Relative Homology

Def. P = one point space $\{p_0\}$. If X is an arbitrary space, then there is a unique continuous (in fact, constant) map $c_X : X \to P$.

Suppose $X \neq \phi$. Then the reduced singular homology group $\tilde{H}_q(X)$ is the kernel of $c_{X*} : H_q(X) \longrightarrow H_q(P)$.

Proposition (1) If $X = P$, then $\tilde{H}_q(X) = 0$ all q.

(2) If $f : A \to B$ is continuous + $A, B \neq \phi$, then $f_* : H_*(A) \longrightarrow H_*(B)$ sends $\tilde{H}_*(A)$ to $\tilde{H}_*(B)$.

(3) If $X \neq \phi$, then $H_*(X) \cong \tilde{H}_*(X) \oplus H_*(P)$.

Proof. (1) c_P is the identity map, so c_{P*} is an isomorphism and its kernel is 0.

(2) Since c_A + c_B are constant maps, $c_B \circ f = c_A$, so $c_{A*} = c_{B*} \circ f_*$. Hence $c_{A*}(w) = 0 \Rightarrow c_{B*}(f_*(w)) = 0$.

(3) Let $x_0 \in X$, and let $r: P \to X$ be defined by $r(p_0) = x_0$. Then $c_X \circ r = $ identity on P, so $id_{H_*(P)} = c_{X*} \circ r_*$. CLAIM $H_q(X) \cong \tilde{H}_q(X) \oplus \mathrm{Im}\, r_*$, and r_* is 1-1 (hence the second summand is isomorphic to $H_q(P)$).

The map r_* is 1-1 because $c_{X*} \circ r_* =$ identity. To see its image is a direct summand, must show $H_q = \tilde{H}_q + \mathrm{Im}\, r_*$ and $0 = \tilde{H}_q \cap \mathrm{Im}\, r_*$. If $u \in H_q$, consider $v = u - r_* c_* u$; this lies in the kernel of c_*, for $c_*(u) = \underbrace{c_* r_*}_{\text{identity}} c_*(u)$ which is 0. If $u \in \tilde{H}_q \cap \mathrm{Im}\, r_*$ then $u \in \tilde{H}_q \Rightarrow c_* u = 0$, and $u = r_* y \Rightarrow y = c_* r_* y = c_* u = 0$, so that $u = r_* 0 = 0$.

Corollary $\tilde{H}_q(X) \cong H_q(X)$ if $q \neq 0$,
$\mathbb{Z} \oplus \tilde{H}_0(X) \cong H_0(X)$.

<u>Another useful fact.</u> $A \subseteq B$ $i =$ inclusion

If $u \in H_0(A)$ and $i_*(u) = 0$, then $u \in \tilde{H}_0(A)$.

<u>Proof</u> $i_*(u) = 0 \Rightarrow c_{B*}\, i_*(u) = (c_B \circ i)_*(u) = c_{A*}(u)$.

APPLICATION TO MAYER-VIETORIS SEQUENCES

$X = U \cup V$, U and V open in X. Then the image of $\Delta : H_1(X) \longrightarrow H_0(U \cap V)$ is contained in $\tilde{H}_0(U \cap V)$.

<u>Proof.</u> By exactness, we need only show that the kernel of $H_0(U \cap V) \xrightarrow{\varphi} H_0(U) \oplus H_0(V)$ has this property, where $\varphi(y) = (i_{U*}\,y, -i_{V*}\,y)$.

But $0 = \varphi(y) \Rightarrow i_{U*}\,y = 0$ and $-i_{V*}\,y = 0$. Hence we can apply the observation at the top of the page.

It follows that the tail end of the MV sequence yields an exact sub-sequence

$$H_1(X) \xrightarrow{\tilde{\Delta}} \tilde{H}_0(U \cap V) \longrightarrow \begin{matrix} \tilde{H}_0(U) \\ \oplus \\ \tilde{H}_0(V) \end{matrix} \longrightarrow \tilde{H}_0(X) \longrightarrow 0$$

because it is a straightforward exercise now to check that

$$0 \longrightarrow \underset{=\mathrm{Im}\,\Delta}{\mathrm{Im}\,\tilde{\Delta}} \longrightarrow \begin{matrix} \mathrm{Ker}\, c_{U*} \\ \oplus \\ \mathrm{Ker}\, c_{V*} \end{matrix} \longrightarrow \mathrm{Ker}\, c_{X*} \longrightarrow 0$$

is exact. Recall this is an ~~exact~~ sub-sequence of

$$0 \longrightarrow \mathrm{Im}\,\Delta \longrightarrow \begin{matrix} H_0(U) \\ \oplus \\ H_0(V) \end{matrix} \longrightarrow H_0(X) \longrightarrow 0.$$