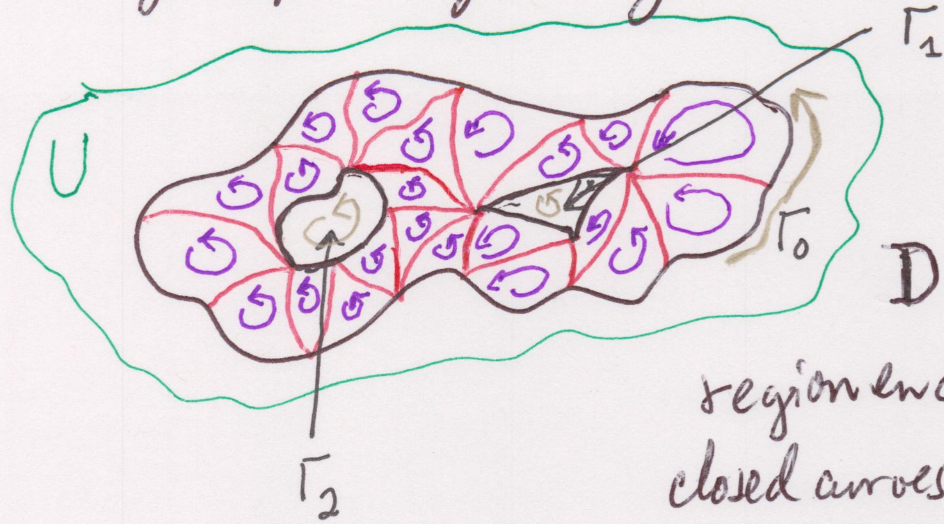
Triangulations

$\boxed{\vec{F} \text{ smooth}}$

<u>Motivation</u> $\vec{F}$ vector field on open set U containing closed, bounded region D whose boundary is finitely many nice curves Γ_i.



D is the region enclosed by the closed curves $\Gamma_0, \Gamma_1, \Gamma_2$

Intuitively, one curve Γ_0 is outermost (need to make this rigorous eventually!). Parametrize Γ_i in the counterclockwise sense.

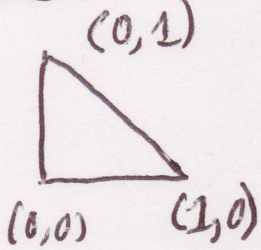
<u>Standard topic in MV calculus</u>

Prove Green's Theorem for D.

$$\vec{F} = \left(P(x,y), Q(x,y)\right)$$

$$\left[\int_{\Gamma_0} - \sum_{i \geq 1} \int_{\Gamma_i}\right] (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

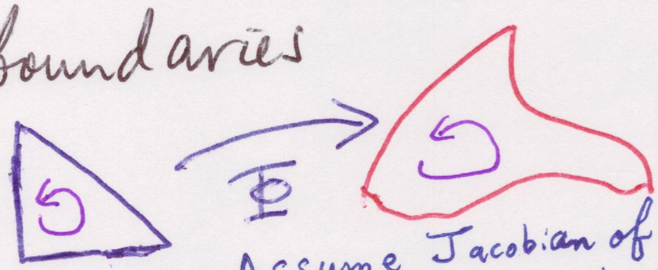
Idea ① Prove for triangular region



(Straightforward)

② Prove for regions with curved boundaries

Prev. case + change of variables for double integrals

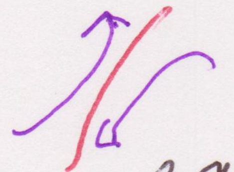


③ Cut D up into triangles as illustrated (needs rigorous justification). Apply ② to each piece and add.

Ⓐ Sum of double integrals = integral over D.

and

(B) If we add the line integrals over the Δs, then the integrals over the inside (red) pieces cancel each other as below



and all that's left is the sum of the line integrals over the boundary pieces.

> Usually this can be done explicitly for any given example, but we would prefer a general proof.

Triangulation (2D)

$X = \cup A_i$

$A_i \cong$ triangle $\{x, y \geq 0,\ x+y \leq 1\}$

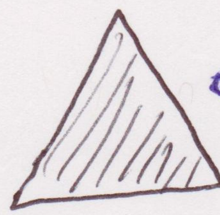
$i \neq j \Rightarrow A_i$ meets A_j in an edge.

Generalize formally to all finite dims.

Building blocks

points

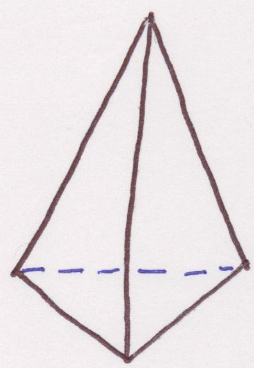
line segments



solid triangular regions

Better than Square regions. Every polygonal region can be cut up into Δr regions

tetrahedra (3D)



3D polyhedral regions can be cut up into tetrahedral regions.

http://faculty.grsu.edu/goldenj/TriangularPrismDissection.html

(interactive picture)

n-dimensions   n-simplex in $\mathbb{R}^n$

$$\{(x_1,\dots,x_n): x_i \geq 0, \sum x_j \leq 1\}$$

More symmetric version

In $\mathbb{R}^{n+1}$, take all points $(x_1, \ldots, x_{n+1})$ s.t. $x_i \geq 0$ and $\sum x_j = 1$

1st $\rightsquigarrow$ 2nd Let $x_{n+1} = 1 - \sum_{i=1}^{n} x_i$.

2nd $\rightsquigarrow$ 1st Drop last coord.

General n-dim simplex in $\mathbb{R}^m$.

Vertices $v_0, \ldots, v_n$ s.t.

$v_1 - v_0, \ldots, v_n - v_0$ are linearly indep.

Take all points $\sum t_i v_i$ s.t. all $t_i \geq 0$ and $\sum t_j = 1$.

<u>Prop.</u> In above setting, $\exists!$ way of writing such a point as $\sum s_i v_i$ with $s_i \geq 0$ and $\sum s_i = 1$.

<u>Proof</u> $x = \sum t_i v_i = \sum s_i v_i \Longrightarrow$

$$x - v_0 = \sum t_i (v_i - v_0) \overset{*}{=} \sum s_i (v_i - v_0)$$

Linear indep now $\Longrightarrow t_i = s_i,\ i>0$

Then show $s_0 = t_0$

$*$ Because $\left(\sum t_i v_i\right) - v_0 =$

$$\sum t_i v_i - \sum t_i v_0 = \sum_{i>0} t_i (v_i - v_0)$$

$\boxed{\sum t_i = 1}$ $\boxed{t_0(v_0 - v_0) = 0}$

Continuation in

<u>math246Anotes2010.pdf</u>

in the course directory.

see Section I.2, pp. 15-21.

<u>Def.</u> A <u>triangulation</u> of a space X is a homeomorphism $P \longrightarrow X$, where P is a simplicial complex. (see p. 17 of above file).