Notes for Unit V

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Supplementary notes for Mathematics 145B, Spring 2015

V. Further topics

One reason for studying advanced mathematical concepts is to see how they shed light on issues which arose in previous courses or on problems of independent interest. We shall consider one example of each type in the final unit of the course, and these notes will be the primary reference.

V.1 : Homotopy of paths and line integrals

(Munkres, 56)

We shall begin by recalling some material from multivariable calculus.

Let U be an open subset of $\mathbb{R}^2$, and let $\Gamma : [a,b] \to U$ be a *regular smooth parametrized curve*; more precisely, we assume that the coordinate functions of Γ all have continuous derivatives and that the tangent vector $\Gamma'(t) = (x'(t), y'(t))$ is never equal to $(0,0)$; the tangent vector will also be called the *derivative* of Γ in numerous discussions. If P and Q are functions defined on U with continuous first partial derivatives, then the line integral of (P,Q) along γ is given by

$$\int_\Gamma P\,dx \;+\; Q\,dy \quad = \quad \int_a^b \big(\, P(\Gamma(t))x'(t) \;+\; Q(\Gamma(t))y'(t)\big)\,dt$$

where the right hand side is known to exist because the integrand is continuous on $[a,b]$.

For the most part, we shall work with a more general class of curves $\Gamma : [a,n] \to U$, where U is open in $\mathbb{R}^n$ and Γ is a *piecewise regular smooth curve*; *i.e.*, there is a partition of $[0,1]$ into subintervals $J_1, \cdots, J_m$ such that the restriction of Γ to each J_α has a continuous derivative (= tangent vector) which is never zero. The boundary curve of a square in the counterclockwise sense is a typical example, and many others arise in ordinary and multivariable calculus. One important point to note about these curves is that if z is a common endpoint of two subintervals J_α and $J_{\alpha+1}$, then the tangent vectors at z coming from the two subintervals need not be equal. However, it is still possible to define a line integral for (P,Q) along such a curve by taking the line integrals over the pieces J_α, on each of which Γ is a regular smooth curve, and adding them together; strictly speaking, this requires a lemma which verifies that the value does not depend upon the way in which the large interval is split into smaller pieces, but there are several standard ways of doing this in a mathematically rigorous treatment of the Riemann integral.

One obvious benefit of considering piecewise smooth curves is that *the concatenation of two regular piecewise smooth curves is also a regular piecewise smooth curve*. The standard additivity properties of Riemann integrals then yield the following observation:

(1) *If the curve γ is obtained by concatenating α and β, then the line integral of $P\,dx + Q\,dy$ over γ is equal to the sum of the corresponding line integrals over α and β.*

We also have the following elementary identities:

(2) *If the curve $-\gamma$ is obtained by reversing the direction of γ with $-\gamma(t) = \gamma(a+b-x)$, then the line integral of $P\,dx + Q\,dy$ over $-\gamma$ is equal to the negative of the corresponding line integral over γ.*

(3) *If* C *is a constant curve, then the line integral of $P\,dx + Q\,dy$ over* C *is zero.*

REMINDER: Concatenation does **NOT** satisfy a commutativity law $\alpha + \beta = \beta + \alpha$ (for example, the concatenation in one order does not imply that the curves can be concatenated in the opposite order) or an associativity law $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$, but we shall see that the construction is associative up to homotopy.

SECOND REMINDER. Some books and papers define $\alpha + \beta$ so that the first part of the curve is β and the last part is α. Each convention has advantages and disadvantages, but in any case it is good to recognize which convention is used in a particular reference in order to avoid misinterpreting some statements.

THIRD REMINDER. We have chosen to use a plus sign (+) for concatenation of curves because of the clear analogies between this concept and concatenation of string variables in some computer languages; since the latter operation is not commutative (*e.g.*, if `A$ = ‘‘a’’` and if `B$ = ‘‘b’’` then `A$ + B$` $\neq$ `B$ + A$`), there is a strong precedent for using a plus sign to denote such a noncommutative operation.

Path independence of line integrals

Let U be an open connected subset of $\mathbb{R}^2$, and let P and Q be real valued functions on U with continuous partial derivatives. In multivariable calculus one learns that certain choices of P and Q the line integrals

$$\int_\Gamma P\,dx \; + \; Q\,dy$$

depend only on the endpoints of Γ. The simplest examples are those for which the integrands satisfy

$$P \;=\; \frac{\partial f}{\partial x}\,, \qquad Q \;=\; \frac{\partial f}{\partial y}$$

for some smooth real valued function f defined on U. In such cases one can use the Fundamental Theorem of Calculus, the chain rule for partial differentiation, and the definition of a line integral to conclude that

$$\int_\Gamma P\,dx \; + \; Q\,dy \;=\; f\circ\Gamma(1) \; - \; f\circ\Gamma(0)\ .$$

More generally, if

$$\frac{\partial P}{\partial y} \;=\; \frac{\partial Q}{\partial x}$$

and P and Q have continuous partial derivatives on U, then the path dependence of this integral is a nontrivial issue in multivariable calculus; although the integral may depend upon path, examples show that the integral is the same for large families of closely related curves.

Standard Examples. Consider the line integral

$$\int_\Gamma \frac{x\,dy - y\,dx}{x^2 + y^2}$$

over the counterclockwise unit circle $(\cos t, \sin t)$ for $0 \le t \le 2\pi$. Direct computation shows that the value obtained is 2π, and in fact one obtains the same answer for every curve which has a polar coordinate parametrization of the form $r = f(\theta), \theta = \theta$, where $\theta \in [0, 2\pi]$ and f is positive valued function with a continuous derivative (we shall prove this later). However, if we consider the corresponding line integral over the counterclockwise circle of radius $\frac{1}{3}$ centered at $(\frac{2}{3}, 0)$ with parametrization

$$x(t) \quad = \quad \frac{2}{3} + \frac{1}{3}\cos t, \qquad y(t) \quad = \quad \frac{1}{3}\sin t \qquad (0 \le t \le 2\pi)$$

then direct computation shows that the integral's value is zero. On the other hand, further study shows that one obtains the same value of 2π for all circular curves in $\mathbb{R}^2 - \{\mathbf{0}\}$ which contain $\mathbf{0}$ in their interior, and one obtains the same value of 0 for all curves which lie in the open half-plane defined by $x > 0$.

It is natural to ask the extent to which the line integral varies with the choice of path, and the basic results in this direction are sometimes stated without proof (or even complete definitions) in some multivariable calculus texts. In fact, a precise formulation and proof of such results involve homotopy classes of curves, so in this section we shall state and prove the basic results on this topic for open subsets in $\mathbb{R}^2$. Similar results also hold in $\mathbb{R}^n$ for $n \ge 3$, but formulating and proving them would require the development of additional background material; for the sake of completeness, we note that Section V.6 in the document

`http://math.ucr.edu/~res/math246A-2012/advancednotes2014.pdf`

summarizes how one can handle higher dimensional cases using more sophisticated techniques.

For our purposes it will be convenient to center the exposition around the following version of the main results:

THEOREM 1. *Let U be a connected open subset of $\mathbb{R}^2$, and let P and Q be smooth functions on U with continuous partial derivatives which satisfy the condition*

$$\frac{\partial P}{\partial y} \quad = \quad \frac{\partial Q}{\partial x}$$

at all points of U. If Γ is a piecewise smooth closed curve which starts and ends at $\mathbf{p}_0 \in U$ which is base point preservingly homotopic to a constant in U, then

$$\int_\Gamma P\,dx + Q\,dy \quad = \quad 0.$$

Before proving this result, we shall state a few alternate versions and show how they follow from Theorem 1.

THEOREM 2. *Let U, P, Q be given as in Theorem 1, but suppose now that Γ and Γ' are two piecewise smooth curves in U with the same endpoints $\mathbf{p}$ and $\mathbf{q}$ such that Γ and Γ' are homotopic by an endpoint preserving homotopy. Then*

$$\int_\Gamma P\,dx + Q\,dy \quad = \quad \int_{\Gamma'} P\,dx + Q\,dy\ .$$

Proof that Theorem 1 implies Theorem 2. In the setting of Theorem 2 the curve $\Gamma' + (-\Gamma)$ is a closed piecewise smooth curve that is homotopic to a constant because $\Gamma \simeq \Gamma'$ implies $[\Gamma' + (-\Gamma)] =$

$[\Gamma + (-\Gamma)] = [\text{constant}]$. Therefore Theorem 1 implies that the line integral over this curve is zero. On the other hand, by the three properties of line integrals listed above, the line integral over $\Gamma' + (-\Gamma)$ is equal to the difference of the line integrals over Γ' and Γ. Combining these observations, we see that the line integrals over Γ' and Γ must be equal.∎

The next result is often also found in multivariable calculus texts.

COROLLARY 3. *If in the setting preceding theorems we also know that the region U is* simply connected, *then the following hold:*

(i) *for every piecewise smooth closed curve Γ in U we have*

$$\int_{\Gamma} P\,dx + Q\,dy \quad = \quad 0$$

(ii) *for every pair of piecewise smooth curves Γ, Γ' with the same endpoints we have*

$$\int_{\Gamma} P\,dx + Q\,dy \quad = \quad \int_{\Gamma'} P\,dx + Q\,dy.$$

The first part of the corollary follows from the triviality of the fundamental group of U, the conclusion of Theorem 1, and the triviality of line integrals over constant curve. The second part follows formally from the first in the same way that the Theorem 2 follows from Theorem 1.∎

Finally, we have the following result, which plays a fundamental role in the theory of functions of one complex variable.

THEOREM 4. *Let U, P, Q be given as in Theorems 1 and 2, but suppose now that Γ and Γ' are two piecewise smooth closed curves in U such that Γ and Γ' are* freely *homotopic. Then*

$$\int_{\Gamma} P\,dx + Q\,dy \quad = \quad \int_{\Gamma'} P\,dx + Q\,dy\ .$$

This proof will require some additional input, so the argument will be postponed until after the proof of Theorem 1 is completed.

Background from multivariable calculus

The following result can be found in many multivariable calculus textbooks.

