Recall from calculus that for any parametric curve, say $\mathbf{y}(t)$, the direction of the tangent line to the curve at a point $\mathbf{y}(t)$ is given by the derivative $\mathbf{y}'(t)$, called the **tangent vector** of the curve. (This derivative is computed entry by entry.)

EXAMPLE 1 Determine how the tangent vector of the quadratic Bézier curve $\mathbf{w}(t)$ is related to the control points of the curve, at $t = 0$ and $t = 1$.

SOLUTION Write the weights in equation (1) as simple polynomials

$$\mathbf{w}(t) = (1 - 2t + t^2)\mathbf{p}_0 + (2t - 2t^2)\mathbf{p}_1 + t^2\mathbf{p}_2$$

Then, because differentiation is a linear transformation on functions,

$$\mathbf{w}'(t) = (-2 + 2t)\mathbf{p}_0 + (2 - 4t)\mathbf{p}_1 + 2t\mathbf{p}_2$$

So

$$\begin{aligned} \mathbf{w}'(0) &= -2\mathbf{p}_0 + 2\mathbf{p}_1 = 2(\mathbf{p}_1 - \mathbf{p}_0) \\ \mathbf{w}'(1) &= -2\mathbf{p}_1 + 2\mathbf{p}_2 = 2(\mathbf{p}_2 - \mathbf{p}_1) \end{aligned}$$

The tangent vector at $\mathbf{p}_0$, for instance, points from $\mathbf{p}_0$ to $\mathbf{p}_1$, but it is twice as long as the segment from $\mathbf{p}_0$ to $\mathbf{p}_1$. Notice that $\mathbf{w}'(0) = \mathbf{0}$ when $\mathbf{p}_1 = \mathbf{p}_0$. In this case, $\mathbf{w}(t) = (1-t^2)\mathbf{p}_1 + t^2\mathbf{p}_2$, and the graph of $\mathbf{w}(t)$ is the line segment from $\mathbf{p}_1$ to $\mathbf{p}_2$. ■

Connecting Two Bézier Curves

Two basic Bézier curves can be joined end to end, with the terminal point of the first curve $\mathbf{x}(t)$ being the initial point $\mathbf{p}_2$ of the second curve $\mathbf{y}(t)$. The combined curve is said to have G^0 *geometric continuity* (at $\mathbf{p}_2$) because the two segments join at $\mathbf{p}_2$. If the tangent line to curve 1 at $\mathbf{p}_2$ has a different direction than the tangent line to curve 2, then a "corner," or abrupt change of direction, may be apparent at $\mathbf{p}_2$. See Figure 2.

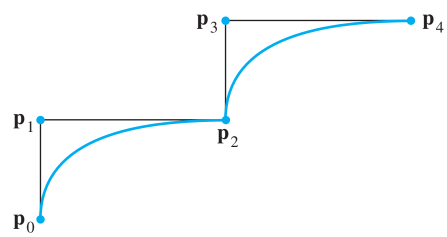


FIGURE 2 G^0 continuity at $\mathbf{p}_2$.

To avoid a sharp bend, it usually suffices to adjust the curves to have what is called G^1 *geometric continuity*, where both tangent vectors at $\mathbf{p}_2$ point in the same direction. That is, the derivatives $\mathbf{x}'(1)$ and $\mathbf{y}'(0)$ point in the same direction, even though their magnitudes may be different. When the tangent vectors are actually equal at $\mathbf{p}_2$, the tangent vector is continuous at $\mathbf{p}_2$, and the combined curve is said to have C^1 continuity, or C^1 *parametric continuity*. Figure 3 shows G^1 continuity in (a) and C^1 continuity in (b).

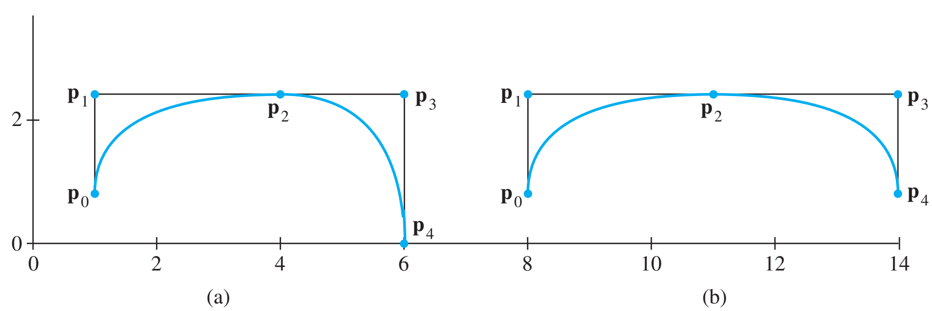


FIGURE 3 (a) G^1 continuity and (b) C^1 continuity.

EXAMPLE 2 Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ determine two quadratic Bézier curves, with control points $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$ and $\{\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$, respectively. The curves are joined at $\mathbf{p}_2 = \mathbf{x}(1) = \mathbf{y}(0)$.

a. Suppose the combined curve has G^1 continuity (at $\mathbf{p}_2$). What algebraic restriction does this condition impose on the control points? Express this restriction in geometric language.

b. Repeat part (a) for C^1 continuity.

