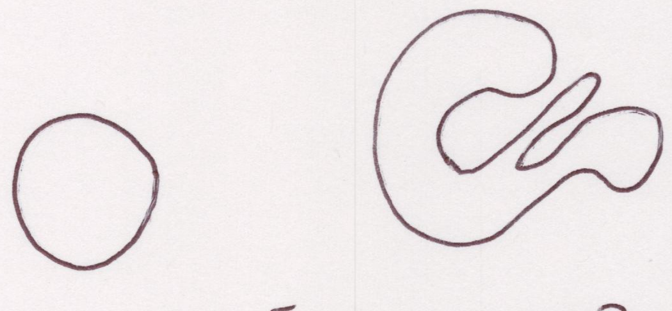
Classical Applications of Homology

Easiest one to formulate:

JORDAN CURVE THEOREM: Let $C \subseteq \mathbb{R}^2$ be homeomorphic to S^1. Then $\mathbb{R}^2 - C$ has two components, and C is the boundary of both.



EASY TO SEE HARDER

STILL HARDER — fishmaze.pdf or

a curve that is highly nonsmooth like some fractal curve.

Special cases have been understood since prehistoric times, and some were tacitly used in Greek geometry but not treated in a logically rigorous manner (with maybe a few exceptions). Proper tools for formulating the result in a mathematically precise fashion did not get developed until the 18th – 19th centuries, and the first attempt to prove the result seems to be due to C. Jordan (1890's or so). The first correct proof was due to O. Veblen (1900's), one of the earliest American mathematicians to do "world class" research.

Reduced Homology

X nonempty $\Longrightarrow$ unique cont map

$c : X \to \{pt\}$. $\quad \rho : \{pt\} \to X \quad \rho(pt.) = x_0$

$\Longrightarrow c \circ \rho = 1_{\{pt\}} \Longrightarrow$

Prop. $H_*(X) \cong \underset{\substack{\uparrow \\ 1-1}}{\rho_*} H_*(pt.) \oplus \operatorname{Ker} c_*$

$\tilde{H}_*(X)$
reduced homology.

Prop. $f : X \to Y$ cont $\Rightarrow$

f_* maps $\tilde{H}_*(X)$ to $\tilde{H}_*(Y)$.

Proof $c_Y \circ f = c_X$, so if $u \in \tilde{H}_*(X)$,

then $c_{X*} u = 0 \Longrightarrow c_{Y*}(f_*(u)) = 0 \Longrightarrow$

$f_*(u) \in \tilde{H}_*(Y)$.

Reduced Mayer-Vietoris Sequence

$X = U \cup V$, $U \cap V \neq \phi$, U & V open in X

Then the MV sequence for $(X; U, V)$ contains a reduced exact sequence

$$\tilde{H}_{q+1}(X) \longrightarrow \tilde{H}_q(U \cap V) \longrightarrow \begin{matrix} \tilde{H}_q(U) \\ \oplus \\ \tilde{H}_q(V) \end{matrix} \longrightarrow \tilde{H}_q(X) \longrightarrow$$

For details, see reducedMVseq.pdf

$$\tilde{H}_q(X) \cong H_q(X) \text{ if } q \neq 0$$

$$\tilde{H}_0(X) \oplus \mathbb{Z} \cong H_0(X).$$

X contractible $\Rightarrow \tilde{H}_*(X) = 0.$

<u>Prop.</u> If $X \subseteq S^n$ is homeomorphic to D^k (say $k \leq n$), then $\tilde{H}_*(S^n - X) = 0$.

<u>Proof</u> $k = 0$: $S^n - X \cong \mathbb{R}^n$.

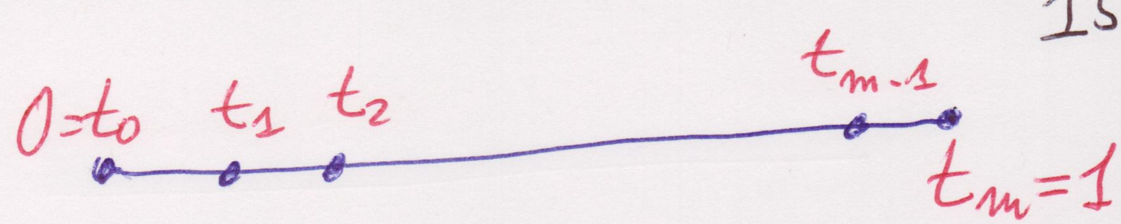
Suppose we know this for $X_0 \cong D^k$ and let $X \cong D^{k+1}$. By results on convex bodies, $X \cong D^k \times [0,1]$. If $A \subseteq [0,1]$ let $X_A \longleftrightarrow D^k \times A$, and let $X_t = X_{\{t\}}$.

<u>I</u>nduction hypothesis $\Longrightarrow$

$\tilde{H}_*(S^n - X_t) = 0$ all t.

Let $u \in \tilde{H}_*(S^n - X)$.

