Realizing $\mathbb{RP}^n$ as a subset of some Euclidean space

One way of defining the real projective space $\mathbb{RP}^n$ is to say that it is the quotient of S^n by the equivalence relation $x \sim y \iff x = \pm y$; in particular, this is explained in the solutions to the Additional Exercises for Section V.1. It follows immediately that $\mathbb{RP}^n$ is a compact space, and one would also like to know that it is Hausdorff. Since the quotient of a metric space is not necessarily Hausdorff, some effort is needed to prove that $\mathbb{RP}^n$ is indeed Hausdorff. The purpose of this document is to prove the following stronger result:

EMBEDDING THEOREM. *For each $n \geq 1$ there is some positive integer M such that $\mathbb{RP}^n$ is homeomorphic to a subset of $\mathbb{R}^M$.*

We shall derive this fact from the next result.

PROPOSITION. *For each $n \geq 2$ there is a continuous mapping $g_n : \mathbb{R}^{n+1} \to \mathbb{R}^M(n)$, where $M(n)$ is some large positive integer, such that the following hold:*

(i) For all $u, v \in \mathbb{R}^{n+1}$ we have $g_n(u) = g_n(v)$ if and only if $u = \pm v$.

(ii) We also have $g_n(x) = \mathbf{0}$ if and only if $x = \mathbf{0}$.

Proof that the Proposition implies the Embedding Theorem. Let $f_n = g_n|S^n$, and let $p_n : S^n \to \mathbb{RP}^n$ be the quotient projection. Then $f_n(x) = f_n(-x)$ implies that f_n factors as a composite $h_n \circ p_n$ for some continuous mapping $h_n : \mathbb{RP}^n \to R^{M(n)}$. By Theorem III.1.9 in `gentopnotes2014.pdf`, it suffices to prove that h_n is 1–1. Suppose we have equivalence classes $[x]$ and $[y]$ in $\mathbb{RP}^n$ such that $h_n([x]) = h_n([y])$. By the definition of h_n this implies that $g_n(x) = g_n(y)$, and by property (i) in the proposition it follows that $y = \pm x$. Since this implies $[x] = [y]$, it follows that h_n is 1–1 as required.■

COROLLARY. *For all integers $n \geq 1$, the space $\mathbb{RP}^n$ is homeomorphic to a metric space; in particular, $\mathbb{RP}^n$ is Hausdorff.*■

Proof of the Proposition. We shall construct the mappings g_n recursively. If $n = 1$ then we can take $g_n(x, y) = (x^2 - y^2, 2xy)$, which sends $z \in \mathbb{C} \cong \mathbb{R}^2$ to the complex number z^2. Suppose now that we have constructed $g_{n-1} : \mathbb{R}^n \to \mathbb{R}^{M(n-1)}$. We can then construct

$$g_n : \mathbb{R} \times \mathbb{R}^{n-1} \cong \mathbb{R}^n \longrightarrow \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{M(n-1)}$$

by the formula

$$g_n(t, z) = \left(t^2, tz, g_{n-1}(z)\right) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{M(n-1)} .$$

To complete the proof of the proposition, we need to verify that g_n satisfies (i) and (ii). We shall start with the second property. If $(t, z) = (0, 0)$ then one can check directly that $g_n(0, 0) = (0, 0, 0)$. Conversely, if $g_n(t, z) = (0, 0, 0)$ then by taking coordinates we see that $t^2 = 0$, $tz = 0$ and $g_{n-1}(z) = 0$. The first equation implies that $t = 0$, and the induction hypotheses on g_{n-1} imply that $z = 0$; therefore (ii) is satisfied.

We must now verify that g_n satisfies property (i). By construction we have $g_n(t, z) = g_n(-t, -z)$. Conversely, suppose that $g_n(t, z) = g_n(s, w)$. Equating coordinates, we see that the latter implies the equations $t^2 = s^2$, $tz = sw$ and $g_{n-1}(z) = g_{n-1}(w)$. The first equation implies that $t = \alpha\, s$ where $\alpha = \pm 1$, and the induction hypotheses on g_{n-1} imply that $z = \beta\, w$ where

$\beta = \pm 1$. If we can choose α and β such that $\alpha = \beta$, then g_n will satify property (i), and this will prove the proposition.

At this point the argument splits into cases depending upon whether or not $t = 0$ and whether or not $z = 0$.

CASE 1: Suppose that $t = z = 0$. — Then we have $g_n(t, z) = 0$ and we can combine $g_n(t, z) = g_n(s, w)$ with the already verified property (ii) to conclude that $(s, w) = (0, 0) = (t, z)$, and hence we have $t = \alpha\, s$ and $z = \alpha\, w$ where $\alpha = \beta = 1$.

CASE 2: Suppose that $t = 0$ but $z \neq 0$. — The first coordinates of $g_n(t, z)$ and $g_n(s, w)$ are t^2 and s^2 respectively, and hence $t = 0$ and $g_n(t, z) = g_n(s, w)$ imply that $0 = t^2 = s^2$, so that $s = 0$. We know that $z = \beta\, w$ where $\beta = \pm 1$, and since $t = s = 0$ we trivially have $t = \beta\, s$.

CASE 3: Suppose that $z = 0$ but $t \neq 0$. — The third coordinates of $g_n(t, z)$ and $g_n(s, w)$ are $g_{n-1}(z) = 0$ and $g_{n-1}(w)$ respectively, and hence $z = 0$ and $g_n(t, z) = g_n(s, w)$ imply that $0 = g_{n-1}(z) = g_{n-1}(w)$, so that $w = 0$. Therefore if $t = \alpha\, s$ where $\alpha = \pm 1$, then we also have $z = \alpha\, w$.

CASE 4: Suppose that both $z \neq 0$ and $t \neq 0$. — Then $t = \alpha\, s$ and $z = \beta\, w$ imply that both s and w are nonzero. In this case the second coordinates of $g_n(t, z)$ and $g_n(s, w)$ are tz and sw respectively, and hence $g_n(t, z) = g_n(s, w)$ implies that $tz = sw$; as in the first senteence, we know that all the factors and products in this equation are nonzero. If we combine the second coordinate equation with the two equations in the preceding sentence, we obtain the identity $\alpha\beta sw = sw$, and since sw is nonzero it follows that $\alpha\beta = 1$. Since both α and β are ± 1, it follows that $\alpha = \beta$, so that $t = \alpha\, s$ and $z = \alpha\, w$.■