CHAPTER VI

Background and Motivation for Homology Theory

§1. Introduction

Homology theory is a subject whose development requires a long chain of definitions, lemmas, and theorems before it arrives at any interesting results or applications. A newcomer to the subject who plunges into a formal, logical presentation of its ideas is likely to be somewhat puzzled because he will probably have difficulty seeing any motivation for the various definitions and theorems. It is the purpose of this chapter to present some explanation, which will help the reader to overcome this difficulty. We offer two different kinds of material for background and motivation. First, there is a summary of some of the most easily understood properties of homology theory, and a hint at how it can be applied to specific problems. Second, there is a brief outline of some of the problems and ideas which led certain mathematicians of the nineteenth century to develop homology theory.

It should be emphasized that the reading of this chapter is *not* a logical prerequisite to the understanding of anything in later chapters of this book.

§2. Summary of Some of the Basic Properties of Homology Thoery

Homology theory assigns to any topological space X a sequence of abelian groups $H_0(X)$, $H_1(X)$, $H_2(X)$, ..., and to any continuous map $f: X \to Y$ a sequence of homomorphisms

$$f_* : H_n(X) \to H_n(Y), \quad n = 0, 1, 2, \ldots .$$

$H_n(X)$ is called the *n-dimensional homology group of* X, and f_* is called *the homomorphism induced by* f. We will list in more or less random order some of the principal properties of these groups and homomorhisms.

(a) If $f: X \to Y$ is a homeomorphism of X onto Y, then the induced homomorphism $f_*: H_n(X) \to H_n(Y)$ is an isomorphism for all n. Thus, the algebraic structure of the groups $H_n(X)$, $n = 0, 1, 2, \ldots$, depends only on the topological type of X. In fact, an even stronger statement holds: if f is a homotopy equivalence, then f_* is an isomorphism. Thus, the structure of $H_n(Y)$ only depends on the homotopy type of X. Two spaces of the same homotopy type have isomorphic homology groups (for the definition of these terms, the reader is referred to Chapter II, §4 and §8.

(b) If two maps $f_0, f_1 : X \to Y$ are homotopic, then the induced homomorphisms f_{0*} and $f_{1*}: H_n(X) \to H_n(Y)$ are the same for all n. Thus, the induced homomorphism f_* only depends on the homotopy class of f. By its use, we can sometimes prove that certain maps are *not* homotopic.

(c) For any space X, the group $H_0(X)$ is free abelian, and its rank is equal to the number of arcwise connected components of X. In other words, $H_0(X)$ has a basis in 1–1 correspondence with the set of arc components of X. Thus, the structure of $H_0(X)$ has to do with the arcwise connectedness of X. By analogy, the groups $H_1(X)$, $H_2(X)$, ... have something to do with some kind of higher connectivity of X. In fact, one can look on this as one of the principal purposes for the introduction of the homology grups: to express what may be called the higher connectivity properties of X.

(d) If X is an arcwise-connected space, the 1-dimensional homology group, $H_1(X)$, is the abelianized fundamental group. In other words, $H_1(X)$ is isomorphic to $\pi(X)$ modulo its commutator subgroup.

(e) If X is a compact, connected, orientable n-dimensional manifold, then $H_n(X)$ is infinite cyclic, and $H_q(X) = \{0\}$ for all $q > n$. In some vague sense, such a manifold is a prototype or model for nonzero n-dimensional homology groups.

(f) If X is an open subset of Euclidean n-space, then $H_q(X) = \{0\}$ for all $q \geq n$.

We have already alluded to the fact that sometimes it is possible to use homology theory to prove that two continuous maps are not homotopic. Analogously, homology groups can sometimes be used to prove that two spaces are not homeomorphic, or not even of the same homotopy type. These are rather obvious applications. In other cases, homology theory is used in less obvious ways to prove theorems. A nice example of this is the proof of the Brouwer fixed-point theorem in Chapter VIII, §2. More subtle examples are the Borsuk–Ulam theorem in Chapter XV, §2 and the Jordan–Brouwer separation theorem in Chapter VIII, §6.

§3. Some Examples of Problems Which Motivated the Development of Homology Theory in the Nineteenth Century

The problems we are going to consider all have to do with line integrals, surface integrals, etc., and theorems relating these integrals, such as the well-known theorems of Green, Stokes, and Gauss. We assume the reader is familiar with these topics.