THEOREM 5. *Let U be a rectangular open subset of the coordinate plane of the form $(a_1, b_1) \times (a_2, b_2)$ where each factor is an open interval in the real line, let P and Q be functions on U with continuous partial derivatives on U such that*

$$\frac{\partial P}{\partial y} \quad = \quad \frac{\partial Q}{\partial x}$$

and let Γ and Γ' be two piecewise smooth curves in U with the same endpoints. Then

$$\int_{\Gamma} P\,dx + Q\,dy \quad = \quad \int_{\Gamma'} P\,dx + Q\,dy.$$

Sketch of proof. The underlying idea behind the proof is to construct a function f such that $\nabla f = (P, Q)$; if this can be done then as before the result will follow from the Fundamental Theorem of Calculus and the chain rule for partial differentiation. We start with an arbitrary point (x_0, y_0) in U; given $(x, y) \in U$, consider the following two broken line curves in U:

(HV) Take the horizontal line segment curve from (x_0, y_0) to (x, y_0) and concatenate it with the vertical line segment from (x, y_0) to (x, y). If either $x_0 = x$ or $y_0 = y$ then the corresponding line segment curve is a constant curve.

(VH) Take the the vertical line segment from (x_0, y_0) to (x_0, y) and concatenate it with the horizontal line segment from (x_0, y) to (x, y). If either $x_0 = x$ or $y_0 = y$ then the corresponding line segment curve is a constant curve.

The curve VH$+(-$HV$)$ traces the boundary of a solid rectangle contained in U, and thus we can use the condition on partial derivatives along with Green's Theorem to conclude that the line integral along this curve is zero. This means that

$$\int_{\mathrm{VH}} P\,dx + Q\,dy \quad = \quad \int_{\mathrm{HV}} P\,dx + Q\,dy \ .$$

Define $f(x, y)$ to be the common value of these two integrals. One can now use Green's Theorem to derive the identity $\nabla f(x, y) = (\,P(x, y),\ Q(x, y)\,)$ for $(x, y) \in U$.■

Broken line inscriptions

We begin by reviewing some standard definitions. Given two points $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$ in $\mathbb{R}^n$, the *closed straight line segment* joining them is the curve $[\mathbf{p}, \mathbf{q}]$ defined by $(1-t)\mathbf{p} + t\mathbf{q}$ over the interval $[0, 1]$.

A broken line curve corresponding to an ordered sequence of points

$$\mathbf{p}_0, \ \mathbf{p}_1, \ \cdots \ \mathbf{p}_m$$

is obtained by joining $\mathbf{p}_0$ to $\mathbf{p}_1$ by a straight line segment $[\mathbf{p}_0\mathbf{p}_1]$, then joining $\mathbf{p}_1$ to $\mathbf{p}_2$ by a straight line segment $[\mathbf{p}_1\mathbf{p}_2]$, and so on. The points $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$, *etc.* are called the *vertices* of the broken line curve. One technical problem with this involves the choices of linear parametrizations for the pieces. However, since line integrals for such curves do not depend upon such parametrizations and in fact we have

$$\int_C P\,dx + Q\,dy \quad = \quad \sum \int_{[\mathbf{p}_{i-1}, \mathbf{p}_i]} P\,dx + Q\,dy$$

we shall not worry about the specific choice of parametrization. Filling in the details will be left as an exercise to a reader who is interested in doing so; this is basically elementary but tedious.

We shall be considering broken line approximations to a piecewise smooth curve, and this requires a few more definitions. A *partition* of the interval $[a, b]$ is a sequence of points

$$\Delta : a \quad = \quad t_0 \ < \ t_1 < \ \cdots \ < t_m \quad = \quad b$$

and the *mesh* of Δ, written $|\Delta|$, is the maximum of the differences $t_i - t_{i-1}$ for $1 \leq i \leq m$. Given a piecewise smooth curve Γ defined on $[a, b]$, the *broken line inscription* $\mathsf{Lin}\,(\Gamma, \Delta)$ is the broken line curve with vertices

$$\Gamma(a) \ = \ \Gamma(t_0), \ \Gamma(t_1), \ \cdots \ \Gamma(t_m) \ = \ \Gamma(b) \ .$$

We are now ready to prove one of the key technical steps of the proof of the main result.

LEMMA 6. *Let U, P, Q, Γ be as usual, where Γ is defined on $[a,b]$ and P and Q satisfy the condition*

$$\frac{\partial P}{\partial y} \quad = \quad \frac{\partial Q}{\partial x} \ .$$

Then there is a positive constant $\delta > 0$ such that the following hold for all partitions Δ of $[a,b]$ with $|\Delta| < \delta$:

(i) *The curves Γ and $\mathsf{Lin}\,(\Gamma, \Delta)$ are endpoint preservingly homotopic.*

(ii) *We have a line integral identity of the form*

$$\int_\Gamma P\,dx + Q\,dy \quad = \quad \int_{\mathsf{Lin}\,(\Gamma,\Delta)} P\,dx + Q\,dy \ .$$

Proof. If K is the image of Γ then K is a compact subset of the open set U, and therefore there is an $\varepsilon > 0$ so that if $\mathbf{x} \in \mathbb{R}^2$ satisfies $|\mathbf{x} - \mathbf{v}| < \varepsilon$ for some $\mathbf{v} \in K$ then $\mathbf{x} \in U$. It follows that if $\mathbf{v} \in K$ then the inner region for the square centered at $\mathbf{v}$ with sides parallel to the coordinate axes of length $\varepsilon\sqrt{2}$ lies entirely in U.

By uniform continuity there is a $\delta > 0$ so that if $s, t \in [a,b]$ satisfy $|s - t| < \delta$ then

$$|\Gamma(s) - \Gamma(t)| \quad < \quad \frac{\varepsilon\sqrt{2}}{2} \ .$$

Let Δ be a partition of $[a,b]$ whose mesh is less than δ. Then for all i the restriction of Γ to $[t_{i-1}, t_i]$ lies in the open disk of radius $\frac{1}{2}\varepsilon\sqrt{2}$. It follows that both this restriction and the closed straight line segment joining $\Gamma(t_{i-1})$ to $\Gamma(t_i)$ lie in the open square region centered at $\Gamma(t_{i-1})$ with sides parallel to the coordinate axes of length of length $\varepsilon\sqrt{2}$. The first conclusion of the lemma now follows because the image of the straight line homotopy from Γ to $\mathsf{Lin}\,(\Gamma, \Delta)$ will be contained in U.

We shall now prove the second conclusion in the lemma. Since the open squares lie entirely in U. it follows that P and Q are defined on these square regions. Therefore, by Theorem 5 we have

$$\int_{\Gamma|[t_{i-1},t_i]} P\,dx + Q\,dy \quad = \quad \int_{[\Gamma(t_{i-1}),\Gamma(t_i)]} P\,dx + Q\,dy$$

for each i. But the line integral over Γ is the sum of the line integrals over the curves $\Gamma|[t_{i-1}, t_i]$, and the line integral over the broken line inscription is the sum of the line integrals over the line segments $[\Gamma(t_{i-1}), \Gamma(t_i)]$, and therefore it follows that the line integral over Γ is equal to the line integral over the broken line inscription, as required.∎

We shall also need a version of Lemma 6 with weaker hypotheses on Γ.

LEMMA 7. *Let U, P, Q, Γ be as in Lemma 6, but now assume only that Γ is a continuous curve. Then there is a positive constant $\delta > 0$ such that the following hold for all partitions Δ of $[a,b]$ with $|\Delta| < \delta$:*

(i) *The curves Γ and $\mathsf{Lin}\,(\Gamma, \Delta)$ are endpoint preservingly homotopic.*

(*ii*) *The values of the line integral*

$$\int_{\mathsf{Lin}\,(\Gamma,\Delta)} P\,dx + Q\,dy$$

are equal for all choices of Δ *such that* $|\Delta| < \delta$.

This lemma can be used to define a **formal value** for the line integral

$$\int_{\Gamma} P\,dx + Q\,dy$$

even if Γ is not a rectifiable curve, provided P and Q satisfy the condition on partial derivatives. Namely, we can take the line integral of some broken line inscription satisfying the condition in Lemma 7.

WARNING. The formal integral is only defined for integrands of the form $P\,dx + Q\,dy$ such that partial derivatives satisfy $Q_x = P_y$.

Proof of Lemma 7. We can prove the first part exactly as in Lemma 6, so it is only necessary to show that the line integrals over all the broken line approximations $\mathsf{Lin}\,(\Gamma, \Delta)$ are equal if $|Delta| < \delta$.

We shall first prove this if one partition is a refinement of the other; as usual, a partition Δ' is said to be a *refinement* of Δ if every partition point in Δ is also a partition point of Δ'; it follows immediately that the mesh of Δ' is no greater than the mesh of Δ. Every refinement can be viewed as the composite of a sequence of elementary refinements

$$\Delta \;=\; \Delta_0 \;<\; \Delta_1 \;<\; \cdots \;<\; \Delta_m \;=\; \Delta'$$

such that Δ_j is obtained from Δ_{j-1} by adding a single point, and therefore by an inductive argument it suffices to prove that the line integrals over $\mathsf{Lin}\,(\Gamma, \Delta)$ and $\mathsf{Lin}\,(\Gamma, \Delta')$ are equal if Δ' is obtained from Δ by adding a single point.

The additional partition point u lies between t_{j-1} and t_j for some j, and it follows that the difference between the line integral over $\mathsf{Lin}\,(\Gamma, \Delta)$ and $\mathsf{Lin}\,(\Gamma, \Delta')$ is given by

$$D \;=\; \int_{[\Gamma(t_{j-1})\Gamma(t_j)]} \Omega \;-\; \int_{[\Gamma(t_{j-1})\Gamma(u)]} \Omega \;-\; \int_{[\Gamma(u)\Gamma(t_j)]} \Omega$$

where $\Omega = P\,dx + Q\,dy$. Since the mesh of Δ is small, it follows that the image of $[t_{j-1}t_j]$ under Γ is a small open square. Therefore Theorem 5 implies the path independence identity

$$\int_{[\Gamma(t_{j-1})\Gamma(t_j)]} \Omega \;=\; \int_{[\Gamma(t_{j-1})\Gamma(u)]} \Omega \;+\; \int_{[\Gamma(u)\Gamma(t_j)]} \Omega\;.$$

If we combine these observations, we see that $D = 0$, and as noted above this implies that the line integrals over $\mathsf{Lin}\,(\Gamma, \Delta)$ and $\mathsf{Lin}\,(\Gamma, \Delta')$ are equal.

