Working with Homology

We can compute $H_q(P, K)$, but we also want to understand what sorts of information it carries.

Examples 1. Does it only depend on P and not on K?

2. What sorts of useful geometric or topological information does it contain?

3. Are there conceptual tools that simplify the computations in important special cases?

Each question has a positive answer (but maybe not provable in this course).

At last we can answer the last two questions now — <u>reverse order.</u>

$H_q(\{pt.\}$, obvious "decomposition") $=$

$$C_q(\text{DITTO}) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q \neq 0. \end{cases}$$

$H_q(\Delta_n$, standard decomposition$)$.

We want contractible objects to have the same homology as a point. This can be done by defining an algebraic analog of a contracting homotopy ("contracting chain homotopy").

$C^\#_q(\Delta_n) \supseteq C_q(\Delta_n)$, with extra free generators $e_0 e_{i_0} \dots e_{i_{q-1}}$ $0 = i_0 < \dots < i_{q-1}$

Straightforward verifications:

① $C_*(\Delta_n) \overset{i^\#}{\subseteq} C_*^\#(\Delta_n)$ is an inclusion of chain complex (define d on $C_*^\#$ in the usual fashion)

② There ∃ a 1-sided inverse $C_*^\#(\Delta_n) \xrightarrow[\rho]{\text{RHO}} C_*(\Delta_n)$ sending all generators $e_0 e_0 e_{i_1} \dots e_{i_{q-1}}$ to 0, AND it is a chain complex map.

The contracting chain homotopy will be a sequence of maps $D_q : C_q \longrightarrow C_{q+1}$ such that

$$d_{q+1} \circ D_q + D_{q-1} \circ d_q = \begin{cases} 1 & q \neq 0 \\ 1 - \alpha & q = 0 \end{cases}$$

$\alpha(v) = v - e_0$.

Implication:

$$H_*(\{e_0\}, \dots) \xrightarrow{\cong} H_*(\Delta_n, \dots).$$

Derivation

$* = 0$ $\quad H_0(\{e_0\}, \dots) \longrightarrow H_0(\Delta_n, \dots)$

$e_i - e_0 = d(e_0 e_i) \Rightarrow$ RHS gen by class of e_0. No multiple of e_0 is a boundary except 0, for if

~~$ke_0 = d(\sum m_j e_j$~~

$$k e_0 = d\Big(\sum_{i<j} m_{ij} e_i e_j\Big) \text{ then}$$

RHS $= \sum m_{ij} e_j - m_{ij} e_i$, and if we rewrite this as $\sum n_\ell e_\ell$, we see that $\sum n_\ell = 0$.

Higher degrees or dimensions

Suppose that $a \in C_q$, $da = 0$. Then

$$a = dD + Dd(a) = dDa + \underbrace{\cancel{Dda}}_{\substack{\text{zero} \\ (da=0)}}$$
$$= dDa.$$

Hence $H_q = 0$.

<u>Construction of D.</u>

$$D_q(e_{j_0} \cdots e_{j_q}) = \rho(e_0 e_{j_0} \cdots e_{j_q}).$$

Verify that $dD + Dd(e_{j_0} \cdots e_{j_q})$

$= e_{j_0} \cdots e_{j_q}$ if $q > 0$

$e_{j_0} - e_0$ if $q = 0$.

CHECK THIS YOURSELF!

good exercise to test one's understanding of "d".

<u>NEXT STEP</u> $\partial \Delta_n \subseteq \Delta_n$

$=$ everything except top simplex.

Note that $\partial \Delta_n \cong S^{n-1}$.

$n=1 \qquad H_q = \mathbb{Z} \oplus \mathbb{Z} \ (q=0),\ 0 \ (q>0)$

$n=2 \qquad H_q \doteq \mathbb{Z} \ (q=0,1),\ 0 \ (q \neq 0,1)$

$n=3, \ldots$

Introduce relative groups.

$(Q, L) \subseteq (P, K)$ subcomplex.

$$C_q(K, L) = C_q(P, K; Q, L) = C_q(P, K) / C_q(Q, L).$$

quotient chain complex.

$H_q(K, L) =$ homology of $C_*(K, L)$.

Now $C_q(\Delta_n, \partial \Delta_n) = \begin{cases} \mathbb{Z} & q = n-1 \\ 0 & \text{otherwise} \end{cases}$

All boundary maps are trivial, so

$$C_q(\Delta_n, \partial \Delta_n) \cong H_q(\Delta_n, \partial \Delta_n).$$

<u>Obvious next issue:</u> Translate this into information about $H_*(\partial\Delta_n)$.

This will require an algebraic digression. The methods and conclusions have far-reaching importance.