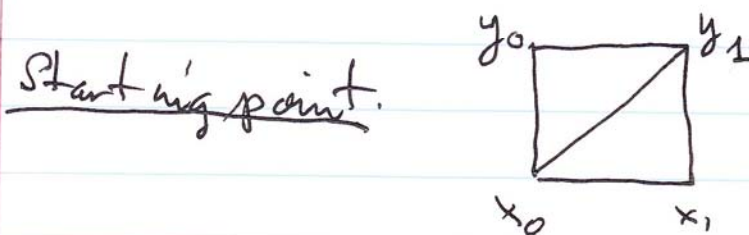


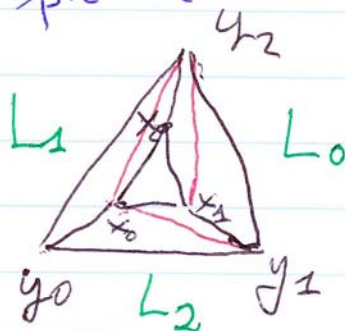
STILL MORE EXAMPLES



Star shaped w.r.t. vertex $x_0 \rightarrow$
 $H_x(\{x_0\}) \xrightarrow{\cong} H_x(\text{Square}) = \begin{cases} \mathbb{Z} & x=0 \\ 0 & \text{otherwise} \end{cases}$

Triangular prism, lateral faces (L)

Flattened out picture



CLAIM Inclusions of $x_0 x_1 x_2$ in L

and $y_0 y_1 y_2$ in L are isomorphisms in homology.

Use Mayer-Vietoris Sequence. Let $L_i =$ lateral side opposite $x_i + y_i$.

Then $L_1 \cup L_2 \cup L_3 = L$, $L_i \cap L_j = \text{edge } x_i y_j$
 $i \neq j$

$$0 = H_q(L_i) \quad q > 0, \quad \mathbb{Z} = H_q(L_i) \quad q = 0.$$

What is $H_*(L_1 \cup L_2)$?

$$0 = H_2(L_1 \cap L_2) \rightarrow \begin{matrix} H_2(L_1) = 0 \\ \oplus \\ H_2(L_2) = 0 \end{matrix} \rightarrow H_2(L_1 \cup L_2)$$

$$\begin{matrix} \leftarrow \\ H_1(L_1 \cap L_2) \end{matrix} \rightarrow \begin{matrix} H_1(L_1) = 0 \\ \oplus \\ H_1(L_2) = 0 \end{matrix} \rightarrow H_1(L_1 \cup L_2)$$

$$\begin{matrix} \leftarrow \\ H_0(L_1 \cap L_2) \end{matrix} \rightarrow \begin{matrix} H_0(L_1) \\ \oplus \\ H_0(L_2) \end{matrix} \rightarrow H_0(L_1 \cup L_2)$$

" diag. $\rightarrow \mathbb{Z} \oplus \mathbb{Z}$

We understand H_0 , given by conn components.

Hence $H_0(L_1 \cap L_2) \cong \mathbb{Z}$ since $L_1 \cup L_2$ connected

but $H_1(L_1 \cup L_2) = H_2(L_1 \cup L_2) \cong 0$.

Now look at $L = L_1 \cup L_2 \cup L_0$ and the

inclusions $x_0 \times x_1 \times x_2 \xrightarrow{j_x} L, \quad y_0 \times y_1 \times y_2 \xrightarrow{j_y} L$

induce the same isomorphism, i.e., if $h: x_0 \times x_1 \times x_2 \xrightarrow{\cong} y_0 \times y_1 \times y_2$ then

$$\begin{array}{ccc} H_*(x_0 \times x_1 \times x_2) & \xrightarrow{j_x} & H_*(L) \\ \downarrow h_* & \searrow & \downarrow j_y \\ H_*(y_0 \times y_1 \times y_2) & \xrightarrow{j_y} & H_*(L) \end{array} \text{ commutes.}$$

Analyze again using M-V.

$$L = L_0 \cup (L_1 \cup L_2)$$

$$L_0 \cap (L_1 \cup L_2) = x_1 y_1 \cup x_2 y_2$$

$$\rightarrow H_q(L_1 \cup L_2) \oplus H_q(L_0) \rightarrow H_q(L) \rightarrow H_{q-1}(\text{int.}) \rightarrow H_{q-1}(L_1 \cup L_2) \oplus H_{q-1}(L_0)$$

The right and left hand sides are 0 if $q \geq 2$ and so is $H_{q-1}(\text{intersection})$.
Hence $H_q(L) = 0$ if $q \geq 2$.

