The chain complex of a graph

$(X, \mathcal{E})$ = graph with decomposition

$C_1(X, \mathcal{E})$ = free abelian group on all E in $\mathcal{E}$

$C_0(X, \mathcal{E})$ = free abelian group on all vertices of edges in E.

Order the vertices $v_1, \ldots, v_n$.

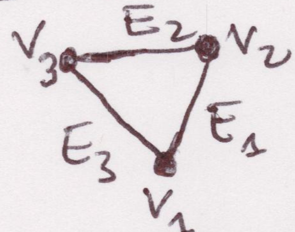
Boundary map

$$d : C_1(X, \mathcal{E}) \longrightarrow C_0(X, \mathcal{E})$$

If E has vertices v_i & v_j with $i < j$, then $d(E) = v_j - v_i$.

cycles = Kernel d = $H_1(X)$ homology.

$d(\sum n_k E_k) = 0$.



$d(E_1 + E_2 - E_3) = 0$

Suppose X is connected

Recall $\chi(X,\xi)$ = #vertices − #edges.

<u>CLAIM:</u> $H_1(X)$ = free abelian gp on $1-\chi(X,\xi)$ generators.

(Do algebra). Independent of ~~cell~~ edge path decomposition, for rank $H_1(X)$ = free rank of $\pi_1(X)$.

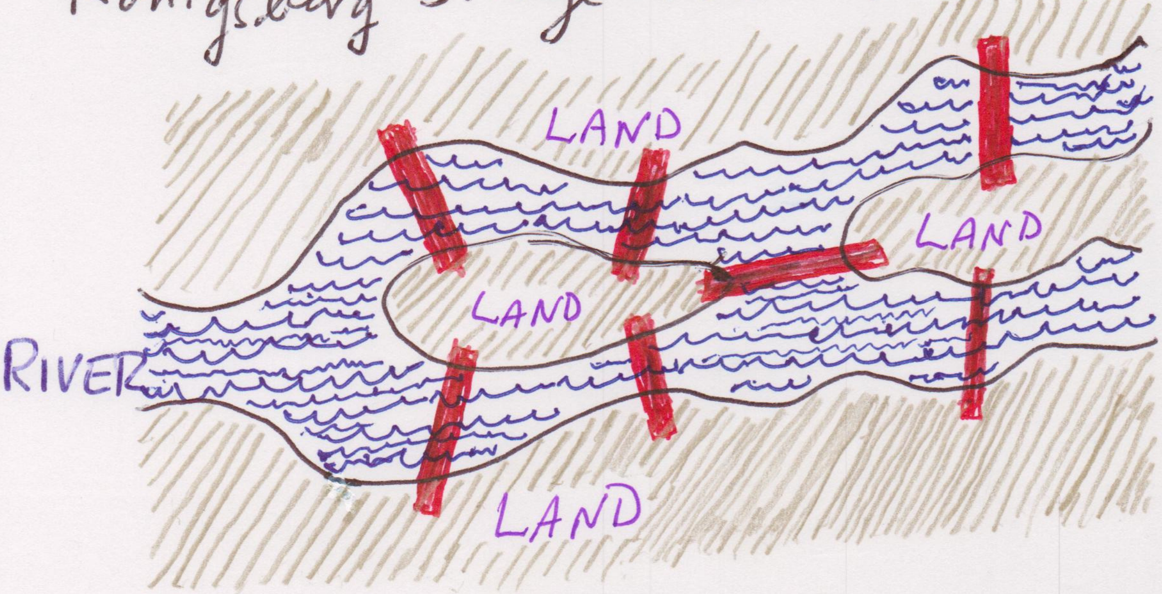
Now suppose X is arbitrary.

<u>CLAIM</u> $H_0(X) \cong$ free abel gp on the set of components of X.

(<u>Hints</u>: Each component contains at least <u>two</u> vertices, and # components $\leq$ # vertices. Two vertices in same component $\Longleftrightarrow$ joined by an edge path.)

We could work instead with $\mathbb{Z}_2$ vector spaces and get similar conclusions.

We shall use this formalism to study the Königsberg Bridge Problem.



Bridges in red.

Question: Is there some path that crosses each bridge exactly once?

Negative answer due to Euler (1734).

Obviously, the first step is to translate this into chain algebra:

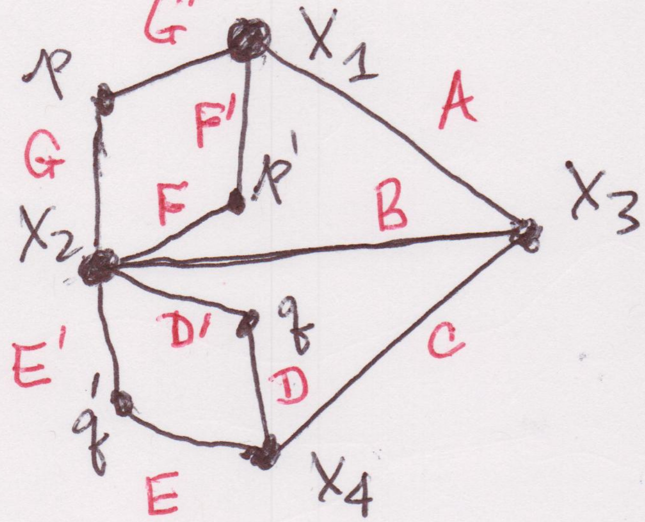
ordering of vertices:

$x_2 < p < p' <$

$x_1 < q < q' <$

$x_4 < x_3$

Need to define the boundary



$x_1, \ldots, x_4$
$\updownarrow$
land masses

bridges $\Longleftrightarrow$ $A, B, C, D+D', E+E', F+F', G+G'$

The path corresponds to a linear comb.

$$\boxed{aA + bB + cC + dD + e(E+E') + f(F+F') + g(G+G')}$$

with all coefficients ± 1

and boundary equal to

(one vertex) − (another vertex, maybe the same)

END LOCATION STARTING LOCATION

Reduce this mod 2. The chain reduces to

$A+B+C+D+D'+E+E'+F+F'+G+G'$

Its mod 2 boundary is a sum of 0 or two vertices.

On the other hand, the boundary mod 2 is

$$\sum_{y \text{ vertex}} m(y)\, y \quad \left\{ \begin{array}{l} (m(y) = \text{number of} \\ \text{edges containing } y) \\ \text{reduced mod } 2 \end{array} \right.$$

So if a path exists, then either

1. $m(y)$ is always even.
2. Two $m(y)$'s are ~~to~~ odd and the others are all even.

For the given graph we have

$$m(x_1) = m(x_3) = m(x_4) = 3$$

$$m(x_2) = 5$$

So neither 1 nor 2 holds. Therefore one $\boxed{\text{cannot}}$ find the desired type of path.