As a first example, consider the following problem which is discussed in most advanced calculus books. Let U be an open, connected set in the plane, and let **V** be a vector field in U (it is assumed that the components of **V** have continuous partial derivatives in U). Under what conditions does there exist a "potential function" for **V**, i.e., a differentiable function $F(x, y)$ such that **V** is the gradient of F? Denote the x and y components of **V** by $P(x, y)$ and $Q(x, y)$ respectively; then an obvious necessary condition is that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

at every point of U. If the set U is convex, then this necessary condition is also sufficient. The standard proof of sufficiency is based on the use of Green's theorem, which asserts that

$$\oint_C P\,dx + Q\,dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Here D is a domain with piecewise smooth boundary C (which may have several component) such that D and C are both contained in U. By using Green's theorem, one can prove that the line integral on the left-hand side vanishes if C is any closed curve in U. This implies that if (x_0, y_0) and (x, y) are any two points of U, and L is any piecewise smooth path in U joining (x_0, y_0) and (x, y), then the line integral

$$\int_L P\,dx + Q\,dy$$

is independent of the choice of L; it only depends on the end points (x_0, y_0) and (x, y). If we hold (x_0, y_0) fixed, and define $F(x, y)$ to be the value of this line integral for any point (x, y) in U, then $F(x, y)$ is the desired potential function.

On the other hand, if the open set U is more complicated, the necessary condition $\partial P/\partial y = \partial Q/\partial x$ may not be sufficient. Perhaps the simplest example to illustrate this point is the following: Let U denote the plane with the origin deleted,

$$P = -\frac{y}{x^2 + y^2} \quad \text{and} \quad Q = \frac{x}{x^2 + y^2}.$$

Then the condition $\partial Q/\partial x = \partial P/\partial y$ is satisfied at each point of U. However, if we compute the line integral

$$\int_C P\,dx + Q\,dy, \tag{6.3.1}$$

where C is a circle with center at the origin, we obtain the value 2π. Since $2\pi \neq 0$, there cannot be any potential function for the vector field $\mathbf{V} = (P, Q)$ in the open set U. It is clear where the preceding proof breaks down in this case: the circle C (with center at the origin) does not bound any domain D such that $D \subset U$.

Since the line integral (1) may be nonzero in this case, we may ask, What are all possible values of this line integral as C ranges over all piecewise smooth closed curves in U? The answer is $2n\pi$, where n ranges over all integers, positive or negative. Indeed, any of these values may be obtained by integrating around the unit circle with center at the origin an appropriate number of times in the clockwise or counterclockwise direction; and an informal argument using Green's theorem should convince the reader that these are the only possible values.

We can ask the same question for any open, connected set U in the plane, and any continuously differentiable vector field $\mathbf{V} = (P, Q)$ in U satisfying the condition $\partial P/\partial y = \partial Q/\partial x$: What are all possible values of the line integral (6.3.1) as C ranges over all piece-wise smooth closed curves in U? Anybody who studies this problem will quickly come to the conclusion that the answer depends on the number of "holes" in the set U. Let us associate with each hole the value of the integral (6.3.1) in the case where C is a closed path which goes around the given hole exactly once and does not encircle any other hole (assuming such a path exists). By analogy with complex function theory, we will call this number the *residue* associated with the given hole. The answer to our problem then is that the value of the inegral (6.3.1) is some finite, integral linear combination of these residues, and any such finite integral linear combination actually occurs as a value.

Next, let us consider the analogous problem in 3-space: we now assume that U is an open, connected set in 3-space, and $\mathbf{V}$ is a vector field in U with components $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$ (which are assumed to be continuously differentiable in U). Furthermore, we assume that curl $\mathbf{V} = 0$. In terms of the components, this means that the equations

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

hold at each point of U. Once again it can be shown that if U is convex, then there exists a function $F(x, y, z)$ such that $\mathbf{V}$ is the gradient of F. The proof is much the same as the previous case, except that now one must use Stokes's theorem rather than Green's theorem to show that the line integral

$$\int P\,dx + Q\,dy + R\,dz$$

is independent of the path.

In case the domain U is not convex, this proof may break down, and it can actually happen that the line integral

$$\oint_C P\,dx + Q\,dy + R\,dz \tag{6.3.2}$$

is nonzero for some closed path C in U. Once again we can ask: What are all possible values of the line integral (2) for all possible closed paths in U? The "holes" in U are again what makes the problem interesting; however, in this case there seem to be different kinds of holes. Let us consider some examples:

(a) Let $U = \{(x, y, z) | x^2 + y^2 > 0\}$, i.e., U is the complement of the z axis. This example is similar to the 2-dimensional case treated earlier. If C denotes a circle in the xy plane with center at the origin, we could call the value of the integral (6.3.2) with this choice of C the residue corresponding to the hole in U. Then the value of the integral (6.3.2) for any other choice of C in U would be some integral multiple of this residue; the reader should be able to convince himself of this in any particular case by using Stokes's theorem.

