Proof of the Long Exact Sequence Thm.

Six parts (paired)

Exactness at $H_q(B)$ $\quad \operatorname{Im} i_* \subseteq \operatorname{Ker} j_*$

$\operatorname{Ker} j_* \subseteq \operatorname{Im} i_*$

Exactness at $H_q(C)$ $\quad \operatorname{Im} j_* \subseteq \operatorname{Ker}\partial$

$\operatorname{Ker}\partial \subseteq \operatorname{Im} j_*$

Exactness at $H_q(A)$ $\quad \operatorname{Im}\partial \subseteq \operatorname{Ker} i_*$

$\operatorname{Ker} i_* \subseteq \operatorname{Im}\partial.$

$\operatorname{Im} i_* \subseteq \operatorname{Ker} j_*$ $\quad j \circ i = 0 \Rightarrow$

$0 = (j \circ i)_* = j_* i_*$, so $i_* x \in \operatorname{Ker} j_*$.

$\operatorname{Ker} j_* \subseteq \operatorname{Im} i_*$ Suppose $j_*(u) = 0$, and write $u = [x]$ for $x \in B_q$. Then $j(x) = dy$ for some $y \in C_q$. Choose Y so $j(Y) = y$. Consider $x - dY \in B_q$; note that $[x - dY] = [x]$. Then $j(x - dY) = j(x) - j\,d(Y) = j(x) - d\,j(Y) = dy - dy = 0.$

Hence $x - j(dY) = i(z)$, some $z \in A_q$.

CLAIM $dz = 0$; if so, then $[x] = i_*[z]$.

But $dz = 0 \iff i\,dz = 0$ and $i\,dz = d\,i(z) = d(x - j\,dY) = dx - d\,j\,dY = 0 - j\,d\,dY = 0 - 0 = 0$.

Image $j_* \subseteq$ Kernel ∂ Let $u \in H_q(B)$, $u = [x]$, and consider $\partial j_*(u) = \partial[jx]$. Since $x \longrightarrow j(x) \in C_q$, we know that this class is given by $[z]$, where $i(z) = dx$. But $u = [x]$ means $dx = 0$, so that $z = 0$ and hence $\partial j_*(u) = 0$ and $j_* u \in \operatorname{Ker} \partial$.

Kernel $\partial \subseteq$ Image j_* Suppose $\partial[x] = 0$ for suitable $x \in C_q$. This means that $z = dw$ for some $w \in A_q$ and hence $dy = i(dw)$, where $y \in B_q \longrightarrow x$.

Modify y to $y' = y - iw$. Then $dy' = 0$ and $j(y') = x \Rightarrow u = [x]$ is $j_*[y']$.

Image $\partial \subseteq$ Ker i_* Let $v \in H_{q+1}(C)$, $v = [x]$, and consider $i_*\partial[v] = i_*[z]$ $= [i(z)] = [dy] = 0$.

Kernel $i_* \subseteq$ Image ∂ Suppose $i_*[x] = 0$ for suitable $x \in A_q$. Then $0 = [i(x)]$ $\Rightarrow i(x) = dy$, some $y \in B_{q+1}$. Let $w = j(y)$. Then $dw = \cancel{jdy} = djy = jdy = ji(x) = 0$, and it follows that $\partial[w] = [x]$ by the definition of ∂.