In general, if we are given two partitions Δ and Δ' there is a third partition Δ^* which is a refinement of both; it suffices to take the partition whose partition points are the union of the partition points for Δ and Δ'. By the preceding paragraph, we then know that the line integrals over both $\mathsf{Lin}\,(\Gamma, \Delta)$ and $\mathsf{Lin}\,(\Gamma, \Delta')$ are equal to the line integral over $\mathsf{Lin}\,(\Gamma, \Delta^*)$, and hence it follows that the first two line integrals are equal, which is what we wanted to prove.■

We shall prove these in order.

Proof of Theorem 1. Let $H : [0,1] \times [0,1] \to U$ be a continuous map such that $H(s,0) = \Gamma(s)$ for all s and H is constant on both $[0,1] \times \{1\}$ and $\{0,1\} \times [0,1]$.

If L is the image of H then L is a compact subset of the open set U, and as in the proof of the lemma there is an $\varepsilon' > 0$ so that if $\mathbf{x} \in \mathbb{R}^2$ satisfies $|\mathbf{x} - \mathbf{v}| < \varepsilon'$ for some $\mathbf{v} \in L$ then $\mathbf{x} \in U$. It follows that if $\mathbf{v} \in L$ then the inner region for the square centered at $\mathbf{v}$ with sides parallel to the coordinate axes of length $\varepsilon'\sqrt{2}$ lies entirely in U.

By uniform continuity there is a $\delta' > 0$ so that if $\mathbf{s}, \mathbf{t} \in [0,1] \times [0,1]$ satisfy $|\mathbf{s} - \mathbf{t}| < \delta'$ then

$$|H(\mathbf{s}) - H(\mathbf{t})| \quad < \quad \frac{\varepsilon'\sqrt{2}}{2} \; .$$

Without loss of generality we may assume that δ' is no greater than the δ in the previous lemma. Let Δ be a partition of $[a,b]$ whose mesh is less than $\frac{1}{2}\delta'\sqrt{2}$, and choose a positive integer N such that

$$\frac{1}{N} \quad < \quad \frac{\delta'\sqrt{2}}{2}.$$

Then for all i such that $1 \leq i \leq m$ and all j such that $1 \leq j \leq N$ the restriction of H to $[t_{i-1}, t_i] \times [\frac{j-1}{N}, \frac{j}{N}]$ lies in an open disk of radius $\frac{1}{2}\varepsilon'\sqrt{2}$.

A special case. To motivate the remainder of the argument, we shall first specialize to the case where H extends to a map on an open set containing the square $[0,1] \times [0,1]$ and has continuous partial derivatives on this open set. For each i such that $0 \leq i \leq m$ and each j such that $1 \leq j \leq N$ let $A(i,j)$ be the broken line curve in the square with vertices

$$(0, \tfrac{j-1}{N}), \; \ldots \; (t_i, \tfrac{j-1}{N}), \; (t_i, \tfrac{j}{N}), \; \ldots \; (1, \tfrac{j}{N}) \; .$$

In other words, this curve is formed by starting with a horizontal line segment from $(0, \frac{j-1}{N})$ to $(t_i, \frac{j-1}{N})$, then concatenating with a vertical line segment from $(t_i, \frac{j-1}{N})$ to $(t_i, \frac{j}{N})$, and finally concatenating with a horizontal line segment from $(t_i, \frac{j}{N})$ to $(1, \frac{j}{N})$. If $W(i,j)$ denotes the composite $H \circ A(i,j)$, then it follows that $W(i,j)$ is a piecewise smooth closed curve in U. Furthermore, $W(m,1)$ is just the concatenation of Γ with a constant curve and $W(0,N)$ is just a constant curve, so the proof of the main result reduces to showing that the line integrals of the expression $P\,dx + Q\,dy$ over the curves $W(m,1)$ and $W(0,N)$ are equal. We claim this will be established if we can show the following hold for all i and j:

(1) The corresponding line integrals over the curves $W(0, j-1)$ and $W(m,j)$ are equal.

(2) The corresponding line integrals over the curves $W(i-1, j)$ and $W(i,j)$ are equal.

To prove the claim, first note that (2) implies that the value of the line integral over $W(i,j)$ is a constant z_j that depends only on j, and then note that (1) implies $z_{j-1} = z_j$ for all j. Thus the two assertions combine to show that the line integrals over all the curves $W(i,j)$ have the same value.

We begin by verifying (1). Since H is constant on $\{0,1\} \times [0,1]$, it follows that $W(m,j)$ is formed by concatenating $H|[0,1] \times \{\frac{j}{m}\}$ and a constant curve (in that order), while $W(0, j-1)$ is formed by concatenating a constant curve and $H|[0,1] \times \{\frac{j}{m}\}$ (again in the given order). Thus the line integrals over both $W(0, j-1)$ and $W(m, j)$ are equal to the line integral over $H|[0,1] \times \{\frac{j}{m}\}$, proving (1).

Turning to (2), since the broken line curves $A(i,j)$ and $A(i-1,j)$ differ only by one vertex, it follows that the difference

$$\int_{W(i,j)} P\,dx + Q\,dy \;-\; \int_{W(i-1,j)} P\,dx + Q\,dy$$

is equal to

$$\int_{V(i,j)} P\,dx + Q\,dy \;-\; \int_{V'(i,j)} P\,dx + Q\,dy$$

where $V(i,j)$ is the composite of H with the broken line curve with vertices

$$\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad \left(t_i, \tfrac{j-1}{N}\right), \quad \left(t_i, \tfrac{j}{N}\right)$$

and $V'(i,j)$ is the composite of H with the broken line curve with vertices

$$\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad \left(t_{i-1}, \tfrac{j}{N}\right), \quad \left(t_i, \tfrac{j}{N}\right).$$

Our hypotheses imply that both of these curves lie in an open disk of radius $\frac{1}{2}\varepsilon'\sqrt{2}$ and thus also in the open square centered at $\mathbf{v}$ with sides parallel to the coordinate axes of length $\varepsilon'\sqrt{2}$; by construction the latter lies entirely in U. Therefore by the previously quoted result from multivariable calculus we have

$$\int_{V(i,j)} P\,dx + Q\,dy \;=\; \int_{V'(i,j)} P\,dx + Q\,dy$$

for each i and j, so that the difference of the line integrals vanishes. Since this difference is also the difference between the line integrals over $W(i,j)$ and $W(i-1,j)$, it follows that the line integrals over the latter two curves must be equal.

The general case. If H is an arbitrary continuous function the preceding proof breaks down because we do not know if the continuous curves $W(i,j)$ are well enough behaved to define line integrals. We shall circumvent this by using broken line approximations to these curves and appealing to the previous lemma to relate the value of the line integrals over these approximations to the value on the original curve. Since the proof is formally analogous to that for the special case we shall concentrate on the changes that are required.

Let $X(i,j)$ denote the broken line curve with vertices

$$H(0, \tfrac{j-1}{N}),\ \ldots\ H(t_i, \tfrac{j-1}{N}),\ H(t_i, \tfrac{j}{N}),\ \ldots\ H(1, \tfrac{j}{N}).$$

By our choice of Δ these broken lines all lie in U, and the constituent segments all lie in suitably small open disks inside U.

We claim that it will suffice to prove that the line integrals over the curves $X(0, j-1)$ and $X(m, j)$ are equal for all j and for each j the corresponding line integrals over the curves $X(i-1, j)$ and $X(i,j)$ are equal. As before it will follow that the line integrals over all the broken line curves $X(i,j)$ have the same value. But $X(m, N)$ is a constant curve, so this value is zero. On the other hand, by construction the curve $X(m, 1)$ is formed by concatenating $\mathsf{Lin}\,(\Gamma, \Delta)$ and a constant curve, so this value is also the value of the line integral over over Γ and $\mathsf{Lin}\,(\Gamma,\Delta)$ are equal, and therefore the value of the line integral over the original curve Γ must also be equal to zero.

The first set of equalities follow from the same sort argument used previously for $W(0, j-1)$ and $W(m, j)$ with the restriction of Γ replaced by the broken line curve with vertices

$$H(0, \tfrac{j}{N}), \ \ldots\ H(1, \tfrac{j}{N}).$$

To verify the second set of equalities, note that the difference between the values of the line integrals over $X(i, j)$ and $X(i-1, j)$ is given by

$$\int_{C(i,j)} P\,dx + Q\,dy \ - \ \int_{C'(i,j)} P\,dx + Q\,dy$$

where $C(i, j)$ is the broken line curve with vertices

$$H\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad H\left(t_i, \tfrac{j-1}{N}\right), \quad H\left(t_i, \tfrac{j}{N}\right)$$

and $C'(i, j)$ is the broken line curve with vertices

$$H\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad H\left(t_{i-1}, \tfrac{j}{N}\right), \quad H\left(t_i, \tfrac{j}{N}\right).$$

By the previously quoted result from multivariable calculus we have

$$\int_{C(i,j)} P\,dx + Q\,dy \quad = \quad \int_{C'(i,j)} P\,dx + Q\,dy$$

for each i and j, and therefore the difference between the values of the line integrals must be zero. Therefore the difference between the values of the line integrals over $X(i, j)$ and $X(i-1, j)$ must also be zero, as required. This completes the proof.■

Before proceeding to the proof of Theorem 4, we shall note the following consequence of the results obtained thus far:

PROPOSITION 8. *Let U be an open connected subset of $\mathbb{R}^2$, let $u_0 \in U$, and let P and Q be functions on U with continuous partial derivatives on U such that*

$$\frac{\partial P}{\partial y} \quad = \quad \frac{\partial Q}{\partial x}\ .$$

Then there is a group homomorphism $\Sigma : \pi_1(U, u_0) \to \mathbb{R}$ (with the additive group structure on $\mathbb{R}$) such that if γ is a base point preserving piecewise smooth curve in U which starts and ends at u_0, then

$$\Sigma([\gamma]) \quad = \quad \int_\gamma P\,dx \ + \ Q\,dy\ .$$

Proof. Lemma 7 shows that every basepoint preserving homotopy class of closed curves x has an representative γ which is piecewise smooth (in fact, one can find a broken line approximation to a given continuous curve). Define $\Sigma(x)$ to be the line integral of $\Omega = P\,dx + Q\,dy$ over γ. By Theorem 1 of this section, the value of the line integral does not depend upon the choice of representative, so the mapping is well-defined. The additivity of Σ follows from the general properties of line integrals at the beginning of this unit.■