(b) Let U be the complement of the origin in $\mathbf{R}^3$. If Σ is any piecewise smooth orientable surface in U with boundary C consisting of one or more piecewise smooth curves, then according to Stokes's theorem,

$$\oint_C P\,dx + Q\,dy + R\,dz = \iint_\Sigma \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

We leave it to the reader to convince himself that any piecewise smooth closed curve c in U is the boundary of such a surface Σ, hence by Stokes's theorem, the integral around such a curve is zero (the integral on the right-hand side is identically zero). Thus, the same argument applies as in the case where U is convex to show that any vector fixed $\mathbf{V}$ in U such that curl $\mathbf{V} = 0$ in U is of the form $\mathbf{V} = \text{grad } F$ for some function F. The existence of the hole in U does not matter in this case.

(c) It is easy to give other examples of domains in 3-space with holes in them such that the hole does not matter. The following are such examples: let $U_1 = \{(x, y, z) | x^2 + y^2 + z^2 > 1\}$; let U_2 be the complement of the upper half ($z \geq 0$) of the z axis; and let U_3 be the complement of a finite set of points in 3-space. In each case, if $\mathbf{V}$ is a vector field in U_i such that curl $\mathbf{V} = 0$, then $\mathbf{V} = \text{grad } F$ for some function F. The basic reason is that any closed curve C in U_i is the boundary of some oriented surface Σ in U_i in each of the cases $i = 1, 2$, or 3.

There is another problem for 3-dimensional space which involves closed surfaces rather than closed curves. It may be phrased as follows: Let U be a connected open set in $\mathbf{R}^3$ and let $\mathbf{V}$ be a continuously differentiable vector field in U such that div $\mathbf{V} = 0$. Is the integral of (the normal component of) $\mathbf{V}$ over any closed, orientable piecewise smooth surface Σ in U equal to 0? If not, what are the possible values of the integral of $\mathbf{V}$ over any such closed surface? If U is a convex open set, then any such integral of 0. One proves this by the use of Gauss's theorem (also called the divergence theorem):

$$\iint_{\Sigma} \mathbf{V} = \iiint_{D} (\operatorname{div} \mathbf{V})\, dx\, dy\, dz.$$

Here D is a domain in U with piecewise smooth boundary Σ (the boundary may have several components). The main point is that a closed orientable surface Σ contained in a convex open set U is always the boundary of a domain D contained in U. However, if the open set U has holes in it, this may not be true, and the situation is more complicated. For example, suppose that U is the complement of the origin in 3-space, and $\mathbf{V}$ is the vector field in U with components $P = x/r^3$, $Q = y/r^3$, and $R = z/r^3$, where $r = (x^2 + y^2 + z^2)^{1/2}$ is the distance from the origin. It is readily verified that div $\mathbf{V} = 0$; on the other hand, the integral of $\mathbf{V}$ over any sphere with center at the origin is readily calculated to be $\pm 4\pi$; the sign depends on the orientation conventions. The set of all possible values of the surface integral $\iint_{\Sigma} \mathbf{V}$ for all closed, orientable surfaces Σ in U is precisely the set of all integral multiples of 4π.

On the other hand, if U is the complement of the z axis in 3-space, then the situation is exactly the same as in the case where U is convex. The reason is that any closed, orientable surface in U bounds a domain D in U; the existence of the hole in U does not matter.

There is a whole series of analogous problems in Euclidean spaces of dimension four or more. Also, one could consider similar problems on curved submanifolds of Euclidean space. Although there would doubtless be interesting new complications, we have already presented enough examples to give the flavor of the subject.

At some point in the nineteenth century certain mathematicians tried to set up general procedures to handle problems such as these. This led them to introduce the following terminology and definitions. The closed curves, surfaces, and higher-dimensional manifolds over which one integrates vector fields, etc., were called *cycles*. In particular, a closed curve is a 1-dimensional cycle, a closed surface is a 2-dimensional cycle, and so on. To complete the picture, a 0-dimensional cycle is a point. It is understood, of course, that cycles of dimension > 0 always have a definite orientation, i.e., a 2-cycle is an oriented closed surface. Moreover, it is convenient to attach to each cycle a certain integer which may be thought of as its "multiplicity." To integrate a vector field over a 1-dimensional cycle or closed curve with multiplicity $+3$ means to integrate it over a path going around the curve 3 times; the result will be three times the value of the integral going around it once. If the multiplicity is -3, then one integrates three times around the curve in the opposite direction. If the symbol c denotes a 1-dimensional cycle, then the symbol $3c$ denotes this cycle with the multiplicity $+3$, and $-3c$ denotes the same cycle with multiplicity -3. It is also convenient to allow formal sums and linear combinations of cycles (all of the same dimension), that is, expressions like $3c_1 + 5c_2 - 10c_3$, where c_1, c_2, and c_3 are cycles. With this definition of addition, the set of all n-dimensional cycles in an open set U of Euclidean space becomes an abelian group; in fact it is a free abelian group. It is customary to denote this group by $Z_n(U)$. There is one further convention that is understood here: If c is the 1-dimensional cycle determined by a certain oriented closed curve, and c' denotes the cycle determined by the same curve with the opposite orientation, then $c = -c'$. This is consistent with the fact that the integral of a vector field over c' is the negative of the integral over c. Of course, the same convention also holds for higher-dimensional cycles.