SOLUTION

a. From Example 1, $\mathbf{x}'(1) = 2(\mathbf{p}_2 - \mathbf{p}_1)$. Also, using the control points for $\mathbf{y}(t)$ in place of $\mathbf{w}(t)$, Example 1 shows that $\mathbf{y}'(0) = 2(\mathbf{p}_3 - \mathbf{p}_2)$. G^1 continuity means that $\mathbf{y}'(0) = k\mathbf{x}'(1)$ for some positive constant k. Equivalently,

$$\mathbf{p}_3 - \mathbf{p}_2 = k(\mathbf{p}_2 - \mathbf{p}_1), \quad \text{with } k > 0 \tag{3}$$

Geometrically, (3) implies that $\mathbf{p}_2$ lies on the line segment from $\mathbf{p}_1$ to $\mathbf{p}_3$. To prove this, let $t = (k+1)^{-1}$, and note that $0 < t < 1$. Solve for k to obtain $k = (1-t)/t$. When this expression is used for k in (3), a rearrangement shows that $\mathbf{p}_2 = (1-t)\mathbf{p}_1 + t\mathbf{p}_3$, which verifies the assertion about $\mathbf{p}_2$.

b. C^1 continuity means that $\mathbf{y}'(0) = \mathbf{x}'(1)$. Thus $2(\mathbf{p}_3 - \mathbf{p}_2) = 2(\mathbf{p}_2 - \mathbf{p}_1)$, so $\mathbf{p}_3 - \mathbf{p}_2 = \mathbf{p}_2 - \mathbf{p}_1$, and $\mathbf{p}_2 = (\mathbf{p}_1 + \mathbf{p}_3)/2$. Geometrically, $\mathbf{p}_2$ is the midpoint of the line segment from $\mathbf{p}_1$ to $\mathbf{p}_3$. See Figure 3. ■

Figure 4 shows C^1 continuity for two cubic Bézier curves. Notice how the point joining the two segments lies in the middle of the line segment between the adjacent control points.

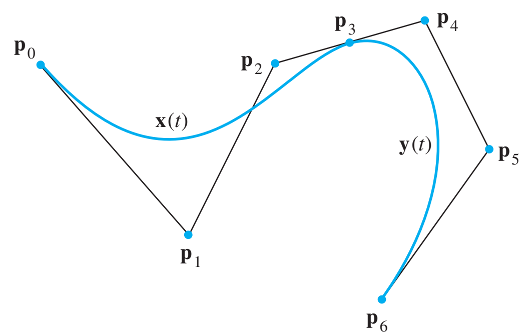


FIGURE 4 Two cubic Bézier curves.

Two curves have C^2 (parametric) continuity when they have C^1 continuity and the *second* derivatives $\mathbf{x}''(1)$ and $\mathbf{y}''(0)$ are equal. This is possible for cubic Bézier curves, but it severely limits the positions of the control points. Another class of cubic curves, called *B-splines*, always have C^2 continuity because each pair of curves share three control points rather than one. Graphics figures using B-splines have more control points and consequently require more computations. Some exercises for this section examine these curves.

Surprisingly, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ join at $\mathbf{p}_3$, the apparent smoothness of the curve at $\mathbf{p}_3$ is usually the same for both G^1 continuity and C^1 continuity. This is because the magnitude of $\mathbf{x}'(t)$ is not related to the physical shape of the curve. The magnitude reflects only the mathematical parameterization of the curve. For instance, if a new vector function $\mathbf{z}(t)$ equals $\mathbf{x}(2t)$, then the point $\mathbf{z}(t)$ traverses the curve from $\mathbf{p}_0$ to $\mathbf{p}_3$ twice as fast as the original version, because $2t$ reaches 1 when t is .5. But, by the chain rule of calculus, $\mathbf{z}'(t) = 2 \cdot \mathbf{x}'(2t)$, so the tangent vector to $\mathbf{z}(t)$ at $\mathbf{p}_3$ is twice the tangent vector to $\mathbf{x}(t)$ at $\mathbf{p}_3$.

In practice, many simple Bézier curves are joined to create graphics objects. Typesetting programs provide one important application, because many letters in a type font involve curved segments. Each letter in a PostScript® font, for example, is stored as a set of control points, along with information on how to construct the "outline" of the letter using line segments and Bézier curves. Enlarging such a letter basically requires multiplying the coordinates of each control point by one constant scale factor. Once the outline of the letter has been computed, the appropriate solid parts of the letter are filled in. Figure 5 illustrates this for a character in a PostScript font. Note the control points.

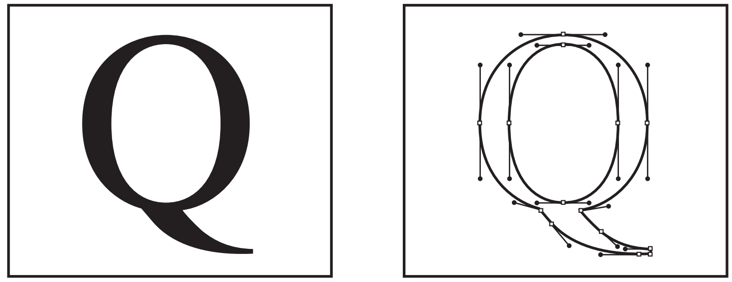


FIGURE 5 A PostScript character.