Proof of Theorem 4. Suppose that α and β are freely homotopic curves in U, and suppose that they start and end at p and q respectively. By Theorems IV.4.3 and IV.4.4 (in `notes4.4.pdf`) there is a curve ω joining p to q such that the class $[\alpha+\omega+(-\beta)+(-\omega)]$ in $\pi_1(U,p)$ is trivial. By Lemmas 6 and 7 there are broken line inscriptions $L(\alpha),\ L(\beta),\ L(\omega)$ of $\alpha,\ \beta,\ \omega$ such that $L(\xi)$ is endpoint preservingly homotopic to ξ for $\xi=\alpha,\beta,\omega$. Combining these observations, we see that

$$[\alpha+\omega+(-\beta)+(-\omega)] \quad = \quad [L(\alpha)+L(\omega)+(-L(\beta))+(-L(\omega))] \quad = \quad 1 \quad \in \quad \pi_1(U,p)$$

and therefore the basic algebraic properties of line integrals and Theorem 1 imply that

$$0 \quad = \quad \int_{L(\alpha)+L(\omega)+(-L(\beta))+(-L(\omega))} P\,dx \ + \ Q\,dy \quad =$$

$$\int_{L(\alpha)} P\,dx \ + \ Q\,dy \ + \ \int_{L(\omega)} P\,dx \ + \ Q\,dy \ - \ \int_{L(\beta)} P\,dx \ + \ Q\,dy \ - \ \int_{L(\omega)} P\,dx \ + \ Q\,dy \quad =$$

$$\int_{L(\alpha)} P\,dx \ - \ \int_{L(\beta)} P\,dx \ + \ Q\,dy$$

and hence the line integrals over $L(\alpha)$ and $L(\beta)$ are equal. But Theorem 1 implies that the line integrals of $L(\xi)$ and ξ are equal for $\xi=\alpha,\beta$, and if we combine this with the preceding sentence we see that the line integrals over α and β are equal.■

Line integrals of complex analytic functions

The material in this subsection is not part of the official course coverage, but the implications for integration in complex analysis and the Fundamental Theorem of Algebra are topics which seem worth including, particularly for students who have already taken an undergraduate level course in complex variables.

For the sake of completeness, we recall that the *Fundamental Theorem of Algebra* states that every nonconstant polynomial $p(z)$ over the complex numbers has a root; *i.e.*, there is some complex number c such that $p(c)=0$.

As noted on pp. 353–354 of Munkres, there are many proofs of the Fundamental Theorem of Algebra, and ultimately they all require some input that is intrinsically nonalgebraic and involves the geometry or topology of the complex plane (so this is really a theorem **about** algebra and not a theorem **of** algebra. In particular, a standard approach using the theory of functions of one complex variable is mentioned at the top of page 354. If one looks carefully at the proofs of the Fundamental Theorem of Algebra in many complex variables texts, issues about the completeness of the arguments often arise. Usually these concern path independence properties of line integrals. A logically rigorous approach to these issues normally requires some information about homotopy classes of closed curves in open subsets of the plane (the same input which appears explicitly in Munkres' proof). Therefore we shall discuss some issues involving line integrals of complex analytic functions over rectifiable curves before looking at the proof of the Fundamental Theorem of Algebra.

One immediate complication involves the definition of an analytic function; in some references it is defined as a complex valued function f defined on an open subset $U\subset\mathbb{C}$ such that f' exists and is continuous on U, and in other references it is taken to be a function f for which f' exists, with no *a priori* assumption of its continuity. In fact, the two notions are equivalent, for the existence of f' guarantees its continuity, but this is a nontrivial result. We shall consider both cases here, beginning with the easier one in which f' is assumed to be continuous.

Suppose we know that f' exists and is continuous. Suppose that we are given a piecewise smooth (or, more generally, a rectifiable continuous) curve γ. Write the function f in the form $f = u + v\mathbf{i}$, where u and v are functions with continuous partial derivatives satisfying the Cauchy-Riemann equations. Then the line integral $\int_\gamma f(z)\,d(z)$ is equal to

$$\int_\gamma u\,dx \;-\; v\,dy \;+\; \mathbf{i}\cdot\int_\gamma v\,dx \;+\; u\,dy\ .$$

Assume now that the region U in the complex plane is rectangular with sides parallel to the coordinate axes (all $x + y\,i$ such that $a \leq x \leq b$ and $c \leq y \leq d$). We claim that the given line integral depends only upon the initial and final points of γ. This is shown using corresponding results from multivariable calculus about path independence. By Green's Theorem, a line integral $\int_\gamma P\,dx + Q\,dy$ over a rectangular region is path independent if we have

$$\frac{\partial Q}{\partial x} \quad = \quad \frac{\partial P}{\partial y}$$

and using the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$, we see that the displayed relation holds for the integrands in the real and imaginary parts of $\int_\gamma f(z)\,dz$. The material in this section then leads to the following basic result:

PROPOSITION 9. *Let f be an analytic function on the open set $U \subset \mathbb{C}$ in the stronger sense (f' is continuous), and let α and β be continuous rectifiable curves in U such that α and β are freely homotopic. Then $\int_\alpha f(z)\,dz = \int_\beta f(z)\,dz$.*■

Now suppose we know that f' exists but we are not given any information regarding its continuity (the weaker definition of analytic function that is found in many texts). We can use the preceding approach PROVIDED we can show that if U is a rectangular region then $\int_\gamma f(z)\,dz$ only depends upon its endpoints. This is done in many complex variables books; for example, it appears on pp. 109–115 of the book by Ahlfors, Section 9.2 of the book by Curtiss, and Section 2.3 of the book by Fisher, all of which are listed below:

L. V. Ahlfors. *Complex Analysis* (3^{rd} Ed.), *McGraw-Hill, New York*, 1979.

J. H. Curtiss. *Introduction to Functions of a Complex Variable* (Pure and Applied Math., Vol. 44). Marcel Dekker, New York, 1978.

S. D. Fisher. *Complex Variables* (2^{nd} Ed.), *Dover, New York*, 1990.

The notion of homotopy also leads to a definitive version of the Cauchy Integral Formula for an analytic function f defined near the complex number a:

$$f(a) \quad = \quad \frac{1}{2\pi\,i}\cdot\int_\gamma \frac{f(z)}{z-a}\,dz$$

The point is that we can give and explicit description of the type of curve γ for which the formula is valid; namely, if f is defined on the open set U and $a \in U$, then we can take γ to be an arbitrary continuous rectifiable curve in $U - \{a\}$ which is homotopic to a counterclockwise circle of arbitrary radius centered at a.

We shall conclude this section by showing deriving the Fundamental Theorem of Algebra. The first step is a familiar limit formula.

LEMMA 10. *If $p(z)$ is a nonconstant monic polynomial in the complex plane then*

$$\lim_{z \to \infty} |p(z)| = \infty \ ,$$

Sketch of proof. Use the identity

$$p(z) \quad = \quad z^n \cdot \left(1 + \frac{c_{n-1}}{z} + \frac{c_{n-2}}{z^2} + \cdots + \frac{c_1}{z^{n-1}} + \frac{c_0}{z^n}\right)$$

and the fact that the limit of the term inside the parentheses is zero.∎

COROLLARY 11. *In the setting above there is an $r > 0$ such that $R \geq r$ implies that $p(z)$ is never zero on a circle C_R of radius R about the origin.*∎

Now let $\Gamma(p, R)$ be the closed curve given by $p(R \cdot \exp(2\pi\, i\, t))$ for $0 \leq t \leq 1$, so that $\Gamma(p, R)$ just describes the behavior of the polynomial p on the circle of radius R about the origin. Consider the so-called *winding number integral*

$$\int_{\Gamma(p,R)} \frac{x\,dy - y\,dx}{x^2 + y^2} .$$

The proof of the Fundamental Theorem of Algebra has two remaining steps.

(1) *If $p(z) \neq 0$ for all z satisfying $|z| \leq R$, then the winding number integral is zero.*

(2) *If p has degree n then the winding number integral is equal to n.*

Proof of first statement. By construction $\Gamma(p, R)$ lies in the punctured plane $\mathbb{C} - \{0\}$. Since p has no zero points it follows that p also defines a map into the punctured plane, and the restriction of p to the solid disk of radius R defines a basepoint preserving homotopy from $\Gamma(p, R)$ to the constant curve. Therefore by homotopy invariance we know that the corresponding line integrals over $\Gamma(p, R)$ and the constant curve are equal. Since the latter integral is zero it follows that the original winding number integral is also zero.

Proof of second statement. First of all, if $p(z) = z^n$ then it follows that the winding number is n by direct calculation. It will suffice to show that for R sufficiently large the closed curves $\Gamma(p, R)$ and z^n are homotopic, for then we can use the modified form of the main result to show that the line integrals associated to the two polynomials are equal.

To prove this we use the identity

$$p(z) \quad = \quad z^n \cdot \left(1 + \frac{c_{n-1}}{z} + \frac{c_{n-2}}{z^2} + \cdots + \frac{c_1}{z^{n-1}} + \frac{c_0}{z^n}\right)$$

to conclude that

$$\lim_{z \to \infty} \frac{p(z)}{z^n} \quad = \quad 1.$$

In particular, there is an $S > 0$ such that $R > S$ implies that

$$\left|\frac{p(z)}{z^n} - 1\right| \quad < \quad \frac{1}{2}.$$

This in turn implies that if $|z| = R$ then the line segment joining 1 to

$$\frac{p(z)}{z^n}$$

lies entirely in the punctured plane. If $h(z, t)$ is this straight line homotopy on the circle $|z| = R$ then $z^n h(z, t)$ defines a homotopy between $\Gamma(p, R)$ and the closed curve defined by the restriction of z^n to the circle of radius R. As noted before, this completes the proof of the second statement and of the Fundamental Theorem of Algebra.■

V.2 : Graph complexes

(Munkres, 64; Crossley, 7.1)

In this unit we shall study a special class of spaces which are built out of very simple pieces but turn out to be important in many branches of mathematics, and in some sense are "toy models" for the sorts of objects usually studied in algebraic and geometric topology. More precisely, these spaces (called *finite graph complexes*, *edge-path graphs* or more simply just *graphs*) are excellent test cases for applying the methods and results of this course.