It is important to point out that 1-dimensional cycles are only assumed to be closed curves; they are not assumed to be *simple* closed curves. Thus, they may have various self-intersections or singularities. Similarly, a 2-dimensions cycle in U is an oriented surface in U which is allowed to have various self-intersections or singularities. It is really a continuous (or differentiable) mapping of a compact, connected, oriented 2-manifold into U. Because of the possible existence of self-intersections or singularities, these cycles are often called *singular* cycles.

Once one knows how to define the integral of a vector field (or differential form) over a cycle, it is obvious how to define the integral over a formal linear combination of cycles. If $c_1, \ldots, c_k$ are cycles in U and

$$z = n_1 c_1 + \cdots + n_k c_k,$$

where $n_1, n_2, \ldots, n_k$ are integers, then

$$\int_z \mathbf{V} = \sum_{i=1}^{k} n_i \int_{c_i} \mathbf{V}$$

for any vector field $\mathbf{V}$ in U.

The next step is to define an equivalence relation between cycles. This equivalence relation is motivated by the following considerations. Assume that U is an open set in 3-space.

(a) Let u and w be 1-dimensional cycles in U, i.e., u and w are elements of the goups $Z_1(U)$. Then we wish to define $u \sim w$ so that this implies

$$\int_u \mathbf{V} = \int_w \mathbf{V}$$

for *any* vector field $\mathbf{V}$ in U such that curl $\mathbf{V} = 0$.

(b) Let u and w be elements of the group $Z_2(U)$. Then we wish to define $u \sim w$ so that this implies

$$\int_u \mathbf{V} = \int_w \mathbf{V}$$

for any vector field $\mathbf{V}$ in U such that div $\mathbf{V} = 0$.

Note that the condition

$$\int_u \mathbf{V} = \int_w \mathbf{V}$$

can be rewritten as follows, in view of our conventions:

$$\int_{u-w} \mathbf{V} = 0.$$

Thus, $u \sim w$ if and only if $u - w \sim 0$.

In case (a), Stokes's theorem suggests the proper definition, while in case (b) the divergence theorem points the way.

We will discuss case (a) first. Suppose we have an oriented surface in U whose boundary consists of the oriented closed curves $c_1, c_2, \ldots, c_k$. The orientations of the boundary curves are determined according to the conventions used in the statement of Stokes's theorem. Then the 1-dimensional cycle

$$z = c_1 + c_2 + \cdots + c_k$$

is defined to be *homologous to zero*, written

$$z \sim 0.$$

More generally, any linear combination of cycles homologous to zero is also defined to be homologous to zero. The set of all cycles homologous to zero is a subgroup of $Z_1(U)$ which is denoted by $B_1(U)$. We define z and z' to be homologous (written $z \sim z'$) if and only if $z - z' \sim 0$. Thus, the set of equivalence classes of cycles, called *homology classes*, is nothing other than the quotient group

$$H_1(U) = Z_1(U)/B_1(U)$$

which is called the 1-dimensional *homology group* of U.

Analogous definitions apply to case (b). Let D be a domain in U whose boundary consists of the connected oriented surfaces $s_1, s_2, \ldots, s_k$. The orientation of the boundary surfaces is determined by the conventions used for the divergence theorem. Then the 2-dimensional cycle

$$z = s_1 + s_2 + \cdots + s_k$$

is by definition homologous to zero, written $z \sim 0$. As before, any linear combination of cycles homologous to zero is also defined to be homologous to 0, and the set of cycles homologous to 0 constitutes a subgroup, $B_2(U)$, of $Z_2(U)$. The quotient group

$$H_2(U) = Z_2(U)/B_2(U)$$

is called the 2-dimensional *homology group* of U.