Matrix Equations for Bézier Curves

Since a Bézier curve is a linear combination of control points using polynomials as weights, the formula for $\mathbf{x}(t)$ may be written as

$$\begin{aligned}\mathbf{x}(t) &= \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} \begin{bmatrix} 1-3t+3t^2-t^3 \\ 3t-6t^2+3t^3 \\ 3t^2-3t^3 \\ t^3 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}\end{aligned}$$

The matrix whose columns are the four control points is called a **geometry matrix**, G. The 4×4 matrix of polynomial coefficients is the **Bézier basis matrix**, M_B. If $\mathbf{u}(t)$ is the column vector of powers of t, then the Bézier curve is given by

$$\mathbf{x}(t) = GM_B\mathbf{u}(t) \tag{4}$$

Other parametric cubic curves in computer graphics are written in this form, too. For instance, if the entries in the matrix M_B are changed appropriately, the resulting curves are B-splines. They are “smoother” than Bézier curves, but they do not pass through any of the control points. A **Hermite** cubic curve arises when the matrix M_B is replaced by a Hermite basis matrix. In this case, the columns of the geometry matrix consist of the starting and ending points of the curves and the tangent vectors to the curves at those points.[1]

The Bézier curve in equation (4) can also be “factored” in another way, to be used in the discussion of Bézier surfaces. For convenience later, the parameter t is replaced

[1] The term *basis matrix* comes from the rows of the matrix that list the coefficients of the *blending* polynomials used to define the curve. For a cubic Bézier curve, the four polynomials are $(1-t)^3$, $3t(1-t)^2$, $3t^2(1-t)$, and t^3. They form a basis for the space $\mathbb{P}_3$ of polynomials of degree 3 or less. Each entry in the vector $\mathbf{x}(t)$ is a linear combination of these polynomials. The weights come from the rows of the geometry matrix G in (4).

by a parameter s:

$$\mathbf{x}(s) = \mathbf{u}(s)^T M_B^T \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} 1 & s & s^2 & s^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$= \begin{bmatrix} (1-s)^3 & 3s(1-s)^2 & 3s^2(1-s) & s^3 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} \tag{5}$$

This formula is not quite the same as the transpose of the product on the right of (4), because $\mathbf{x}(s)$ and the control points appear in (5) without transpose symbols. The matrix of control points in (5) is called a **geometry vector**. This should be viewed as a 4×1 block (partitioned) matrix whose entries are column vectors. The matrix to the left of the geometry vector, in the second part of (5), can be viewed as a block matrix, too, with a scalar in each block. The partitioned matrix multiplication makes sense, because each (vector) entry in the geometry vector can be left-multiplied by a scalar as well as by a matrix. Thus, the column vector $\mathbf{x}(s)$ is represented by (5).

Bézier Surfaces

A 3D bicubic surface patch can be constructed from a set of four Bézier curves. Consider the four geometry matrices

$$\begin{bmatrix} \mathbf{p}_{11} & \mathbf{p}_{12} & \mathbf{p}_{13} & \mathbf{p}_{14} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{p}_{21} & \mathbf{p}_{22} & \mathbf{p}_{23} & \mathbf{p}_{24} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{p}_{33} & \mathbf{p}_{34} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{p}_{41} & \mathbf{p}_{42} & \mathbf{p}_{43} & \mathbf{p}_{44} \end{bmatrix}$$

and recall from equation (4) that a Bézier curve is produced when any one of these matrices is multiplied on the right by the following vector of weights:

$$M_B\mathbf{u}(t) = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}$$

Let G be the block (partitioned) 4×4 matrix whose entries are the control points $\mathbf{p}_{ij}$ displayed above. Then the following product is a block 4×1 matrix, and each entry is a Bézier curve:

$$GM_B\mathbf{u}(t) = \begin{bmatrix} \mathbf{p}_{11} & \mathbf{p}_{12} & \mathbf{p}_{13} & \mathbf{p}_{14} \\ \mathbf{p}_{21} & \mathbf{p}_{22} & \mathbf{p}_{23} & \mathbf{p}_{24} \\ \mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{p}_{33} & \mathbf{p}_{34} \\ \mathbf{p}_{41} & \mathbf{p}_{42} & \mathbf{p}_{43} & \mathbf{p}_{44} \end{bmatrix} \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}$$

In fact,

$$GM_B\mathbf{u}(t) = \begin{bmatrix} (1-t)^3\mathbf{p}_{11} + 3t(1-t)^2\mathbf{p}_{12} + 3t^2(1-t)\mathbf{p}_{13} + t^3\mathbf{p}_{14} \\ (1-t)^3\mathbf{p}_{21} + 3t(1-t)^2\mathbf{p}_{22} + 3t^2(1-t)\mathbf{p}_{23} + t^3\mathbf{p}_{24} \\ (1-t)^3\mathbf{p}_{31} + 3t(1-t)^2\mathbf{p}_{32} + 3t^2(1-t)\mathbf{p}_{33} + t^3\mathbf{p}_{34} \\ (1-t)^3\mathbf{p}_{41} + 3t(1-t)^2\mathbf{p}_{42} + 3t^2(1-t)\mathbf{p}_{43} + t^3\mathbf{p}_{44} \end{bmatrix}$$

Now fix t. Then $GM_B\mathbf{u}(t)$ is a column vector that can be used as a geometry vector in equation (5) for a Bézier curve in another variable s. This observation produces the **Bézier bicubic surface**:

$$\mathbf{x}(s,t) = \mathbf{u}(s)^T M_B^T G M_B \mathbf{u}(t), \quad \text{where } 0 \leq s,t \leq 1 \tag{6}$$

The formula for $\mathbf{x}(s,t)$ is a linear combination of the sixteen control points. If one imagines that these control points are arranged in a fairly uniform rectangular array, as in Figure 6, then the Bézier surface is controlled by a web of eight Bézier curves, four in the "s-direction" and four in the "t-direction." The surface actually passes through the four control points at its "corners." When it is in the middle of a larger surface, the sixteen-point surface shares its twelve boundary control points with its neighbors.