Informally, a graph can be constructed by taking a finite collection of closed intervals and identifying their endpoints in a suitable fashion; following geometric intuition, the images of the intervals are called *edges* and the images of their endpoints are called *vertices*. Note that *these are* NOT *graphs as defined and studied in coordinate geometry and calculus*, but the name has stuck and become standard usage, both in mathematics and in its applications to numerous other subjects such as computer science, physics, chemistry, industrial engineering, the biological sciences and even to other areas of knowledge where it is useful to look at chains of relationships or passage from one state of a system to another. A fairly simple application of graph theory to a problem about relationships is given in the following online video:

`http://www.youtube.com/watch?v=b3lbjoiEAyA`

In the next section we shall use the decomposition of a graph into edges as the basis for defining an abelian **chain group** which corresponds to curves constructed from these edges. Results beyond the scope of this course relate certain subquotients of these chain groups to π_0 and π_1 of the underlying topological space for the graph. In the final section of this unit we shall show how these chain groups can be applied to analyze a problem about paths along edges called the *Königsberg Bridge Problem*, which was solved by L. Euler (pronounced OY-ler) in the 18^{th} century and can be viewed as one of the first results in algebraic topology.

Basic definitions

Since we have already described finite graphs intuitively, we shall proceed to the formal description.

Definition. A **finite edge-path graph complex** (more simply a *finite graph*) is a pair $(X, \mathcal{E})$ consisting of a compact Hausdorff space X and a finite family $\mathcal{E}$ of closed subsets with the following properties:

(1) Each subset $E \in \mathcal{E}$ is homeomorphic to the closed interval $[0, 1]$.

(2) The space X is the union of all the subspaces E in the family $\mathcal{E}$.

(3) If E_1 and E_2 are distinct subsets of $\mathcal{E}$, then either $E_1 \cap E_2$ is empty or else it is a single point corresponding to a vertex of each interval E_i.

COMMENTS ON THE DEFINITION. The endpoints of a set homeomorphic to $[0, 1]$ are topologically characterized by the fact that their complements are connected; for all other points, the complement has two components. As above, we shall say that a subset of $\mathcal{E}$ is an edge and an endpoint of an *edge* will be called a *vertex*.

The setting in Chapter 14 of Munkres is more general and includes examples where the set of edges is infinite but each vertex lies on only finitely many edges. We are restricting attention to examples with finitely many edges in order to simplify the discussion.

Examples. It is easy to draw many examples of graphs, and such drawings are extremely useful for understanding this concept. The file `graphpix1.pdf` contains a few examples, including some that will appear later in this course.

An alternate definition

Our definition of a graph assumes that two edges meet in just one endpoint, but in some situations it is convenient to consider examples for which the intersection of two edges is also allowed to be both vertices of the two edges as in the following illustration:

D

(Two vertices at the corners, two edges have these endpoints.)

We shall prove that every object of this more general type can be expressed as a graph in the sense of our definition.

LEMMA 1. *Let Γ be a system satisfying the conditions for an finite edge-vertex graph except that two edges may have both of their vertices, and let $\mathcal{E}$ be the collection of edges for this system. Then there is another family of closed subsets $\mathcal{E}'$ such that the following hold:*

(i) *The family $\mathcal{E}'$ is a collection of edges for a graph structure on Γ.*

(ii) *Each element of $\mathcal{E}'$ is contained in a unique element of $\mathcal{E}$ such that one endpoint of $\mathcal{E}'$ is also an endpoint for $\mathcal{E}$ but another is not, and each edge in $\mathcal{E}$ is a union of two edges in $\mathcal{E}'$.*

(iii) *The intersection of two distinct edges in $\mathcal{E}'$ is a single point which is a common vertex.*

Proof. For each edge $E \in \mathcal{E}$, pick a point $b_E \in E$ that is not an endpoint. It follows that $E - \{b_E\}$ has two connected components, each of which contains exactly one endpoint of E. If x

is an endpoint of E define the set $[x, E]$ to be the closure of the component of $E - \{b_E\}$ which contains x. If $\mathcal{E}'$ denotes the set of all such subsets $[x, E]$, then it follows immediately that $\mathcal{E}'$ has the properties stated in the lemma. Note that by construction the endpoints of a given edge $[x, E]$ are x and b_E.■

The family $\mathcal{E}'$ is frequently called the *derived* graph structure associated to $\mathcal{E}$.

As noted in one of the exercises, many examples of edge-vertex graphs are suggested by ordinary letters and numerals.

Subgraphs

Definition. Let $(X, \mathcal{E})$ be a finite edge-path graph. A *subgraph* $(X_0, \mathcal{E}_0)$ is given by a subfamily $\mathcal{E}_0 \subset \mathcal{E}$ such that X_0 is the union of all the edges in $\mathcal{E}_0$. It is said to be a *full subgraph* if two vertices $\mathbf{v}$ and $\mathbf{w}$ lie in X_0 and there is an edge $E \in \mathcal{E}$ joining them, then $E \in \mathcal{E}_0$.

PROPOSITION 2. *Let $(X, \mathcal{E})$ be a finite edge-path graph, and let $(X_0, \mathcal{E}_0)$ be a subgraph. Then the derived graph $(X_0, \mathcal{E}_0')$ is a full subgraph of $(X, \mathcal{E}')$.*

Proof. Suppose we are given an edge K in $(X, \mathcal{E}')$, so that its vertices must have the form $\mathbf{y}$, $\mathbf{m}_L$ where L is an edge in $(X, \mathcal{E})$ that has $\mathbf{y}$ as one of its endpoints and $\mathbf{m}_L$ is a non-vertex point in L. If both of these vertices belong to X_0, then the latter contains a point of L which is not a vertex, and since $(X_0, \mathcal{E}_0)$ is a subgraph it follows that L must be entirely contained in X_0. But this automatically implies that the edge in the derived complex with endpoints $\mathbf{y}$ and $\mathbf{m}_L$ must be contained in X_0.■

Connectedness

One immediate consequence of the definitions is that every point of a graph lies in the arc component of some vertex; specifically, if x lies on the edge E and the vertices of the latter are a and b, then x lies in the same arc component as both a and b. In fact, one can prove much stronger conclusions:

PROPOSITION 3. *If $(X, \mathcal{E})$ is a finite edge-path graph, then X is connected if and only if for each pair of distinct vertices $\mathbf{v}$ and $\mathbf{w}$ there is an* edge-path sequence *$E_1, \cdots, E_n$ such that $\mathbf{v}$ is one vertex of E_1, $\mathbf{w}$ is one vertex of E_n, for each k satisfying $1 < k \leq n$ the edges E_k and E_{k-1} have one vertex in common, and $\mathbf{v}$ and $\mathbf{w}$ are the "other" vertices of E_1 and E_n. Furthermore, X is a union of finitely many components, each of which is a full subgraph.*

IMPORTANT: In a general edge-path sequence defined as in the statement of the proposition, we do **NOT** make any assumptions about whether or not these two vertices are equal. If they are, then we shall say that the edge-path sequence is *closed* or that it is a *circuit* or **cycle**.

Proof. First of all, since every point lies on an edge, it follows that every point lie in the connected component of some vertex. In particular, there are only finitely many connected components. Define a binary relation on the set of vertices such that $\mathbf{v} \sim \mathbf{w}$ if and only if the two vertices are equal or there is an edge-path sequence as in the statement of the proposition. It is elementary to check that this is an equivalence relation, and that vertices in the same equivalence class determine the same connected component in X.

Given a vertex $\mathbf{v}$, let $Y_{\mathbf{v}}$ denote the union of all edges containing vertices which are equivalent to $\mathbf{v}$ in the sense of the preceding paragraph. If we choose one vertex $\mathbf{v}$ from each equivalence class, then we obtain a finite, pairwise disjoint family of closed connected subsets whose union is X, and it follows that these sets are must be the connected components of X. In fact, by construction each of these connected component is a full subgraph of $(X, \mathcal{E})$.■

Frequently it is convenient to look at edge-path sequences that are *minimal* or *simple* in the sense that one cannot easily extract shorter edge-path sequences from them. Here is a more precise formulation:

Definition. Let $E_1 , \cdots , E_n$ be an edge-path sequence such that the vertices of E_i are $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$. This sequence is said to be *reduced* if $\mathbf{v}_1 , \cdots , \mathbf{v}_n$ are distinct and either $n \neq 2$ or else $\mathbf{v}_0 \neq \mathbf{v}_2$ (if $n = 2$ and $\mathbf{v}_0 = \mathbf{v}_2$, then the edge-path is just a sequence with $E_2 = E_1$, physically corresponding to going first along E_1 in one direction and then back in the opposite direction).

We then have the following result:

PROPOSITION 4. *If two distinct vertices* $\mathbf{x}$ *and* $\mathbf{y}$ *can be connected by an edge-path sequence, then they can be connected by a reduced sequence.*

Proof. Take a sequence with a minimum number of edges. We claim it is automatically reduced. If not, then there is a first vertex which is repeated, and a first time at which it is repeated. In other words, there is a minimal pair (i, j) such that $i < j$ and $\mathbf{v}_i = \mathbf{v}_j$, which means that if (p, q) is any other pair with this property we have $p \geq i$ and $q > j$. If we remove E_{i+1} through E_j from the edge-path sequence, we obtain a shorter sequence which joins the given two vertices.■

There may be several different reduced sequences joining a given pair of vertices. For example, take X to be a triangle graph in the plane whose vertices are the three noncollinear points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and whose edges are the three line segments joining these pairs of points. Then $\mathbf{ab}$, $\mathbf{bc}$ and $\mathbf{ac}$ are two reduced edge-path sequences joining $\mathbf{a}$ to $\mathbf{c}$.

Definition. A circuit (or cycle) $E_1 , \cdots , E_n$ is called a a *simple circuit* or *simple cycle* if it is reduced.

COROLLARY 5. *Every simple circuit in a graph contains at least three edges.*■

Further topological properties of graphs

By definition and construction, a finite edge-path graph is compact Hausdorff, and in fact one can say considerably more:

PROPOSITION 6. *A finite edge-path graph is homeomorphic to a subset of* $\mathbb{R}^n$ *for some* n.