If $q=1$ we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_1(L) & \longrightarrow & H_0(x_1 y_1 \cup x_2 y_2) & \longrightarrow & H_0(L_1 \cup L_2) \\ & & \uparrow & & \uparrow \cong & & \oplus \\ & & & & \mathbb{Z} & & H_0(L_0) \\ & & & & \uparrow \cong & & \uparrow \mathbb{Z} \\ 0 & \longrightarrow & H_1(x_0 x_1 x_2) & \longrightarrow & H_0(\{x_1, x_2\}) & \longrightarrow & H_0(\{x_1, x_2\}) \\ & & & & & & \oplus \\ & & & & & & H_0(x_0 x_1 \cup x_2 x_0) \\ & & & & & & \oplus \\ & & & & & & H_0(x_1 x_2) \\ & & & & & & \mathbb{Z} \oplus \mathbb{Z} \end{array}$$

So we need to look at the map

$$\mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$$

$$\begin{array}{ccc} [x_1] & [x_2] & \uparrow \downarrow \\ & & (x_0 x_1 \cup x_2 x_0) \end{array} \quad \begin{array}{c} \uparrow \\ x_1 x_2 \end{array}$$

induced by taking components.

$$[x_1], [x_2] \longrightarrow [x_3] = [x_2] \text{ in first coord.}$$

same in second

So $\mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ has the form
 $(u_1, u_2) \mapsto (v_1, v_2) = (u_1 + u_2, -v_1 - v_2)$

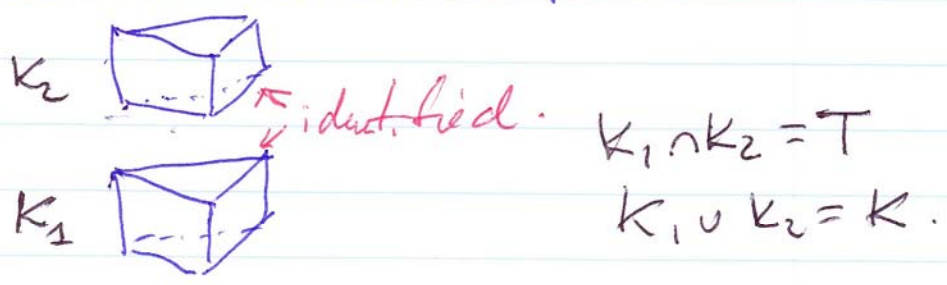
Hence the map has a non-trivial
 (∞ cyclic) kernel, which means $H_1(U) \cong \mathbb{Z}$.

The diagram also yields an iso

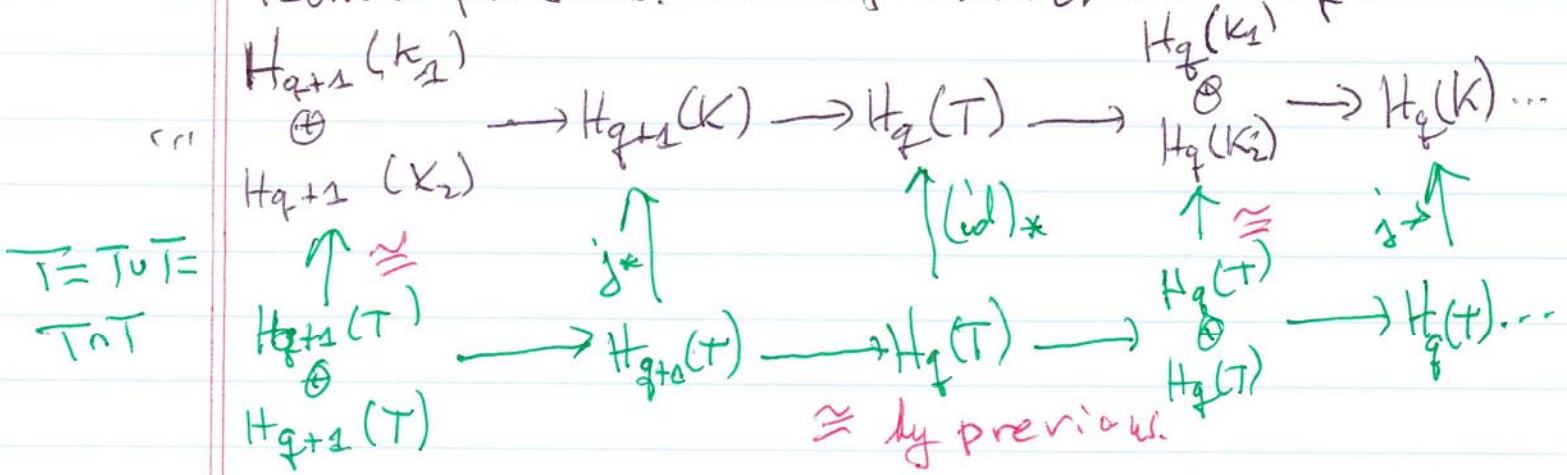
$$H_* (x_0 \times x_1 \times x_2) \xrightarrow{\cong} H_* (L).$$