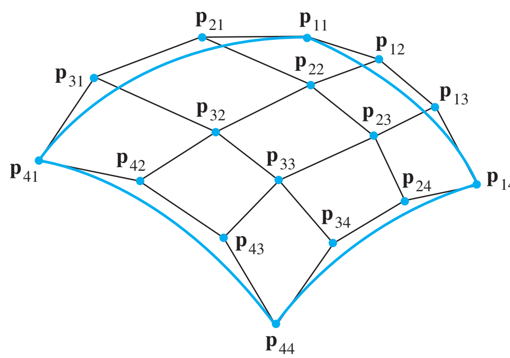


FIGURE 6 Sixteen control points for a Bézier bicubic surface patch.

Approximations to Curves and Surfaces

In CAD programs and in programs used to create realistic computer games, the designer often works at a graphics workstation to compose a "scene" involving various geometric structures. This process requires interaction between the designer and the geometric objects. Each slight repositioning of an object requires new mathematical computations by the graphics program. Bézier curves and surfaces can be useful in this process because they involve fewer control points than objects approximated by many polygons. This dramatically reduces the computation time and speeds up the designer's work.

After the scene composition, however, the final image preparation has different computational demands that are more easily met by objects consisting of flat surfaces and straight edges, such as polyhedra. The designer needs to *render* the scene, by introducing light sources, adding color and texture to surfaces, and simulating reflections from the surfaces.

Computing the direction of a reflected light at a point $\mathbf{p}$ on a surface, for instance, requires knowing the directions of both the incoming light and the *surface normal*—the vector perpendicular to the tangent plane at $\mathbf{p}$. Computing such normal vectors is much easier on a surface composed of, say, tiny flat polygons than on a curved surface whose normal vector changes continuously as $\mathbf{p}$ moves. If $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are adjacent vertices of a flat polygon, then the surface normal is just plus or minus the cross product $(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_2 - \mathbf{p}_3)$. When the polygon is small, only one normal vector is needed for rendering the entire polygon. Also, two widely used shading routines, Gouraud shading and Phong shading, both require a surface to be defined by polygons.

As a result of these needs for flat surfaces, the Bézier curves and surfaces from the scene composition stage now are usually approximated by straight line segments and polyhedral surfaces. The basic idea for approximating a Bézier curve or surface is to divide the curve or surface into smaller pieces, with more and more control points.

Recursive Subdivision of Bézier Curves and Surfaces

Figure 7 shows the four control points $\mathbf{p}_0, \ldots, \mathbf{p}_3$ for a Bézier curve, along with control points for two new curves, each coinciding with half of the original curve. The "left" curve begins at $\mathbf{q}_0 = \mathbf{p}_0$ and ends at $\mathbf{q}_3$, at the midpoint of the original curve. The "right" curve begins at $\mathbf{r}_0 = \mathbf{q}_3$ and ends at $\mathbf{r}_3 = \mathbf{p}_3$.

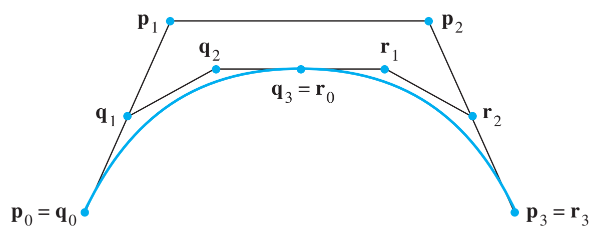


FIGURE 7 Subdivision of a Bézier curve.

Figure 8 shows how the new control points enclose regions that are "thinner" than the region enclosed by the original control points. As the distances between the control points decrease, the control points of each curve segment also move closer to a line segment. This *variation-diminishing property* of Bézier curves depends on the fact that a Bézier curve always lies in the convex hull of the control points.

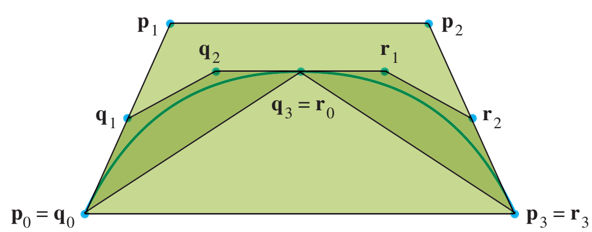


FIGURE 8 Convex hulls of the control points.

The new control points are related to the original control points by simple formulas. Of course, $\mathbf{q}_0 = \mathbf{p}_0$ and $\mathbf{r}_3 = \mathbf{p}_3$. The midpoint of the original curve $\mathbf{x}(t)$ occurs at $\mathbf{x}(.5)$ when $\mathbf{x}(t)$ has the standard parameterization,

$$\mathbf{x}(t) = (1 - 3t + 3t^2 - t^3)\mathbf{p}_0 + (3t - 6t^2 + 3t^3)\mathbf{p}_1 + (3t^2 - 3t^3)\mathbf{p}_2 + t^3\mathbf{p}_3 \tag{7}$$

for $0 \le t \le 1$. Thus, the new control points $\mathbf{q}_3$ and $\mathbf{r}_0$ are given by

$$\mathbf{q}_3 = \mathbf{r}_0 = \mathbf{x}(.5) = \tfrac{1}{8}(\mathbf{p}_0 + 3\mathbf{p}_1 + 3\mathbf{p}_2 + \mathbf{p}_3) \tag{8}$$