At the end of this section we shall prove that a graph is always homeomorphic to a subset of $\mathbb{R}^3$.

Proof. Suppose that the vertices are $\mathbf{v}_1, \cdots , \mathbf{v}_n$. Consider the graph in $\mathbb{R}^n$ whose vertices are the standard unit vectors $\mathbf{e}_i$ and whose edges are the closed line segments $A_{i,j}$ joining these vertices; the resulting compact subspace of $\mathbb{R}^n$ is a graph because two of these segments intersect in at most a common endpoint (use linear independence of the unit vectors to prove this). Define a continuous map f from original the graph X to the new graph Y such that if E is an edge with vertices $\mathbf{v}_i$ and $\mathbf{j}$ for $i < j$ and E is a homeomorphism from $[0, 1]$ to E such that $\mathbf{v}_i$ corresponds to 0 and $\mathbf{v}_j$ corresponds to 1, then $t \in [0, 1]$ is sent to

$$t\,\mathbf{e}_j \;+\; (1-t)\,\mathbf{e}_i$$

(since E is homeomorphic to $[0,1]$ and endpoints are topologically characterized by the property that their complements are connected, it follows that either $\mathbf{v}_i$ corresponds to 0 and $\mathbf{v}_j$ corresponds to 1 or vice versa; in the second case, if we compose the original homeomorphism with the reflection on $[0,1]$ sending s to $1-s$, then we obtain a homeomorphism for which the first alternative holds).

It is a routine exercise to verify that f is continuous and 1–1, and therefore it maps X homeomorphically onto its image.■

In some contexts it is useful to know the smallest n for which this is possible. The methods of point set topology show that a connected compact subset of $\mathbb{R}$ is an interval, and for all other graphs the minimum value of n is at most 3. There are many obvious examples for which the minimum value is 2 (one can draw many physical models in the plane), but there are other examples — for example, a network joining three houses to gas, water and electrical utilities — for which the minimum value is 3. A fairly straightforward proof that every graph is homeomorphic to a subset of $\mathbb{R}^3$ appears in Unit III of the notes

`http://math.ucr.edu/∼res/math205B-2012/algtop-notes.pdf`

but a proof of the statement about the gas–water–electricity network requires more sophisticated techniques. One proof is given in Section 64 of Munkres; this argument requires a minimum of background but it is fairly complicated. Another proof is given in Section VII.4 of the notes cited above; the argument in those notes is arguably less opaque than the argument in Munkres, but it involves a very substantial amount of background material.

V.3 : Chains, homology and fundamental groups

(Crossley, 9.1)

We shall begin with the following question:

> *Suppose that $(X, \mathcal{E})$ is a connected graph in the sense of the preceding section. Do the graph data provide enough information to describe the fundamental group of $(X, \mathbf{v})$, where (say) $\mathbf{v} \in X$ is some vertex?*

By the results of Section IV.4, we know that the isomorphism type of $\pi_1(X, \mathbf{v})$ does not depend upon the choice of vertex.

The following result shows that the answer to the question is emphatically yes.

FUNDAMENTAL GROUPS OF CONNECTED GRAPHS. *If $(X, \mathcal{E})$ is a connected graph and $\mathbf{v} \in X$ is some vertex, then $\pi_1(X, \mathbf{v})$ is a* free group *on $1 + E - V$ generators, where E denotes the number of edges in $\mathcal{E}$ and V denotes the number of vertices in $\mathcal{E}$. In other words, there are elements x_i in the $\pi_1(X, \mathbf{v})$ (where $1 \leq i \leq 1 + E - V$) such that every nontrivial element of the group has a unique expression as a monomial product*

$$x_{i(1)}^{n(1)} \cdots x_{i(k)}^{n(k)}$$

where $k \geq 1$, each exponent $n(i)$ is a nonzero integer, and $x_{i(i)} \neq x_{i(i+1)}$ for $i = 1, \cdots, k-1$.

A proof of this result is given in Section III.3 of the previously cited algebraic topology notes:

This theorem shows that one can retrieve a great deal of important information about a graph from its combinatorial structure.

Similar questions in higher dimensions

Motivated by the theorem on the fundamental groups of connected graphs, it is natural to pose the following more general question:

> *Suppose that X is a space with a decomposition $\mathbf{K}$ into well behaved examples of closed subsets A_i which are at most n-dimensional in an appropriate sense (for example, suppose that X has a* triangulation *as described on page 122 of Crossley). Are there algebraic constructions based on $\mathbf{K}$ which yield insight into the topological structure of X?*

In order to study such a question it is necessary to have a more refined idea of the structures that are potentially worth analyzing. We shall start with open subsets of $\mathbb{R}^3$. The fundamental group gives us some insight into the sorts of 1-dimensional configurations which lie inside an open subspace $U \subset \mathbb{R}^3$, and motivated by the importance of surface integrals in U it is natural to look for some object which provides insight into the sorts of surfaces that somehow lie in U. A closely related question involves the possible maps of a sphere S^k into such an open set U, and this is discussed at considerable length in Chapter 8 of Crossley; in particular, the latter shows how one can describe homotopy groups $\pi_n(X, x)$, whose elements are base point preserving homotopy classes of maps from s^n to X, for every $n \geq 2$. Although these groups can be defined abstractly almost as easily as the fundamental group itself, they have two drawbacks. First, there are many situations in which wants to look at k-surfaces other than spheres; for example, if U is open in $\mathbb{R}^3$ one may well wish to consider torus surfaces which are homeomorphic to $S^1 \times S^1$. Second, there are serious obstacles to computing these groups precisely. Results from the nineteen fifties imply two strong general results for the class of simply connected spaces with triangulations in the sense mentioned above:

(*i*) (J.-P. Serre) *The groups $\pi_n(X, x)$ are finitely generated abelian groups.*

(*ii*) (E. H. Brown) *Each of these groups is finitely computable.*

Unfortunately, the groups $\pi_n(X, x)$ are often extremely difficult to compute, even with the dramatic advances in computer technology over the past 70 years. For example, our understanding of the groups $\pi_p(S^q)$ (where $p > q$) is highly incomplete.

These drawbacks suggest that it might be useful to consider some type of rough approximation or analog to homotopy groups. The most basic objects of this sort are called *homology groups*. Although it is not feasible to give a mathematically complete account of homology theory in an undergraduate course, many key points of the subject are presented in Chapters 9 and 10 of Crossley. The latter indicates how one can define such groups for arbitrary topological spaces and how one can retrieve the groups for spaces with triangulations in a relatively straightforward manner.

A graph $(X, \mathcal{E})$ can be viewed as a special case of a triangulation, and in these notes we shall only describe the construction of homology groups for graphs. In Section 4 we shall illustrate some of their uses in a discussion of the previously mentioned Königsberg Bridge Problem.

Algebraic chains for graphs

Let $(X, \mathcal{E})$ be a finite graph complex, and let ω denote a linear ordering of the vertices (since the set of vertices is finite, the existence of such orderings is immediate); we shall often denote the combined graph structure and vertex ordering by symbolism such as $\mathcal{E}^\omega$. If $\mathbb{F}$ is a field, then the *simplicial chain groups with coefficients in* $\mathbb{F}$, which will be denoted by $C_q(X, \mathcal{E}^\omega; \mathbb{F})$ for $q \in \mathbb{Z}$, are defined such that

$C_1(X, \mathcal{E}^\omega; \mathbb{F})$ is a vector space over $\mathbb{F}$ with a basis given by the edges in $\mathcal{E}$,

$C_0(X, \mathcal{E}^\omega; \mathbb{F})$ is a vector space over $\mathbb{F}$ with a basis given by the vertices in $\mathcal{E}$,

$C_q(X, \mathcal{E}^\omega; \mathbb{F})$ is trivial if $q \neq 0, 1$.

The associated **graph chain complex** $C_*(X, \mathcal{E}^\omega; \mathbb{F}; d)$ consists of these simplicial chain groups together with **boundary homomorphisms**

$$d_q : C_q(X, \mathcal{E}^\omega; \mathbb{F}) \longrightarrow C_{q-1}(X, \mathcal{E}^\omega; \mathbb{F})$$

such that $d_q = 0$ unless $q = 1$ and d_1 on a basis element corresponding to an edge E is given as follows: One of the two vertices of E precedes the other, and if $\partial_- E$ precedes $\partial_+ E$ let

$$d_1(E) \quad = \quad \partial_+ E \ - \ \partial_- E \ .$$

Since the edges form a basis for the 1-dimensional chain group, this map of basis elements extends to a homomorphism of chain groups.

Chains are often visualized as unions of simple edge paths in the graph, with $\pm E$ corresponding to an oriented path starting at $\partial_- E$ and ending at $\partial_+ E$.

The use of the word "chain" can be motivated geometrically as follows: Let $E_1, \cdots, E_n$ be an edge-path sequence such that the vertices of E_i are $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$. Then we can associate a 1-chain to this path by the formula

$$c(E) \quad = \quad \sum_i \varepsilon_i \, E_i$$

where $\varepsilon_i \in \{\pm 1\}$ is defined to be $+1$ if $\mathbf{v}_{i-1}$ precedes $\mathbf{v}_i$ and -1 if $\mathbf{v}_i$ precedes $\mathbf{v}_{i-1}$. The signs have been chosen so that $d_1(c(E)) = \mathbf{v}_n - \mathbf{v}_0$. If the edge path is a **cycle** in the sense that the initial and final points agree, it follows that d_1 sends that chain to 0, and conversely if d_1 sends the chain for an edge path to zero then the edge path is a cycle. More generally, we define the **1-cycles** to be those 1-dimensional chains (or 1-chains) c such that $d_1(c) = 0$. Similarly, we say that a 0-dimensional chain b is a **boundary** if $b = d_1(c)$ for some 1-chain c. The **homology groups** $H_q(X, \mathcal{E}^\omega)$ are then given as follows:

$H_1(X, \mathcal{E}^\omega; \mathbb{F})$ is isomorphic to the kernel of d_1.

$H_0(X, \mathcal{E}^\omega; \mathbb{F})$ is isomorphic to the quotient of $C_0(X, \mathcal{E}^\omega)$ by the image of d_1.