Then $u \to 0$ in $\tilde{H}_*(S^n - X_t)$ for all t.
Choose compact sets $K \subseteq L$ such that $K \subseteq S^n - X$, $L \subseteq S^n - X_t$, $u \in$ Image $\tilde{H}_*(K) \longrightarrow \tilde{H}_*(S^n - X)$, $u \to 0$ in $\tilde{H}_*(L)$. (COMPACT SUPPORTS). Now choose $\varepsilon(t) > 0$ so that $X_{[t-\varepsilon(t), t+\varepsilon(t)] \cap [0,1]}$ is disjoint from L (note that X_t and L have disjoint open neighborhoods!). Then $u \to 0$ in $\tilde{H}_*(S^n - X_{[t-\varepsilon(t), t+\varepsilon(t)]})$.
The intervals $[0,1] \cap (t - \varepsilon(t), t + \varepsilon(t))$ form an open covering of $[0, 1]$. By compactness we can construct a finite family of intervals



such that $w \to 0$ in $\tilde{H}_*(S^n - X_{[t_{i-1},t_i]})$

Let $X_i = X_{[0,t_i]}$. Will prove by induction that $w \to 0$ in $\tilde{H}_*(S^n - X_i)$.

Known if $i=1$ by previous.

Assume known ~~for~~ $i = \cancel{k}j-1$, and try to prove when $i = \cancel{k}j$

Look at the reduced MV sequence:

$$\tilde{H}_{q+1}(S^n - X_{t_{j-1}}) \to \tilde{H}_q(S^n - X_j) \to \begin{matrix} \tilde{H}_q(S^n - X_{j-1}) \\ \oplus \\ \tilde{H}_q(S^n - X_{[t_{j-1},t_j]}) \end{matrix}$$

$\tilde{H}_{q+1}(S^n - X_{t_{j-1}}) = 0$ by induction on k = disk dim.

Image of w

$[t_{j-1}, t_j]$

$$u \to 0 \text{ in } \tilde{H}_q(S^n - X_{j-1})$$

induction on j

$$u \to 0 \text{ in } \tilde{H}_q(S^n - X_{[t_{j-1}, t_j]})$$

by construction.

Hence $u \to 0$ in $\tilde{H}_q(S^n - X_j)$.
When $j = m$, we get $u \Rightarrow 0$ in $\tilde{H}_q(S^n - X)$, so that $u = 0$.

Jordan-Brouwer Separation Thm.

Suppose $A \subseteq S^n$, $A \cong S^{n-1}$.
Then $S^n - A$ has two components and A is the boundary of each one.

<u>Proof</u> Induction on k.

$k = 0 \Rightarrow \cancel{S^k}\; S^n - A = S^n - \{p,q\} \cong$

$\mathbb{R}^n - \{pt.\} \cong S^{n-1} \times \mathbb{R}$. WILL PROVE

$$\widetilde{H}_*(S^n - A) \cong \widetilde{H}_*(S^{n-k-1}).$$

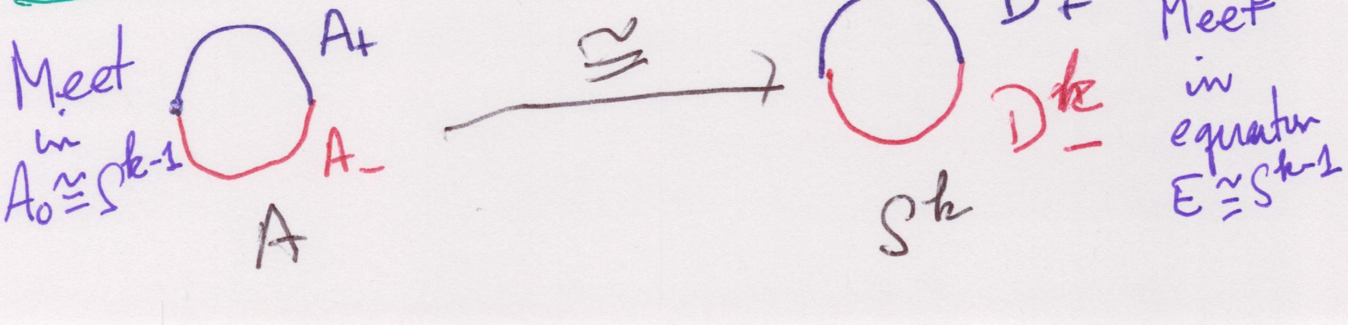
If so, then $k = n-1 \Rightarrow$

$\widetilde{H}_*(S^n - A) \cong \widetilde{H}_*(S^0) \Rightarrow$

$H_0(S^n - A) \cong \widetilde{H}_0(S^0) \oplus \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$

Hence $S^n - A$ has two components.

ASSUME WE KNOW THE FORMULA FOR $k-1$ AND TRY TO PROVE IT FOR k.



Look at the reduced MV sequence for $(S^n - A_0; S^n - A_+, S^n - A_-)$

$$\begin{matrix} \widetilde{H}_{q+1}(S^n - A_+) \\ \oplus \\ \widetilde{H}_{q+1}(S^n - A_-) \end{matrix} \longrightarrow \widetilde{H}_{q+1}(S^n - A_0) \longrightarrow \widetilde{H}_q(S^n - A) \longrightarrow$$

0 (prev)

also 0.

So $\widetilde{H}_q(S^n - A) \cong \widetilde{H}_{q+1}(S^n - A_0)$

$$\cong \widetilde{H}_{q+1}(S^{n-k})$$

$$= \begin{cases} \mathbb{Z} & q+1 = n+k \\ 0 & \text{otherwise.} \end{cases}$$

This translates directly into the formula in the Theorem.

To complete the proof, must show every $x \in A$ is a frontier point for <u>both</u> components of $S^n - A$. (OFTEN NOT DONE IN TEXTS!)