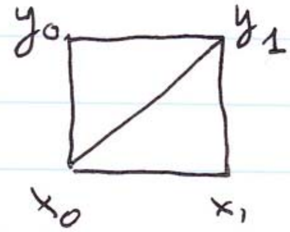
STILL MORE EXAMPLES

Starting point.

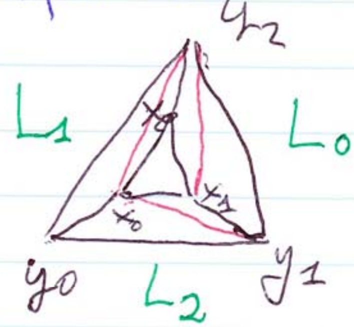


Star shaped w.r.t. vertex $x_0 \Rightarrow$

$$H_*(\{x_0\}) \xrightarrow{\cong} H_*(\text{Square}) = \begin{cases} \mathbb{Z} & *=0 \\ 0 & \text{otherwise.} \end{cases}$$

Triangular prism, lateral faces (L)

Flattened out picture



CLAIM Inclusions of $x_0x_1x_2$ in L and $y_0y_1y_2$ in L are isomorphisms in homology.

Use Mayer-Vietoris Sequence. Let L_i = lateral side opposite x_i & y_i.

Then $L_1 \cup L_2 \cup L_3 = L$, $\underset{i \neq j}{L_i \cap L_j}$ = edge $x_k y_k$

$0 = H_q(L_i)$ $q > 0$, $\mathbb{Z} = H_q(L_i)$ $q = 0$.

What is $H_*(L_1 \cup L_2)$?

$$0 = H_2(L_1 \cap L_2) \to \begin{matrix} H_2(L_1) = 0 \\ \oplus \\ H_2(L_2) = 0 \end{matrix} \to H_2(L_1 \cup L_2)$$

$$\to H_1(L_1 \cap L_2) \longrightarrow \begin{matrix} H_1(L_1) = 0 \\ \oplus \\ H_1(L_2) = 0 \end{matrix} \longrightarrow H_1(L_1 \cap L_2)$$

$$H_1(L_1 \cap L_2) \overset{\shortparallel}{} 0$$

$$\to H_0(L_1 \cap L_2) \longrightarrow \begin{matrix} H_0(L_1) \\ \oplus \\ H_2(L_2) \end{matrix}$$

$$\mathbb{Z} \xrightarrow{\text{diag.}} \mathbb{Z} \oplus \mathbb{Z}$$

We understand H_0, given by conn components.

Hence $H_0(L_1 \cap L_2) \cong \mathbb{Z}$ since $L_1 \cup L_2$ connected

but $H_1(L_1 \cup L_2) = H_2(L_1 \cup L_2) \cong 0$.

Now look at $L = L_1 \cup L_2 \cup L_0$ and the inclusions $x_0 x_1 x_2 \xrightarrow{J_x} L$, $y_0 y_1 y_2 \xrightarrow{J_y} L$ induce the same isomorphism; i.e., if $h: x_0 x_1 x_2 \xrightarrow{\cong} y_0 y_1 y_2$ then

$$\begin{tikzcd}
H_*(x_0 x_1 x_2) \arrow[d, "h_*"'] \arrow[dr, "J_{x*}"] & \\
H_*(y_0 y_1 y_2) \arrow[r, "J_{y*}"'] & H_*(L)
\end{tikzcd}$$

commutes.

Analyze again using M-V.

$$L = L_0 \cup (L_1 \cup L_2)$$

$$L_0 \cap (L_1 \cup L_2) = x_1 y_1 \cup x_2 y_2$$

$$\to H_q(L_1 \cup L_2) \oplus H_q(L_0) \longrightarrow H_q(L) \longrightarrow H_{q-1}(\text{int.}) \longrightarrow H_{q-1}(L_1 \cup L_2) \oplus H_{q-1}(L_0)$$

The right and left hand sides are 0 if $q \geq 2$ and so is H_{q-1} (intersection). Hence $H_q(L) = 0$ if $q \geq 2$.

If $q = 1$ we have

$$\begin{tikzcd}
0 \arrow[r] & H_1(L) \arrow[r] & H_0(x_1y_1 \cup x_2y_2) \arrow[r] & \overset{\mathbb{Z}}{H_0(L_1 \cup L_2)} \oplus H_0(L_0) \\
0 \arrow[r] & H_1(x_0x_1x_2) \arrow[r] \arrow[u] & H_0(\{x_1, x_2\}) \arrow[r] \arrow[u, "\mathbb{Z}", "\cong"'] & \cancel{H_0(\ldots)} \oplus H_0(x_0x_1 \cup x_2x_0) \oplus H_0(x_1x_2) \arrow[u, "\mathbb{Z}"']
\end{tikzcd}$$

$$\mathbb{Z} \oplus \mathbb{Z}$$

So we need to look at the map

$$\underset{[x_1] \quad [x_2]}{\mathbb{Z} \oplus \mathbb{Z}} \longrightarrow \underset{(x_0x_1 \cup x_2x_0) \quad x_1x_2}{\mathbb{Z} \oplus \mathbb{Z}}$$

induced by taking components.