By construction the homology groups are finite dimensional vector spaces over $\mathbb{F}$.

We now have the following results relating the chain groups to the topological structure of the graph:

THEOREM 1. *Let $(X, \mathcal{E}^\omega; \mathbb{F})$ be a graph with a linear ordering of its vertices.*

(i) If the connected components of X are given by $(X_i, \mathcal{E}_i)$, then the chain groups $C_*(X, \mathcal{E}^\omega; \mathbb{F})$ and homology groups $H_*(X, \mathcal{E}^\omega; \mathbb{F})$ are isomorphic to direct sums of the corresponding groups for $(X_i, \mathcal{E}_i^\omega)$.

(ii) If X is connected then $H_0(X, \mathcal{E}^\omega; \mathbb{F}) \cong \mathbb{F}$ and $H_1(X, \mathcal{E}^\omega; \mathbb{F})$ has dimension $1 - \chi(X, \mathcal{E})$, where $\chi(X, \mathcal{E})$ is the Euler characteristic given by number of vertices minus the number of edges.

This result implies that the structure of the homology groups is completely determined by the homotopy type of the underlying space X. Note that the dimension of H_1 in the second case is equal to the number of free generators for $\pi_1(X, \mathbf{v})$ in the statement of the theorem on the structure of that group. This is not a coincidence, but in this course we shall not be able to explain the mathematical relationship between these two facts; for further information on this topic, see Section III.1 of the following notes:

`http://math.ucr.edu/~res/math246A-2012/advanced-notes.pdf`

Proof of Theorem 1. We begin with the first part. Since every edge of a graph is contained in an arc component, we know that the graph has finitely many arc components, and by the results of the preceding sections the connected components are the same as the arc components. Since every edge of the graph is contained in a unique component, the two vertices of the edge are also contained in that component. This means that the boundary map d_1 for the graph can be decomposed as a direct sum of boundary maps for the individual components

$$(d_1)_i : C_q(X_i, \mathcal{E}_i^\omega; \mathbb{F}) \longrightarrow C_0(X_i, \mathcal{E}_i^\omega; \mathbb{F})$$

and it follows that one has a similar decomposition for the homology groups of $(X, \mathcal{E}^\omega)$ with coefficients in $\mathbb{F}$.

Assume now that X is connected. The first step in proving (ii) is to prove the assertion about H_0. Define a map

$$\varepsilon : C_0(X, \mathcal{E}_i^\omega; \mathbb{F}) \longrightarrow \mathbb{F}$$

(called an *augmentation homomorphism*) such that its value at each vertex generator is equal to $+1$.

CLAIM 1. *The kernel of ε is equal to the image of d_1.*

To see that the kernel of ε contains the image of d_1 it suffices to show that $\varepsilon \circ d_1 = 0$, and since chain groups are vector spaces it suffices to check this for a basis element corresponding to an arbitrary edge E. Since

$$\varepsilon \circ d_1(E) \;=\; \varepsilon\partial_+(E) - \varepsilon\partial_-(E) \;=\; 1 - 1 \;=\; 0\,.$$

Suppose now that the chain $a = \Sigma_i \; n_i \, \mathbf{v}_i$ lies in the kernel of d_i, so that $\Sigma_i \; n_i = 0$ and

$$n_1 \;=\; -\sum_{i>1} n_i \qquad .$$

Since X is connected, for each $i > 1$ there is an edge path starting at $\mathbf{v}_1$ and ending at $\mathbf{v}_i$, and by the discussion above we have 1-chains c_i such that $d_1(c_i) = \mathbf{v}_i - \mathbf{v}_0$. Therefore we have

$$\sum_{i>1} n_i \, c_i \;=\; \sum_{i>1} n_i \; (\mathbf{v}_i - \mathbf{v}_0) \;=\;$$

$$\sum_{i>1} n_i \, \mathbf{v}_i \;-\; \left(\sum_{i>1} n_i \right) \mathbf{v}_0$$

and since $\Sigma_{i \geq 1} \; n_i = 0$ the last expression is equal to a, so that a lies in the image of d_1.

The preceding discussion implies that H_0 is 1-dimensional and spanned by the class of $\mathbf{v}$, where the latter is an arbitrary vertex of the graph, and the image of d_1 is the subspace generated by $\mathbf{w} - \mathbf{v}^*$, where $\mathbf{v}^*$ is a fixed vertex and $\mathbf{w}$ runs through the remaining vertices. This means that the image of d_1 is a $(V-1)-$dimensional vector space, where V denotes the set of vertices in the graph. Therefore d_1 defines a surjective homomorphism from C_1, which is an $E-$ dimensional vector space, to a $(V-1)-$dimensional vector space. The conclusion about the structure of H_1 will then be a consequence of the following algebraic result:

CLAIM 2. *If A and B are vector spaces of dimensions α and β respectively and $\varphi : A \to B$ is onto, then the kernel of φ is $(\beta - \alpha)-$dimensional.*

This is a standard exercise in linear algebra.

As noted above, this completes the proof of Theorem 1.■

An application

We shall conclude this section with a criterion for finding cycles in a connected graph using homology. It is also possible to prove this without using homology, but the argument is more complicated (we can view the use of homology as substitute for the more complicated argument, so this is not really getting something for nothing).

Definition. If $(X, \mathcal{E})$ is a graph and v is a vertex of $\mathcal{E}$, then the *valency* or *degree* of v is the number of edges which have v as one of their vertices, and it will be denoted by $d(v, \mathcal{E})$ or more simply by $d(v)$ if there is no ambiguity about the graph under consideration.

PROPOSITION 2. *If E is the number of edges in $(X, \mathcal{E})$ then we have*

$$2\,E \quad = \quad \sum_v \, d(v)$$

where the sum ranges over all the vertices of the graph.

Proof. Consider all pairs (v, e) where e is an edge of the graph and v is a vertex of e. Since each edge has two vertices there are $2\,E$ such pairs. On the other hand, we can also count these pairs by adding the numbers $d(v)$ for all vertices v, and this sum is just the right hand side of the displayed equation. Since the number of pairs is the same no matter how we split things up, it follows that the two expressions in the display must be equal.■

We can now derive the promised application.

PROPOSITION 3. *Suppose that $(X, \mathcal{E})$ is a connected graph and $d(v) \geq 2$ for each vertex v. Then there is a (nontrivial) cycle/circuit in $(X, \mathcal{E})$.*

Proof. Let V be the number of vertices. Then $d(v) \geq 2$ for all vertices v implies that the right hand side is at least $2\,V$, and therefore Proposition 2 implies that $2E \geq 2V$ or equivalently $E \geq V$. If $\mathbb{Z}_2$ is the integers mod 2, then Theorem 1 implies that $\dim H_1(X, \mathcal{E}^\omega; \mathbb{Z}_2) = 1 + E - V$, and since $E \geq V$ this expression is positive. This implies the existence of a nonzero **algebraic** cycle $z = \sum F_j$ where each F_j is an edge and $d_1(z) = 0$ (note that this summation does not necessrily run through all the edges!). Since there is some nonempty set of edges whose algebraic sum is a cycle, there must be set $\mathbf{C}$ with the least positive number of edges, say v.

To conclude the proof, we need to show that the existence of the nonzero algebraic cycle in the preceding sentence implies the existence of a circuit in the graph. Let $L_1 = K_1$, so that

$0 \neq d_1(L_1) = u_1 + u_0$, where u_0 and u_1 are the vertices of L_1; recall that we are working with $\mathbb{Z}_2$ coefficients, so there is no difference between $v + w$ and $v - w$. Since $d(L_1) \neq 0$ it follows that $\mathbf{C}$ contains at least one other edge. Furthermore, since $\sum_j \; K_j = 0$ there must be some $L_2 \in \mathbf{C} - \{L_1\}$ such that $d_1(L_2) = u_1 + u_2$ for some u_2. Since $L_2 \neq L_1$ we must have $u_2 \neq u_0$ and hence $d_1(L_1 + L_2) = u_2 + u_0 \neq 0$. For the same reasons as before, there must be some $L_3 \neq L_2$ such that $d_1(L_3) = u_3 + u_2$. In this case we know that u_1, u_2, u_3 are distinct, but we might have $u_0 = u_3$. In the latter case the edge sequence L_1, L_2, L_3 defines a circuit and the proof is complete, but if not we have to continue the argument.

In fact, the preceding observations form the basis for an inductive argument. Suppose that we have distinct edges $L_1, \; \cdots \; , L_k$ in $\mathbf{C}$ such that $d_1(L_j) = j_j + u_{j-1}$ and the vertices $u_0, \; \cdots \; , u_k$ are distinct. Then we have $d_1(L_1 + \; \cdots + L_k) \neq 0$ and therefore $k < r$. Since the larger summation $\sum \; K_j$ is a cycle, it follows that there must be some $L_{k+1} \in \mathbf{C}$ such that $L_{k+1} \neq L_k$ and $d(L_{k+1}) = u_{k+1} + u_k$ for some u_{k+1}. Furthermore, since u_k is not a vertex of $L_1, \; \cdots \; , L_{k-1}$ it follows that the edges $L_1, \; \cdots \; , L_{k+1}$ are distinct and u_{k+1} is distinct from the vertices $u_0, \; \cdots \; , u_k$. Once again, if $k + 1 = r$ then we are finished, and if not then we can repeat this procedure to obtain additional edges L_{k+2} *etc.* until we reach r and use up all the edges in $\mathbf{C}$.■

NOTE. In some cases the subfamily $\mathbf{C}$ constructed in the proof is a proper subfamily of $\mathbf{F} = \{F_1, \; \cdots \; , F_r\}$. In particular, if we the graph defined by the edges of a cube as in `cube-graph.pdf`, take then we have the algebraic cycle

$$(AB \;+\; BC \;+\; CD \;-\; AD) \;+\; (EF \;+\; FH \;+\; GH \;-\; FH)$$

and either of the expressions $AB + BC + CD - AD$ or $EF + FG + GH - FH$ is a cycle with a minimum number of summands (see the drawing in `cubegraph.pdf`).

