Homology of simplicial complexes

(P, K) simplicial complex. Choose a linear ordering of the vertices. Represent a simplex by its vertices in order.

__Chain groups__ $C_q(P, K, \text{ordering}) = C_q(P, K)$ for short.

$C_q(P, K) =$ free abelian group on the q-simplices $x_0 \cdots x_q$.

Boundary:

OMIT X_i

$$d(x_0 \dots x_q) = \sum_{i=0}^{q} (-1)^i x_0 \dots \hat{x_i} \dots x_q =$$

$$\sum_{i=0}^{q} (-1)^i \partial_i (x_0, \dots, x_q).$$

ith face (opposite the ith vertex)

Reasons for this:

$$d(x_0 x_1) = x_1 - x_0.$$

See ~~[illegible]~~ chainboundary.pdf for additional motivation.

Fundamental identity:

$$d_{q-1} \circ d_q = 0, \text{ so}$$

$$\text{Kernel } d_{q-1} \supseteq \text{Image } d_q.$$

Homology groups: $H_q(P, K) = \dfrac{\text{Kernel } d_q}{\text{Image } d_{q+1}}.$

Kernel d_q = cycles, Image d_{q+1} = boundaries.

Generalizes earlier definition in the 1-dim case.

It is useful to define a very abstract version of simplicial chains:

<u>Def.</u> A <u>chain complex</u> over a ring R is a pair (C_*, d_*) where C_q ~~is an~~ are R-modules and $d_q : C_q \longrightarrow C_{q-1}$ s.t. $d_{q-1} \circ d_q = 0$.

Homology $H_q(C) = \dfrac{\operatorname{Ker} d_q}{\operatorname{Image} d_{q+1}}$

Map of chain complexes:

$f_q : C_q \longrightarrow D_q$ s.t.

$$\begin{tikzcd} C_q \arrow[r, "f_q"] \arrow[d, "d_q^C"'] & D_q \arrow[d, "d_q^D"] \\ C_{q-1} \arrow[r, "f_{q-1}"] & D_{q-1} \end{tikzcd}$$

commutes.

$$\boxed{f_{q-1} \circ d_q^C = d_q^D \circ f_q}$$

<u>Example</u> (Q, L) subcomplex (P, K)
[$\Leftrightarrow$ if simplex $\sigma \in L$ then so are all its faces]. $Q \subseteq P$.

Then inclusion $C_*(Q, L) \to C_*(P, K)$ is a chain map (chain subcomplex).

<u>Lemma</u> $f_\#: C_\# \to D_\#$ chain map $\Rightarrow$
$\exists$ homomorphisms ~~[illegible]~~

$$f_*: H_*(C) \longrightarrow H_*(D)$$

s.t. if $u \in C_q$ represents $x \in H_q(C)$, then $f_\#(u)$ represents $f_*(x) \in H_q(D)$.

<u>Quotient complex.</u>

$$(A_*, d^A_*) \subseteq (B_*, d^B_*) \rightsquigarrow \exists \text{ chain map}$$

$$(B_*, d^B_*) \xrightarrow[\text{onto}]{} (B_*/A_*, d^{B/A}_*).$$

Proof of Lemma Must show

① $0 = d_q x \Longrightarrow 0 = d_q f(x)$ $[= f(\underbrace{d_q(x)}_{=0})]$

② $dx = dx' = 0$ and $x - x' = dy$

implies $df(y) = f(dy) = f(x) - f(x')$.

③ module hom.

x reps u
x' reps u' $\Longrightarrow$ $x + x'$ reps $u + u'$
$c \in R$ $\quad$ cx reps cu.

Further properties (functoriality)

$(1_C)_* =$ identity on $H_*(C)$

$(gf)_* = g_* f_*$

[$\Rightarrow$ if f is an iso, then so is f_*].

COMPUTING HOMOLOGY

Assume each C_* is fin gen free abelian, $C_q = 0$ for all but finitely many q.

$\boxed{R = \text{principal ideal domain}}$

Also assume we are given ordered free bases $\mathcal{B}_q$ for C_q and that we can easily find the matrix for d_q w.r.t. $\mathcal{B}_q$ and $\mathcal{B}_{q-1}$ (clearly true for simplicial chains). Then there is a direct way of computing $H_q(C)$.

See computingHq.pdf

Back to the general form of Green's Theorem

In terms of the concepts presented, the assumptions behind the general form of Green's Theorem are:

(1) We can triangulate D (by P, K) so that its boundary corresponds to a subcomplex (the big cx is 2D).

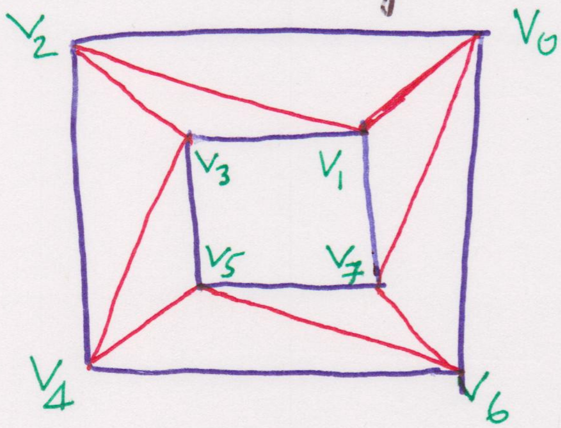
(2) If $\alpha_1, \dots, \alpha_M$ are the free generators of $C_2(P,K)$, then $\exists\, \varepsilon_i = \pm 1$ s.t.

$$d\left(\sum \varepsilon_i \alpha_i\right) = \gamma_0 - \sum_{j>0} \gamma_j$$

where $\gamma_j \in C_1(P,K)$ corresponds to going around boundary curve γ_j in the counterclockwise sense.

** Need the ε_i ~~so that~~ because the standard identification of Δ_2 with a triangle in K (order preserving on vertices) may have negative Jacobian.

<u>Example</u> Look at the region between two squares with the indicated triangulation



The counter clockwise inner and outer boundary curves are given by

$$V_1V_3 + V_3V_5 + V_5V_7 - V_1V_7 \quad \text{(inner)}$$

$$V_0V_2 + V_2V_4 + V_4V_6 - V_0V_6 \quad \text{(outer)}$$

and (outer) − (inner) is the boundary of

$$\boxed{\begin{aligned} &- V_0V_1V_2 + V_1V_2V_3 - V_2V_3V_4 + V_3V_4V_5 \quad \text{(continued)} \\ &- V_4V_5V_6 + V_5V_6V_7 - V_0V_6V_7 + V_0V_1V_7 \end{aligned}}$$

Finally, we note that the assumptions 1 and 2 can be verified, but this requires more machinery than we can develop in the present course.