The formulas for the remaining "interior" control points are also simple, but the derivation of the formulas requires some work involving the tangent vectors of the curves. By definition, the tangent vector to a parameterized curve $\mathbf{x}(t)$ is the derivative $\mathbf{x}'(t)$. This vector shows the direction of the line tangent to the curve at $\mathbf{x}(t)$. For the Bézier curve in (7),

$$\mathbf{x}'(t) = (-3 + 6t - 3t^2)\mathbf{p}_0 + (3 - 12t + 9t^2)\mathbf{p}_1 + (6t - 9t^2)\mathbf{p}_2 + 3t^2\mathbf{p}_3$$

for $0 \le t \le 1$. In particular,

$$\mathbf{x}'(0) = 3(\mathbf{p}_1 - \mathbf{p}_0) \quad \text{and} \quad \mathbf{x}'(1) = 3(\mathbf{p}_3 - \mathbf{p}_2) \tag{9}$$

Geometrically, $\mathbf{p}_1$ is on the line tangent to the curve at $\mathbf{p}_0$, and $\mathbf{p}_2$ is on the line tangent to the curve at $\mathbf{p}_3$. See Figure 8. Also, from $\mathbf{x}'(t)$, compute

$$\mathbf{x}'(.5) = \tfrac{3}{4}(-\mathbf{p}_0 - \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \tag{10}$$

Let $\mathbf{y}(t)$ be the Bézier curve determined by $\mathbf{q}_0, \ldots, \mathbf{q}_3$, and let $\mathbf{z}(t)$ be the Bézier curve determined by $\mathbf{r}_0, \ldots, \mathbf{r}_3$. Since $\mathbf{y}(t)$ traverses the same path as $\mathbf{x}(t)$ but only gets to $\mathbf{x}(.5)$ as t goes from 0 to 1, $\mathbf{y}(t) = \mathbf{x}(.5t)$ for $0 \le t \le 1$. Similarly, since $\mathbf{z}(t)$ starts at $\mathbf{x}(.5)$ when $t = 0$, $\mathbf{z}(t) = \mathbf{x}(.5 + .5t)$ for $0 \le t \le 1$. By the chain rule for derivatives,

$$\mathbf{y}'(t) = .5\mathbf{x}'(.5t) \quad \text{and} \quad \mathbf{z}'(t) = .5\mathbf{x}'(.5 + .5t) \quad \text{for } 0 \le t \le 1 \tag{11}$$

From (9) with $\mathbf{y}'(0)$ in place of $\mathbf{x}'(0)$, from (11) with $t = 0$, and from (9), the control points for $\mathbf{y}(t)$ satisfy

$$3(\mathbf{q}_1 - \mathbf{q}_0) = \mathbf{y}'(0) = .5\mathbf{x}'(0) = \tfrac{3}{2}(\mathbf{p}_1 - \mathbf{p}_0) \tag{12}$$

From (9) with $\mathbf{y}'(1)$ in place of $\mathbf{x}'(1)$, from (11) with $t = 1$, and from (10),

$$3(\mathbf{q}_3 - \mathbf{q}_2) = \mathbf{y}'(1) = .5\mathbf{x}'(.5) = \tfrac{3}{8}(-\mathbf{p}_0 - \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \tag{13}$$

Equations (8), (9), (10), (12), and (13) can be solved to produce the formulas for $\mathbf{q}_0, \ldots, \mathbf{q}_3$ shown in Exercise 13. Geometrically, the formulas are displayed in Figure 9. The interior control points $\mathbf{q}_1$ and $\mathbf{r}_2$ are the midpoints, respectively, of the segment from $\mathbf{p}_0$ to $\mathbf{p}_1$ and the segment from $\mathbf{p}_2$ to $\mathbf{p}_3$. When the midpoint of the segment from $\mathbf{p}_1$ to $\mathbf{p}_2$ is connected to $\mathbf{q}_1$, the resulting line segment has $\mathbf{q}_2$ in the middle!

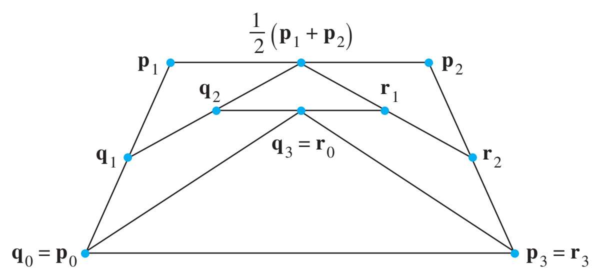


FIGURE 9 Geometric structure of new control points.

This completes one step of the subdivision process. The "recursion" begins, and both new curves are subdivided. The recursion continues to a depth at which all curves are sufficiently straight. Alternatively, at each step the recursion can be "adaptive" and not subdivide one of the two new curves if that curve is sufficiently straight. Once the subdivision completely stops, the endpoints of each curve are joined by line segments, and the scene is ready for the next step in the final image preparation.

A Bézier bicubic surface has the same variation-diminishing property as the Bézier curves that make up each cross-section of the surface, so the process described above can be applied in each cross-section. With the details omitted, here is the basic strategy. Consider the four "parallel" Bézier curves whose parameter is s, and apply the subdivision process to each of them. This produces four sets of eight control points; each set determines a curve as s varies from 0 to 1. As t varies, however, there are eight curves, each with four control points. Apply the subdivision process to each of these sets of four points, creating a total of 64 control points. Adaptive recursion is possible in this setting, too, but there are some subtleties involved.[2]

[2] See Foley, van Dam, Feiner, and Hughes, *Computer Graphics—Principles and Practice*, 2nd Ed. (Boston: Addison-Wesley, 1996), pp. 527–528.

