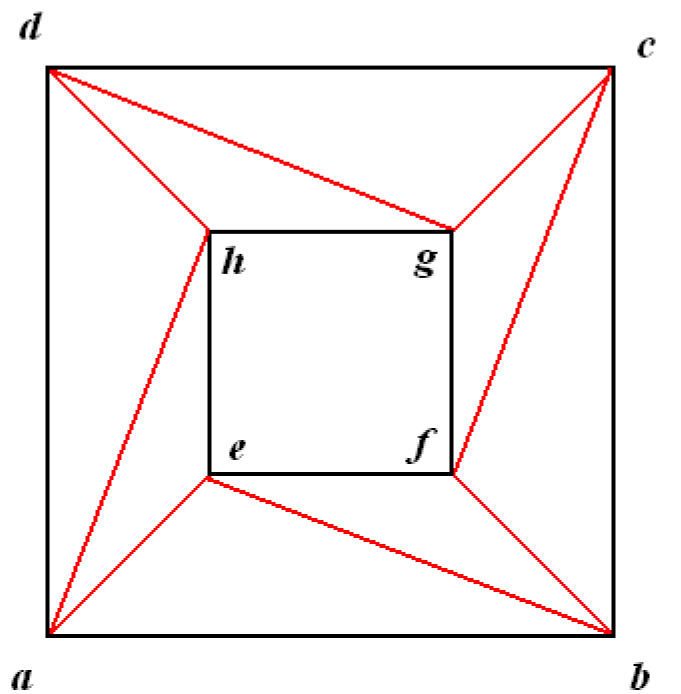
# Green's Theorem for multiply connected regions

Simplicial chains are implicit in the standard derivation of Green's Theorem for closed non – simply connected regions in the plane. For example, consider a region $A$ in the plane bounded by concentric squares as in the illustration below. If we take the norm $|\mathbf{x}|_{\max}$ on $\mathbb{R}^2$ whose value at $\mathbf{x} = (x_1, x_2)$ is the maximum of $|x_1|$ and $|x_2|$, then $A$ is the set of all points satisfying $r < |\mathbf{x}|_{\max} < s$ for a suitable pair of positive real numbers $r$ and $s$. In particular, it follows that $A$ is a deformation retract of either boundary component, and since each of the latter is homeomorphic to $S^1$ we see that $A$ is definitely not simply connected.




If $\mathbf{F} = (P, Q)$ is a vector field which is defined on an open neighborhood of $A$, then the appropriate version of Green's Theorem for the region $A$ is given as follows:

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \;=\; \oint_{abcd} Pdx + Qdy \;-\; \oint_{efgh} Pdx + Qdy$$

The standard explanation of this in multivariable calculus books is that that Green's Theorem depends upon finding a triangulation as in the drawing. Specifically, the arguments in the calculus books yield Green's Theorem for each of the triangular regions in the decomposition, and if we add up the integrals associated to the various regions we obtain the double integral over $A$ on one side of the equation and the difference of the line integrals over the boundary curves on the other side.

This process of adding up integrals over the pieces can be viewed as a statement about simplicial chains for the triangulation in the drawing. Suppose that we order the vertices lexicographically (in simpler language, alphabetically). Then the counterclockwise line integrals over the triangular regions are given by the following table:

| Triangular Region | Counterclockwise boundary curve | Vertex order |
|---|---|---|
| $abe$ | $ab + be - ae$ | + |
| $ebf$ | $bf - ef - eb$ | – |
| $bcf$ | $bc + cf - bf$ | + |
| $fcg$ | $cg - fg - cf$ | – |
| $gcd$ | $cd + dg - cg$ | + |
| $gdh$ | $dh - gh - dg$ | – |
| $hda$ | $ah - dh - ad$ | – |
| $eha$ | $ae + eh - ah$ | + |

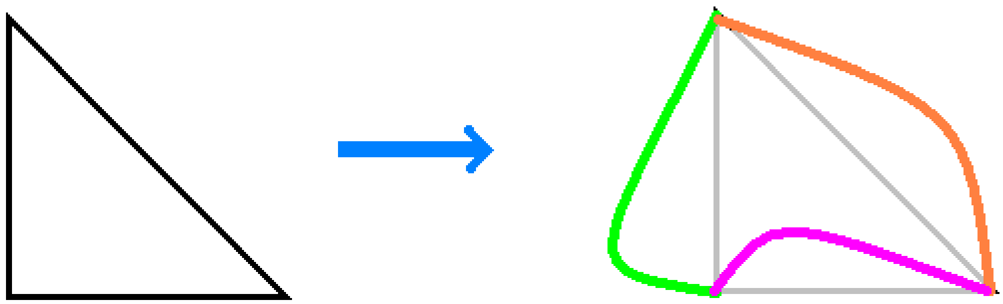
The third column of the table indicates whether an even or odd permutation of three letters is needed to put the vertices of the region into alphabetical order. This potential change of signs is necessary in order to obtain the expressions in the middle columns as the boundaries of the various pieces, and from this viewpoint the basic fact about decompositions is summarized in the following identity:

$$d(abe - bef + bcf - cfg + cdg - dgh - adh + aeh) =$$

$$(ab + bc + cd - ad) - (ef + fg + gh - eh)$$

Here is another explanation of the need for signs. If $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are the vertices of a $2-$ simplex $K$, then there is a standard homeomorphism from the simplex $\Delta$ in the plane with vertices $\mathbf{0}$, $\mathbf{e}_1$, and $\mathbf{e}_2$ to $K$ which sends $\mathbf{0}$, $\mathbf{e}_1$, and $\mathbf{e}_2$ to $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ respectively; explicitly, the point $(s, t)$ goes to $(1 - s - t)\,\mathbf{x} + s\,\mathbf{y} + t\,\mathbf{z}$. When $K$ also lies in the coordinate plane, this map has a Jacobian which will be either positive or negative at all points. If $C$ is the counterclockwise boundary of $\Delta$, then the standard homeomorphism maps $C$ to the boundary of $K$ in a counterclockwise sense if the Jacobian is positive and in a clockwise sense if the Jacobian is negative. In the latter case we must do something in order to ensure that the boundary curve will be counterclockwise, and we do this formally in the simplicial chain group by taking the negative of the simplex $K$.

***All of the preceding goes through for arbitrary closed regions whose boundaries are unions of finitely many regular, simple, piecewise smooth curves.*** The basic idea is to show first that the region can be nicely triangulated as a polygonal region and then to prove that Green's Theorem remains true under suitable changes of variables. A typical example appears on the next page.



(**<u>Note:</u>** The triangular region on the left maps to the region whose boundary curves are drawn in various non-gray colors. It is assumed that the coordinate functions all have continuous partial derivatives of sufficiently high order. One can then combine Green's Theorem for the region on the left with standard change of variables formulas for line integrals and double integrals to prove that Green's Theorem also holds for the region on the right.)

In order to make this discussion rigorous, one needs input from algebraic topology to prove that for an arbitrary polyhedral region one has a simplicial $2-$chain whose algebraic boundary is given by the geometric boundary curves with the "correct" orientations. However, in this course it is not possible to develop all the tools necessary to formulate and prove such a generalization of Green's Theorem.