Let us consider some examples. If U is an open subset of the plane, then $H_1(U)$ is a free abelian group, and it has a basis (or minimal set of generators) in 1–1 correspondence with the holes in U. If U is an open subset of 3-spaces, then both $H_1(U)$ and $H_2(U)$ are free abelian groups, and each hole in U contributes generators to $H_1(U)$ or $H_2(U)$, or perhaps to both. This helps explain the different kinds of holes in this case.

In principle, there is nothing to stop us from generalizing this procedure, and defining for any topological space X and non-negative integer n the group $Z_n(X)$ of n-dimensional cycles in X, the subgroup $B_n(X)$ consisting of cycles which are homologous to zero, and the quotient group

$$H_n(X) = Z_n(X)/B_n(X),$$

called the n-dimensional homology group of X. However, there are difficulties in formulating the definitions rigorously in this generality; the reader may have noticed that some of the definitions in the preceding pages were lacking in precision. Actually, it took mathematicians some years to surmount these difficulties. The key idea was to think of an n-dimensional cycle as made up of small n-dimensional pieces which fit together in the right way, in much the same way that bricks fit together to make a wall. In this book, we will use n-dimensional cycles that consist of n-dimensional cubes which fit together in a nice way. To be more precise, the "singular" cycles will be built from "singular" cubes; a singular n-cube in a topological space X is simply a continuous map $T: I^n \to X$, where I^n denotes the unit n-cube in Euclidean n-space.

There is another complication which should be pointed out. We mentioned in connection with the examples above that if U is an open subset of the plane or 3-space, then the homology groups of U are free abelian groups. However, there exist open subsets U of Euclidean n-space for all $n > 3$ such that the group $H_1(U)$ contains elements of finite order (compare the discussion of the homology groups of nonorientable surfaces in §VIII.4). Suppose that $u \in H_1(U)$ is a homology class of order $k \neq 0$. Let z be a 1-dimensional cycle in the homology class u. Then z is not homologous to 0, but $k \cdot z$ is homologous to 0. This implies that if $\mathbf{V}$ is any vector field in U such that curl $\mathbf{V} = 0$, then

$$\int_z \mathbf{V} = 0.$$

To see this, let $\int_z \mathbf{V} = r$. Then $\int_{kz} \mathbf{V} = k \cdot r$; but $\int_{kz} \mathbf{V} = 0$ since $kz \sim 0$. Therefore $r = 0$. It is not clear that this phenomenon was understood in the nineteenth century; at least there seems to have been some confusion in Poincaré's early papers on topology about this point. Of course, one source of difficulty is the fact that this phenomenon eludes our ordinary geometric intuition, since it dows not occur in 3-dimensional space. Nevertheless it is a phenomenon of importance in algebraic topology.

Before ending this account, we should make clear that we do not claim that the nineteenth century development of homology theory actually proceeded along the lines we have just described. For one thing, the nineteenth centry mathematicians involved in this development were more interested in complex analysis than real analysis. Moreover, many of their false starts and tentative attempts to establish the subject can only be surmised from reading the published papers which have survived to the present. For a fairly readable nineteenth century account of some of these ideas, the reader is referred to the famous book by J. C. Maxwell [6].

The modern development of these same ideas led to De Rham's theorem; see Appendix A.

NOTES

The history of algebraic topology

The early development of what is now called algebraic topology occurred mainly in the nineteenth century. Even in the early part of that century some mathematicians, such as Gauss, foresaw the need for such a development. In those days topology was referred to as "analysis situs." The work of Riemann on complex function theory in the middle of the century was a strong stimulus for the further development of the subject, especially for the topology of surfaces. Unfortunately, Riemann never published his ideas on algebraic topology; the brief "Fragment" [10] published after his death in his collected works seems rather vague and incomprehensible. Riemann contracted tuberculosis in 1862 and spent much of the few remaining years of his life in Italy, trying to regain his health. While there, he discussed his ideas on topology with some Italian mathematicians, especially Professor Enrico Betti of Pisa. Some of Betti's letters to other Italian mathematicians have been published; he writes of the things he has learned from Riemann. Betti published on these topics in a paper [1] after Riemann's death. In 1895 Poincaré tried to further develop the ideas of Riemann and Betti in a long paper entitled "Analysis Situs" [7]. The Danish mathematician P. Heegard in his Copenhagen thesis of 1898 criticized certain aspects of Poincaré's paper. This apparently forced Poincaré to reexamine his ideas, and in subsequent "Complements" to his original paper on analysis situs he changed his point of view and created what was to become homology theory.

Background and motivation for homology theory

The student may find it helpful to read further articles on this subject. Several such articles are listed in the bibliography blow. The books by Blackett [11] and Frechet and Fan [12] have bibliographies which list many additional articles that are helpful and interesting.

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