PRACTICE PROBLEMS

A *spline* usually refers to a curve that passes through specified points. A B-spline, however, usually does not pass through its control points. A single segment has the parametric form

$$\mathbf{x}(t) = \tfrac{1}{6}\big[(1-t)^3\mathbf{p}_0 + (3t^3 - 6t^2 + 4)\mathbf{p}_1 + (-3t^3 + 3t^2 + 3t + 1)\mathbf{p}_2 + t^3\mathbf{p}_3\big] \qquad (14)$$

for $0 \le t \le 1$, where $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$ are the control points. When t varies from 0 to 1, $\mathbf{x}(t)$ creates a short curve that lies close to $\overline{\mathbf{p}_1\mathbf{p}_2}$. Basic algebra shows that the B-spline formula can also be written as

$$\mathbf{x}(t) = \tfrac{1}{6}\big[(1-t)^3\mathbf{p}_0 + (3t(1-t)^2 - 3t + 4)\mathbf{p}_1 + (3t^2(1-t) + 3t + 1)\mathbf{p}_2 + t^3\mathbf{p}_3\big] \qquad (15)$$

This shows the similarity with the Bézier curve. Except for the $1/6$ factor at the front, the $\mathbf{p}_0$ and $\mathbf{p}_3$ terms are the same. The $\mathbf{p}_1$ component has been increased by $-3t + 4$ and the $\mathbf{p}_2$ component has been increased by $3t + 1$. These components move the curve closer to $\overline{\mathbf{p}_1\mathbf{p}_2}$ than the Bézier curve. The $1/6$ factor is necessary to keep the sum of the coefficients equal to 1. Figure 10 compares a B-spline with a Bézier curve that has the same control points.

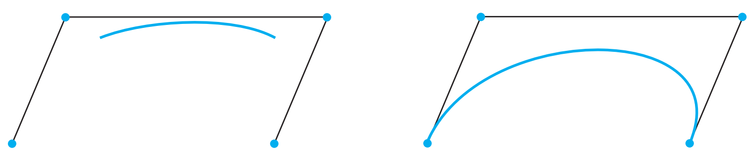


FIGURE 10 A B-spline segment and a Bézier curve.

1. Show that the B-spline does not begin at $\mathbf{p}_0$, but $\mathbf{x}(0)$ is in conv $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$. Assuming that $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ are affinely independent, find the affine coordinates of $\mathbf{x}(0)$ with respect to $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$.
2. Show that the B-spline does not end at $\mathbf{p}_3$, but $\mathbf{x}(1)$ is in conv $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$. Assuming that $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are affinely independent, find the affine coordinates of $\mathbf{x}(1)$ with respect to $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

8.6 EXERCISES

1. Suppose a Bézier curve is translated to $\mathbf{x}(t) + \mathbf{b}$. That is, for $0 \le t \le 1$, the new curve is

$$\mathbf{x}(t) = (1-t)^3\mathbf{p}_0 + 3t(1-t)^2\mathbf{p}_1 + 3t^2(1-t)\mathbf{p}_2 + t^3\mathbf{p}_3 + \mathbf{b}$$

Show that this new curve is again a Bézier curve. [*Hint:* Where are the new control points?]

2. The parametric vector form of a B-spline curve was defined in the Practice Problems as

$$\mathbf{x}(t) = \tfrac{1}{6}\big[(1-t)^3\mathbf{p}_0 + (3t(1-t)^2 - 3t + 4)\mathbf{p}_1 + (3t^2(1-t) + 3t + 1)\mathbf{p}_2 + t^3\mathbf{p}_3\big] \quad \text{for } 0 \le t \le 1,$$

where $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$ are the control points.

a. Show that for $0 \le t \le 1$, $\mathbf{x}(t)$ is in the convex hull of the control points.

b. Suppose that a B-spline curve $\mathbf{x}(t)$ is translated to $\mathbf{x}(t) + \mathbf{b}$ (as in Exercise 1). Show that this new curve is again a B-spline.

3. Let $\mathbf{x}(t)$ be a cubic Bézier curve determined by points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$.

a. Compute the *tangent* vector $\mathbf{x}'(t)$. Determine how $\mathbf{x}'(0)$ and $\mathbf{x}'(1)$ are related to the control points, and give geometric descriptions of the *directions* of these tangent vectors. Is it possible to have $\mathbf{x}'(1) = \mathbf{0}$?

b. Compute the second derivative $\mathbf{x}''(t)$ and determine how $\mathbf{x}''(0)$ and $\mathbf{x}''(1)$ are related to the control points. Draw a figure based on Figure 10, and construct a line segment that points in the direction of $\mathbf{x}''(0)$. [*Hint:* Use $\mathbf{p}_1$ as the origin of the coordinate system.]

4. Let $\mathbf{x}(t)$ be the B-spline in Exercise 2, with control points $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$.

a. Compute the tangent vector $\mathbf{x}'(t)$ and determine how the derivatives $\mathbf{x}'(0)$ and $\mathbf{x}'(1)$ are related to the control points. Give geometric descriptions of the *directions* of these tangent vectors. Explore what happens when both $\mathbf{x}'(0)$ and $\mathbf{x}'(1)$ equal $\mathbf{0}$. Justify your assertions.

b. Compute the second derivative $\mathbf{x}''(t)$ and determine how $\mathbf{x}''(0)$ and $\mathbf{x}''(1)$ are related to the control points. Draw a figure based on Figure 10, and construct a line segment that points in the direction of $\mathbf{x}''(1)$. [*Hint:* Use $\mathbf{p}_2$ as the origin of the coordinate system.]

5. Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be cubic Bézier curves with control points $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ and $\{\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6\}$, respectively, so that $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are joined at $\mathbf{p}_3$. The following questions refer to the curve consisting of $\mathbf{x}(t)$ followed by $\mathbf{y}(t)$. For simplicity, assume that the curve is in $\mathbb{R}^2$.

a. What condition on the control points will guarantee that the curve has C^1 continuity at $\mathbf{p}_3$? Justify your answer.

b. What happens when $\mathbf{x}'(1)$ and $\mathbf{y}'(0)$ are both the zero vector?

6. A B-spline is built out of B-spline segments, described in Exercise 2. Let $\mathbf{p}_0, \dots, \mathbf{p}_4$ be control points. For $0 \le t \le 1$, let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be determined by the geometry matrices $[\,\mathbf{p}_0 \;\; \mathbf{p}_1 \;\; \mathbf{p}_2 \;\; \mathbf{p}_3\,]$ and $[\,\mathbf{p}_1 \;\; \mathbf{p}_2 \;\; \mathbf{p}_3 \;\; \mathbf{p}_4\,]$, respectively. Notice how the two segments share three control points. The two segments do not overlap, however—they join at a common endpoint, close to $\mathbf{p}_2$.

a. Show that the combined curve has G^0 continuity—that is, $\mathbf{x}(1) = \mathbf{y}(0)$.

b. Show that the curve has C^1 continuity at the join point, $\mathbf{x}(1)$. That is, show that $\mathbf{x}'(1) = \mathbf{y}'(0)$.

7. Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be Bézier curves from Exercise 5, and suppose the combined curve has C^2 continuity (which includes C^1 continuity) at $\mathbf{p}_3$. Set $\mathbf{x}''(1) = \mathbf{y}''(0)$ and show that $\mathbf{p}_5$ is completely determined by $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. Thus, the points $\mathbf{p}_0, \dots, \mathbf{p}_3$ and the C^2 condition determine all but one of the control points for $\mathbf{y}(t)$.

8. Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be segments of a B-spline as in Exercise 6. Show that the curve has C^2 continuity (as well as C^1 continuity) at $\mathbf{x}(1)$. That is, show that $\mathbf{x}''(1) = \mathbf{y}''(0)$. This higher-order continuity is desirable in CAD applications such as automotive body design, since the curves and surfaces appear much smoother. However, B-splines require three times the computation of Bézier curves, for curves of comparable length. For surfaces, B-splines require nine times the computation of Bézier surfaces. Programmers often choose Bézier surfaces for applications (such as an airplane cockpit simulator) that require real-time rendering.

9. A quartic Bézier curve is determined by five control points, $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, and $\mathbf{p}_4$:

$$\begin{aligned}\mathbf{x}(t) = (1-t)^4\mathbf{p}_0 + 4t(1-t)^3\mathbf{p}_1 + 6t^2(1-t)^2\mathbf{p}_2 \\ + 4t^3(1-t)\mathbf{p}_3 + t^4\mathbf{p}_4 \quad \text{for } 0 \le t \le 1\end{aligned}$$

Construct the quartic basis matrix M_B for $\mathbf{x}(t)$.

10. The "B" in B-spline refers to the fact that a segment $\mathbf{x}(t)$ may be written in terms of a basis matrix, M_S, in a form similar to a Bézier curve. That is,

$$\mathbf{x}(t) = GM_S\mathbf{u}(t) \quad \text{for } 0 \le t \le 1$$

where G is the geometry matrix $[\,\mathbf{p}_0 \;\; \mathbf{p}_1 \;\; \mathbf{p}_2 \;\; \mathbf{p}_3\,]$ and $\mathbf{u}(t)$ is the column vector $(1, t, t^2, t^3)$. In a *uniform* B-spline, each segment uses the same basis matrix, but the geometry matrix changes. Construct the basis matrix M_S for $\mathbf{x}(t)$.

In Exercises 11 and 12, mark each statement True or False. Justify each answer.

11. a. The cubic Bézier curve is based on four control points.

b. Given a quadratic Bézier curve $\mathbf{x}(t)$ with control points $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$, the directed line segment $\mathbf{p}_1 - \mathbf{p}_0$ (from $\mathbf{p}_0$ to $\mathbf{p}_1$) is the tangent vector to the curve at $\mathbf{p}_0$.

c. When two quadratic Bézier curves with control points $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$ and $\{\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$ are joined at $\mathbf{p}_2$, the combined Bézier curve will have C^1 continuity at $\mathbf{p}_2$ if $\mathbf{p}_2$ is the midpoint of the line segment between $\mathbf{p}_1$ and $\mathbf{p}_3$.

12. a. The essential properties of Bézier curves are preserved under the action of linear transformations, but not translations.

b. When two Bézier curves $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are joined at the point where $\mathbf{x}(1) = \mathbf{y}(0)$, the combined curve has G^0 continuity at that point.

c. The Bézier basis matrix is a matrix whose columns are the control points of the curve.

Exercises 13–15 concern the subdivision of a Bézier curve shown in Figure 7. Let $\mathbf{x}(t)$ be the Bézier curve, with control points $\mathbf{p}_0, \dots, \mathbf{p}_3$, and let $\mathbf{y}(t)$ and $\mathbf{z}(t)$ be the subdividing Bézier curves as in the text, with control points $\mathbf{q}_0, \dots, \mathbf{q}_3$ and $\mathbf{r}_0, \dots, \mathbf{r}_3$, respectively.