$[x_1], [x_2] \longrightarrow [x_1] = [x_2]$ in first coord.

same in second

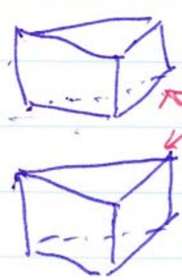
So $\mathbb{Z}\oplus\mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z}$ has the form

$$(w_1, w_2) \quad (v_1, v_2) = (w_1+w_2, -v_1-v_2)$$

Hence the map has a nontrivial (∞ cyclic) kernel, which means $H_1(L)\cong\mathbb{Z}$.
The diagram also yields an iso
$H_*(x_0x_1x_2) \xrightarrow{\cong} H_*(L)$.
We could have used the y's just as well. ∎

Double stacked lateral faces

K_2



identified.

K_1

$K_1 \cap K_2 = T$
$K_1 \cup K_2 = K$.

Here we know $H_*(T) \longrightarrow H_*(K_i)$ are isomorphisms. Set up another MV seq.

$T = T\cup T = T\cap T$

$$\begin{tikzcd}
\cdots H_{q+1}(K_1)\oplus H_{q+1}(K_2) \arrow[r] & H_{q+1}(K) \arrow[r] & H_q(T) \arrow[r] & H_q(K_1)\oplus H_q(K_2) \arrow[r] & H_q(K)\cdots \\
H_{q+1}(T)\oplus H_{q+1}(T) \arrow[r] \arrow[u, "\cong"] & H_{q+1}(T) \arrow[r] \arrow[u, "j_*"] & H_q(T) \arrow[r] \arrow[u, "(id)_*"] & H_q(T)\oplus H_q(T) \arrow[r] \arrow[u, "\cong"] & H_q(T)\cdots \arrow[u, "j_*"]
\end{tikzcd}$$

$\cong$ by previous.

Apply the Five Lemma to show j_* is an isomorphism. Likewise for the top and bottom inclusion of T in K.

Torus Triple stack with top and bottom identified

$X_i \sim W_i \quad i = 0, 1, 2.$

D = torus (doughnut)

G = xy + yz double stack

H = top stack.

$D = G \cup H, \quad G \cap H = x_0 x_1 x_2 \cup z_0 z_1 z_2.$

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{J} \begin{matrix} H_1(G) \; \mathbb{Z} \\ \oplus \\ H_1(H) \; \mathbb{Z} \end{matrix}$$

$$0 = \begin{matrix} H_2(G) \\ \oplus \\ H_2(H) \end{matrix} \longrightarrow H_2(D) \longrightarrow H_1(G \cap H) \longrightarrow \begin{matrix} H_1(G) \\ \oplus \\ H_1(H) \end{matrix}$$

$$\xrightarrow{\Delta} \begin{matrix} H_0(G) \\ \oplus \\ H_0(H) \end{matrix} \longrightarrow H_0(D).$$

onto, Kernel = $\mathbb{Z}$ by looking at components.

$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$

By previous observations, both generators of $H_1(G \cap H)$ go to the same generator in both $H_1(G)$ & $H_1(H)$.

This means that $H_2(D) \cong \mathbb{Z}$, $H_1(D) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Klein bottle Similar, but now

$$x_0 \sim w_0 \qquad K = \frac{S^1 \times [0,1]}{(z,0) \sim (z^{-1},1)}.$$
$$x_1 \sim w_2$$
$$x_2 \sim w_1.$$

Now the map $H_1(G \cap H) \longrightarrow H_1(G) \oplus H_1(H)$
$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$
changes.

$z_0 z_1 z_2$ still goes to the standard gens. of $H_1(G)$ & $H_1(H)$

but now $x_0 x_1 x_2$ goes to the generator $x_0 x_1 x_2$ in $H_1(G)$ and the negative of the generator for $H_1(H)$
$$= -w_0 w_1 w_2.$$

Why? $x_0 x_1 x_2$ goes to the path chain

$$-w_0 w_1 - w_1 w_2 + w_0 w_2.$$

Hence the exact sequence looks like

$$\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{J_1} \mathbb{Z}\oplus\mathbb{Z}$$

$$0 \longrightarrow H_2(K) \longrightarrow H_1(G\cap H) \longrightarrow \begin{matrix} H_1(G) \\ \oplus \\ H_1(H) \end{matrix} \longrightarrow H_1(K)$$

$$H_0(G\cap H) \longrightarrow \begin{matrix} H_0(G) \\ \oplus \\ H_0(H) \end{matrix} \longrightarrow H_1(D) \longrightarrow 0$$

$$\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{J_0} \mathbb{Z}\oplus\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Here J_1 has a matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow$

So $H_2(K)=0$

Kernel $=\{0\}$, Image $\cong \mathbb{Z}_2$. Since $\operatorname{Ker} J_0 = \mathbb{Z}$ we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow H_1(K) \longrightarrow \mathbb{Z} \longrightarrow 0$$

which means that $H_1(K) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$. ∎