The proof of Proposition 3 also yields the following stronger result:

COROLLARY 4. *If $(X, \mathcal{E})$ is a connected graph and $H_1(X, \mathcal{E}^\omega; \mathbb{Z}_2)$ is nontrivial, then $(X, \mathcal{E})$ has a circuit such that the edges are all distinct.*■

V.4 : Euler paths

One particular graph that is historically noteworthy is the Königsberg Bridge Graph, in which the vertices correspond to four land masses in the city of Königsberg (now Kaliningrad, Russia) and the 1-cells (or *edges*) correspond to the seven bridges which joined pairs of land masses in the 18^{th} century (see `koenigsberg.pdf` for drawings). This configuration can be modeled by a graph with vertices $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ representing the land masses and edges representing one bridge each from $\mathbf{w}$ to $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ along with two bridges joining $\mathbf{y}$ to each of $\mathbf{x}$ and $\mathbf{z}$. This configuration is homotopic to a simplicial complex if we add extra vertices $\mathbf{u}_1$ and $\mathbf{u}_2$ on each of the bridges joining $\mathbf{y}$ to $\mathbf{x}$ and $\mathbf{v}_1$ and $\mathbf{v}_2$ on each of the bridges joining $\mathbf{y}$ to $\mathbf{z}$. This will be our graph $(P, \mathbf{K})$, and we shall let C_* denote the ordered chain complex with $\mathbb{Z}_2$ coefficients which associated to some ordering of the vertices; since $1 = -1$ in $\mathbb{Z}_2$, one obtains the same boundary map for every ordering of the vertices.

The problem is to determine whether there is a path on this complex in which each bridge is crossed exactly once, and the first step is to formulate this in terms of the chain complex C_*. What we want is a 1-chain $\sum_{\mathbf{E}} \; \theta(\mathbf{E})\, \mathbf{E}$, where the sum runs over all basis elements of C_1 and $\theta_{\mathbf{E}}$

is nonzero for all $\mathbf{E}$, such that the boundary of this 1-chain has the form $\mathbf{p} + \mathbf{q}$ for two vertices in C_0 (the case $\mathbf{p} = \mathbf{q}$ is allowed). The problem is then to determine if such a 1-chain exists.

Euler's crucial insight into the problem can be stated algebraically as follows:

PROPOSITION 1. *Let $(X, \mathcal{E})$ be a connected graph, let $\gamma \in C_1(X, \mathcal{E}; \mathbb{Z}_2)$ be the 1-chain $\sum_{\mathbf{E}} \mathbf{E}$, where the sum runs over all basis elements of C_1, and write $d_1(\gamma) = \sum_{\mathbf{v}} n(\mathbf{v})\,\mathbf{v}$ for suitable mod 2 integers $n(\mathbf{v})$, where the sum runs over all vertices of $(P, \mathbf{K})$. Then $n(\mathbf{v})$ is the mod 2 reduction the number $m(\mathbf{v})$ of 1-simplices $\mathbf{E}$ that have $\mathbf{v}$ as one of their endpoints.*

Proof. An integer representing $n_{\mathbf{v}}$ is equal to the number (mod 2) of edges containing $\mathbf{v}$ as a vertex.■

COROLLARY 2. *In the preceding setting, if there is a 1-chain γ such that $d(\gamma) = \mathbf{p} - \mathbf{q}$ where $\mathbf{p} = \mathbf{q}$ is possible, then $m_{\mathbf{v}}$ must be even if $\mathbf{v} \neq \mathbf{p}, \mathbf{q}$.*■

Application to the Königsberg Bridge Problem. The impossibility of finding a suitable 1-chain for our Königsberg bridge network now follows by observing that $m = 3$ for $\mathbf{w}$, $\mathbf{x}$ and $\mathbf{z}$, while $m = 5$ for $\mathbf{y}$. In particular, if γ is a chain as in the statement of the theorem, then in $d(\gamma)$ the coefficients of all four of these vertices must be nonzero.■

NOTE. We are not necessarily claiming that one needs to introduce chain groups in order to solve this problem (in fact, when Euler first solved the problem he did not view it as question in mathematics, but he later modified his opinion). The purpose here is to illustrate how the use of chain complexes can provide a framework for finding solutions to this and similar questions.

It is left as an exercise for the reader to show that the homology groups for the Königsberg bridge graph are given by $H_1 \cong \mathbb{F}^4$ and $H_0 \cong \mathbb{F}$.

A general result

The preceding discussion can be modified to yield one direction of the following result:

THEOREM 3. *Let $(X, \mathcal{E})$ be a connected graph. Then the following are equivalent:*

(i) The graph has an Euler path *$E_1 \cdots E_m$ such that each adjacent pair E_j and E_{j+1} have one vertex in common and every edge of $\mathcal{E}$ appears exactly once in the sequence $E_1 \cdots E_m$.*

(ii) The number of odd vertices v (with $d(v)$ odd) is either 2 or 0.

Note that if $E_1 \cdots E_m$ is a closed Euler path (so that E_1 and E_m also have one vertex in common), then $m \geq 3$ and $E_2 \cdots E_m E_1$ is also a closed Euler path.

Proof. $[(i) \implies (ii)]$ If the vertices of E_j are u_{j-1} and u_j, then over $\mathbb{Z}_2$ we have $\sum E_k = u_m + u_0$. If $u_m \neq u_0$ this means that u_m and u_0 are the only odd vertices. If $u_m = u_0$ then all the vertices are even vertices (*i.e.*, $d(v)$ is even).■

$[(ii) \implies (i)]$ Suppose first that there is a graph which does not satisfy (i) but has no vertices of odd degree. Then there is a graph $(X, \mathcal{E})$ of this type with a minimum number of edges. Each vertex of this graph must lie on at least two edges, and therefore Proposition 3.3 and Corollary 4 imply that $(X, \mathcal{E})$ has a circuit whose edges are distinct. Let $Y \subset X$ be such a circuit with maximum length.

We claim that Y contains all the edges of the original graph; suppose that it does not. Let W be the union of the vertices in X with the edges which are not contained in X; then W is a finite union of pairwise disjoint sets which are either isolated points or connected graphs, and at least one of these components contains an edge. This component will be denoted by W_0. Let Z be the union of the other components in W.

The next step in proving the claim is to look at W_0 more closely. First of all, we claim that every vertex of W_0 is even. If a vertex v of W_0 is not a vertex of Y, this is true because every edge of Y which contains v lies in W and hence must lie in the connected component W_0. On the other hand, if v is also a vertex of Y, then Y contains an even number of edges with v as an endpoint; since v is an even vertex of X and the edges of W_0 containing v are the edges which do not lie in Y, it follows that v is also an even vertex of W_0. Furthermore, in this case there must be some edges of W_0 containing v, for otherwise $\{v\}$ would be a connected component of W and we know that the connected component W_0 containing v contains at least one edge. Therefore W_0 is a connected graph with no vertices of odd degree. Since W_0 is a proper subgraph of X and the latter is a minimal example for which (i) is false, it follows that W_0 has an Euler path $K_1 \ \cdots \ K_n$ which contains all the vertices of W_0.

We shall now show that the closed paths $E_1 \ \cdots \ E_m$ and $K_1 \ \cdots \ K_n$ have a common vertex. Suppose this is not the case. Since all the vertices of Y and W_0 are covered by the closed Euler paths, we know that these subgraphs have no vertices in common. Furthermore, since W_0 and Z are disjoint open and closed subsets of W, it follows that there is no edge path in W joining a vertex of W_0 to a vertex of Z. However, since X is connected there is an edge path X joining a vertex in W_0 to a vertex in Y, and there must be a first vertex of this edge path which does not lie in W_0 (since the path ends at a vertex of Y and the latter has no vertices in common with W_0). This first vertex must lie on an edge joining a vertex of W_0 to a vertex not in W_0. By the fourth sentence of this paragraph, the second vertex of such an edge H cannot lie in Y. But if the second vertex belonged to W_0, then H would be an edge in W with one vertex in W_0 and the other in X, and therefore H would be in W_0. Since the latter is a subcomplex, the second vertex of H must also be a vertex of W_0, contradicting the observation in the third sentence of this paragraph. The source of this contradiction is our assumption that Y and W_0 had no vertices in common, and therefore this assumption is false. Therefore Y and W_0 must have a common vertex.

By the observation in the sentence following the statement of the theorem, we can cyclically renumber $E_1 \ \cdots \ E_m$ and $K_1 \ \cdots \ K_n$ so that the common vertex is the initial vertex for both circuits. Therefore we can concatenate the two Euler paths in X to obtain a longer Euler path of the form $E_1 \ \cdots \ E_m K_1 \ \cdots \ K_n$. Since $E_1 \ \cdots \ E_m$ was assumed to be a maximal Euler circuit in X, this yields a contradiction. The source of this contradiction was our assumption that this path did not contain all the edges of X, and therefore this assumption must be false. In other words, the original maximal edge path must contain all the edges of the graph.

The preceding discsussion proves that (ii) implies (i) if there are no odd vertices in $(X, \mathcal{E})$, so all that remains is to prove the same implication if there are two odd vertices. If there are only two vertices then the Euler path can have only one edge, and conversely if there is only one edge there is clearly an Euler path consisting of the edge by itself. Therefore we shall assume henceforth that there are at least two edges in the graph, so that the two odd vertices do not lie on the same edge. In such cases we can expand the graph by adding another edge F to whose endpoints are the two odd vertices, and this yields a new connected graph $(X', \mathcal{E}')$ with no vertices of odd order. By the argument in the preceding paragraphs, this graph has an Euler path $E_1 \ \cdots \ E_m$ whose edges include all the edges of $(X', \mathcal{E}')$. As before, we can cyclically reorder the edges so that the last edge is the additional edge F, and if we remove this edge we obtain an Euler path for $(X, \mathcal{E})$.■

Further references for graph theory

There are many textbooks on the subject, and here are three of them. The book by Bondy and Murty is probably a good first reference at the undergraduate level, while the first two chapters of the book by Harary also cover the basics (later chapters go much further into the theory) and the book by Chartrand has more information on some applications of graph theory, both inside and outside of mathematics.

J. A. Bondy and U. S. R. Murty. *Graph Theory: An Advanced Course*. Springer-Verlag, New York-*etc.*, 2008.

G. Chartrand. *Introductory Graph Theory* [UNABRIDGED]. Dover Publications, New York, 1984.

F. Harary. *Graph Theory*. AddisonWesley, Reading, MA, 1969.