13. a. Use equation (12) to show that $\mathbf{q}_1$ is the midpoint of the segment from $\mathbf{p}_0$ to $\mathbf{p}_1$.

b. Use equation (13) to show that

$$8\mathbf{q}_2 = 8\mathbf{q}_3 + \mathbf{p}_0 + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3.$$

c. Use part (b), equation (8), and part (a) to show that $\mathbf{q}_2$ is the midpoint of the segment from $\mathbf{q}_1$ to the midpoint of the segment from $\mathbf{p}_1$ to $\mathbf{p}_2$. That is, $\mathbf{q}_2 = \frac{1}{2}[\mathbf{q}_1 + \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)]$.

14. a. Justify each equal sign:

$$3(\mathbf{r}_3 - \mathbf{r}_2) = \mathbf{z}'(1) = .5\mathbf{x}'(1) = \tfrac{3}{2}(\mathbf{p}_3 - \mathbf{p}_2).$$

b. Show that $\mathbf{r}_2$ is the midpoint of the segment from $\mathbf{p}_2$ to $\mathbf{p}_3$.

c. Justify each equal sign: $3(\mathbf{r}_1 - \mathbf{r}_0) = \mathbf{z}'(0) = .5\mathbf{x}'(.5)$.

d. Use part (c) to show that $8\mathbf{r}_1 = -\mathbf{p}_0 - \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + 8\mathbf{r}_0$.

e. Use part (d), equation (8), and part (a) to show that $\mathbf{r}_1$ is the midpoint of the segment from $\mathbf{r}_2$ to the midpoint of the segment from $\mathbf{p}_1$ to $\mathbf{p}_2$. That is, $\mathbf{r}_1 = \frac{1}{2}[\mathbf{r}_2 + \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)]$.

15. Sometimes only one half of a Bézier curve needs further subdividing. For example, subdivision of the "left" side is accomplished with parts (a) and (c) of Exercise 13 and equation (8). When both halves of the curve $\mathbf{x}(t)$ are divided, it is possible to organize calculations efficiently to calculate both left and right control points concurrently, without using equation (8) directly.

a. Show that the tangent vectors $\mathbf{y}'(1)$ and $\mathbf{z}'(0)$ are equal.

b. Use part (a) to show that $\mathbf{q}_3$ (which equals $\mathbf{r}_0$) is the midpoint of the segment from $\mathbf{q}_2$ to $\mathbf{r}_1$.

c. Using part (b) and the results of Exercises 13 and 14, write an algorithm that computes the control points for both $\mathbf{y}(t)$ and $\mathbf{z}(t)$ in an efficient manner. The only operations needed are sums and division by 2.

16. Explain why a cubic Bézier curve is completely determined by $\mathbf{x}(0), \mathbf{x}'(0), \mathbf{x}(1)$, and $\mathbf{x}'(1)$.

17. TrueType® fonts, created by Apple Computer and Adobe Systems, use quadratic Bézier curves, while PostScript® fonts, created by Microsoft, use cubic Bézier curves. The cubic curves provide more flexibility for typeface design, but it is important to Microsoft that every typeface using quadratic curves can be transformed into one that uses cubic curves. Suppose that $\mathbf{w}(t)$ is a quadratic curve, with control points $\mathbf{p}_0, \mathbf{p}_1$, and $\mathbf{p}_2$.

a. Find control points $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2$, and $\mathbf{r}_3$ such that the cubic Bézier curve $\mathbf{x}(t)$ with these control points has the property that $\mathbf{x}(t)$ and $\mathbf{w}(t)$ have the same initial and terminal points and the same tangent vectors at $t = 0$ and $t = 1$. (See Exercise 16.)

b. Show that if $\mathbf{x}(t)$ is constructed as in part (a), then $\mathbf{x}(t) = \mathbf{w}(t)$ for $0 \le t \le 1$.

18. Use partitioned matrix multiplication to compute the following matrix product, which appears in the alternative formula (5) for a Bézier curve:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

SOLUTIONS TO PRACTICE PROBLEMS

1. From equation (14) with $t = 0$, $\mathbf{x}(0) \neq \mathbf{p}_0$ because

$$\mathbf{x}(0) = \tfrac{1}{6}[\mathbf{p}_0 + 4\mathbf{p}_1 + \mathbf{p}_2] = \tfrac{1}{6}\mathbf{p}_0 + \tfrac{2}{3}\mathbf{p}_1 + \tfrac{1}{6}\mathbf{p}_2.$$

The coefficients are nonnegative and sum to 1, so $\mathbf{x}(0)$ is in conv $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$, and the affine coordinates with respect to $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$ are $\left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right)$.

2. From equation (14) with $t = 1$, $\mathbf{x}(1) \neq \mathbf{p}_3$ because

$$\mathbf{x}(1) = \tfrac{1}{6}[\mathbf{p}_1 + 4\mathbf{p}_2 + \mathbf{p}_3] = \tfrac{1}{6}\mathbf{p}_1 + \tfrac{2}{3}\mathbf{p}_2 + \tfrac{1}{6}\mathbf{p}_3.$$

The coefficients are nonnegative and sum to 1, so $\mathbf{x}(1)$ is in conv $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$, and the affine coordinates with respect to $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ are $\left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right)$.