We could have used the y 's if just as well. \square

Double stacked lateral faces



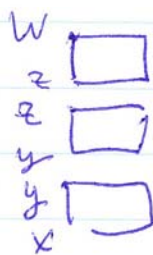
Here we know $H_*(T) \rightarrow H_*(K_i)$ are
 isomorphisms. Set up another MV seq.



$T \cong T \cup T =$
 $T \cap T$

Apply the Five Lemma to show j_* is an isomorphism. Like wise for the top and bottom inclusions of T in K .

Torus Triple stack with top and bottom identified



$$X_i \sim W_i \quad i=0,1,2.$$

$D = \text{torus (doughnut)}$

$G = xytyz$ double stack

$H = \text{top stack.}$

$$D = G \cup H, \quad G \cap H = x_0 x_1 x_2 \cup z_0 z_1 z_2.$$

$$0 = \begin{matrix} H_2(G) \\ \oplus \\ H_2(H) \end{matrix} \rightarrow H_2(D) \rightarrow H_1(G \cap H) \rightarrow \begin{matrix} H_1(G) \cong \mathbb{Z} \\ \oplus \\ H_1(H) \cong \mathbb{Z} \end{matrix}$$

$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{J} \mathbb{Z}$

$$\begin{matrix} H_0(G) \\ \oplus \\ H_0(H) \end{matrix} \xrightarrow{\Delta} H_0(D).$$

onto, Kernel = \mathbb{Z} by looking at components.

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$$

By previous observations, both generators of $H_1(G \cap H)$ go to the same generator in both $H_1(G) \times H_1(H)$.

This means that $H_2(D) \cong \mathbb{Z}$, $H_1(D) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Klein bottle Similar, but now

$$\begin{aligned} x_0 &\sim w_0 \\ x_1 &\sim w_1 \\ x_2 &\sim w_2. \end{aligned}$$

$$K = S^1 \times [0, 1] / (z, 0) \sim (z^{-1}, 1)$$

Now the map $H_2(G \cap H) \rightarrow H_2(G) \oplus H_2(H)$
 $\mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$
 changes.

~~$x_0 x_1 x_2$~~ still goes to the standard basis of $H_2(G) \oplus H_2(H)$
 $\mathbb{Z} \oplus \mathbb{Z}$

but now $x_0 x_1 x_2$ goes to the generator $x_0 x_1 x_2$ in $H_2(G)$ and the negative of the generator for $H_2(H)$
 $= -w_0 w_1 w_2.$

Why? $x_0 x_1 x_2$ goes to the path chain
 ~~$w_0 w_1 w_2 - w_1 w_2 w_0 + w_2 w_0 w_1$~~
 $-w_0 w_1 w_2 - w_1 w_2 w_0 + w_2 w_0 w_1.$

Hence the exact sequence looks like

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{J_1} \mathbb{Z} \oplus \mathbb{Z}$$

$$0 \rightarrow H_2(K) \rightarrow H_1(G \rtimes H) \xrightarrow{\begin{matrix} H_1(G) \\ \oplus \\ H_1(H) \end{matrix}} H_1(K) \rightarrow 0$$

$$\begin{array}{ccccccc} & & & H_0(G) & & & \\ & & & \oplus & & & \\ & & & H_0(H) & \rightarrow & H_1(\mathbb{D}) & \rightarrow 0 \\ & \swarrow & & & & & \\ H_0(G \rtimes H) & \rightarrow & H_0(H) & & & & \\ \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{J_0} & \mathbb{Z} \oplus \mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & 0 \\ & & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & & & & \end{array}$$

Here J_1 has a matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow$
 So $H_2(K) = 0$

Kernel = $\{0\}$, Image $\cong \mathbb{Z}_2$. Since $\text{Ker } J_0 = \mathbb{Z}$
 we have a short exact sequence

$$0 \rightarrow \mathbb{Z}_2 \rightarrow H_1(K) \rightarrow \mathbb{Z} \rightarrow 0$$

which means that $H_1(K) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$. \square