Homology of starshaped complexes

$C_*(P,K) \subseteq C_*^+(P,K)$ [v = star vertex]

$j_\#$

add extra generators in dim $q+1$ of the form $vv_0 \ldots v_q$ if $v_0 = v$.

Then there is a chain map

$$\rho : C_*^+(P,K) \longrightarrow C_*(P,K)$$

s.t. $\rho(x_0 \ldots x_k) = \begin{cases} x_0 \ldots x_k, & x_0 \neq x_1 \\ 0 & x_0 = x_1. \end{cases}$

satisfying $\rho_\# j_\# = \text{identity}$.

The contracting chain homotopy is given by $D_q(v_0 \ldots v_q) = \rho(vv_0 \ldots v_q)$.

<u>Claim</u>: $d_{q+1} D_q(x) + D_{q-1} d_q(x) = \begin{cases} x & q \neq 0 \\ x - \varepsilon(x)v & q = 0. \end{cases}$

As usual, it suffices to show this holds for the standard free generators.

But now we have

$$d_{q+1} D_q (v_0 \ldots v_q) = d_{q+1} \rho (v v_0 \ldots v_q) =$$

$$\rho\, d_{q+1} (v v_0 \ldots v_q) =$$

NOTE SIGN!

$$\rho(v_0 \ldots v_q) - \rho(v v_1 \ldots v_q) + \sum_{i \geq 1} (-1)^{i+1} \rho(v v_0 \ldots \hat{v}_i \ldots v_q)$$

and also

$$D_{q-1}\, d_q (v_0 \ldots v_q) =$$

$$D_{q-1} \left(v_1 \ldots v_q + \sum_{i \geq 1} (-1)^i v_0 \ldots \hat{v}_i \ldots v_q \right) =$$

$$\rho(v v_1 \ldots v_q) + \sum_{i \geq 1} (-1)^i \rho(v v_0 \ldots \hat{v}_i \ldots v_q)$$

if $q > 0$. Add these expressions to see that $dD + Dd (v_0 \ldots v_q) = \rho(v_0 \ldots v_q) = v_0 \ldots v_q$.

Now suppose that $q = 0$, so that $D_{q-1} = 0$.

In that case we have

$$d_{q+1} D_q (v_0) = \rho\, d_{q+1} (v v_0) =$$

$$v_0 - v = v_0 - \varepsilon(v_0) v.$$