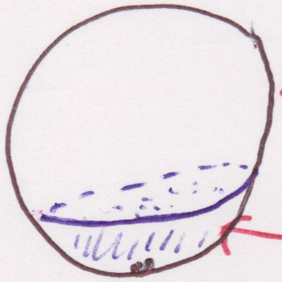
Preliminary observation Every point in S^k has a neighborhood base of open neighborhoods $\{U_j\}$ such that $U_j \cong$ open disk and $S^k - U_j \cong$ closed disk.



$S^k - U_j = \{\text{all } x \in \mathbb{R}^{k+1} \text{ s.t. } x_{k+1} \geq -1+\varepsilon\}$

$U_j = \{\text{all } x \in \mathbb{R}^{k+1} \text{ s.t. } x_{k+1} < -1+\varepsilon\}$

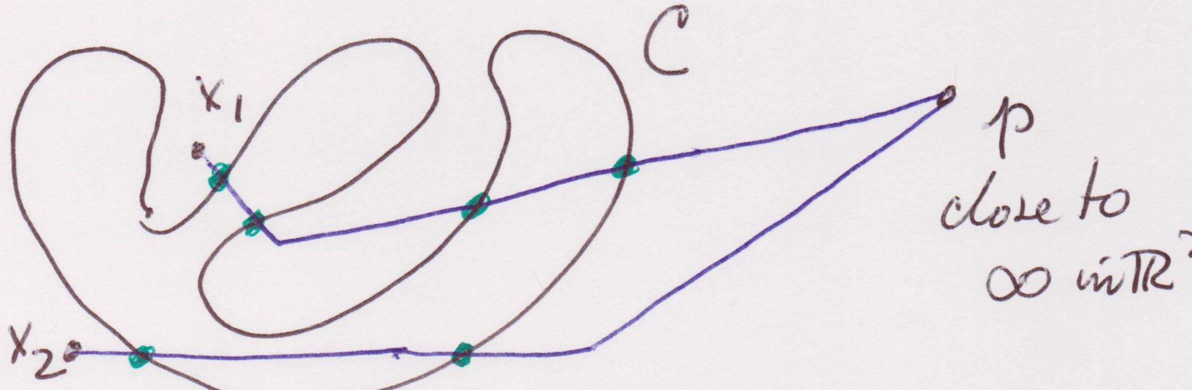
Now suppose $x \in A$. It is enough to show x is a limit point of U (to get the same result for V, switch the roles of U and V in the argument).

Suppose $x \in A$ is not a limit point of U. Since $A \cap U = \phi$, it follows there is an open neighborhood W_0 of x in A such that $A \cap W_0 = \phi$. Let $x \in W \subseteq W_0$ such that $W \cap A$ is an open $(n-1)$-disk and $\overset{=E}{A - W}$ is a closed $(n-1)$-disk. Then U and $V \cup W$ are nonempty open subsets s.t. $U \cap (V \cup W) = \phi$ and $U \cup (V \cup W) = S^n - E$.

Hence $S^n - E$ is disconnected. But $E \cong D^{n-1} \Rightarrow \tilde{H}_*(S^n - E) = 0 \Rightarrow S^n - E$ connected. <u>contradiction</u> Hence x must be a limit point of U.

<u>Question</u> If $C \subseteq \mathbb{R}^2$ is a regular piecewise smooth curve, how do we determine whether a point lies in the bounded or unbounded component of $\mathbb{R}^2 - C$?

Method sketched in exercises



Join p to x by a regular piecewise smooth curve which meets C transversely at smooth points. ~~Eq~~ Transversely means the tangent vectors are lin. independent. Count the number of crossing points

Even $\Longrightarrow$ x in the unbounded component

Odd $\Longrightarrow$ x in the bounded component

At each crossing point, curve jumps from one component of $\mathbb{R}^2 - C$ to the other.

~~See~~ fishmaze2.pdf for a more complex example.

<u>Theorem</u> (Invariance of Domain — <u>Brouwer</u>). Let U be open in $\mathbb{R}^n$, and suppose that $f: U \longrightarrow \mathbb{R}^n$ is continuous and 1-1. Then f is open (and hence is a homeo onto its image, which must also be open).

<u>Paraphrase</u> If $U, A \subseteq \mathbb{R}^n$ and U is open, $U \cong A$, then A is also open (i.e., one is a domain $\Leftrightarrow$ other is).

<u>Proof</u> It suffices to show that if the open ε-disk $N_\varepsilon(x) \subseteq U$, and $\overline{N_\varepsilon(x)} \subseteq U$ too, then f maps $N_\varepsilon(x)$ to an open subset of $\mathbb{R}^n$.

$$\boxed{S^n = \mathbb{R}^n \cup \{\infty\}}$$

This is true because $U = \bigcup_\alpha N_{\varepsilon(\alpha)}(x_\alpha)$ for suitable x_α, $\varepsilon(\alpha) > 0$. Let $D_\varepsilon(x) =$ closure of $N_\varepsilon(x)$, $\partial D_\varepsilon(x) =$ boundary of $N_\varepsilon(x)$.

Given $D_\varepsilon(x) \subseteq U$, let $A = f[\partial D_\varepsilon(x)]$, $B = f[D_\varepsilon(x)]$ and notice $f|A$, $f|B$ are homeomorphisms onto their images. By our previous results, $W = S^n - B$ is connected and $S^n - A$ has two components U and V. Assume $\infty \in V$ (without loss of generality), so $\infty \in W$ & W connected $\Rightarrow$ $W \subseteq V$. CLAIM: They are equal.

We know that $S^n = W \cup A \cup (B - A)$ where the summands are pairwise disjt.

Therefore $S^n - A = W \cup (B-A)$ where W, $B-A$ are disjoint and connected. It suffices to show $B-A \subseteq U$, for then

$$\left.\begin{array}{r}(B-A) \subseteq U \\ W \subseteq V\end{array}\right\} \quad (B-A) \cup W = U \cup V \Rightarrow$$

must have $B-A = U$, $W = V$.

But if $B-A$ is not contained in U, then it is contained in V and hence $S^n - A \subseteq V$, which contradicts $S^n - A = U \cup V$. So we also know $f[N_\varepsilon(x)] = B-A$ is open from this discussion, and hence as above it follows that f is open. $\blacksquare$

Example $\mathbb{R}^\infty$ = all $(x_1, x_2, \ldots)$ with only finitely many x_i nonzero. If H = set where $x_1 = 0$, then $H \cong \mathbb{R}^\infty$ but H is not open in $\mathbb{R}^\infty$.